Initial Algebras and Terminal Coalgebras

The Theory of Fixed Points of Functors

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5 Finitary Iteration in Enriched Settings

Example 5.3.4. Scott's model of the untyped λ-calculus. The formulas t of the λ-calculus have the form

$$t ::= x \mid tt \mid \lambda x.t,$$

where x ranges through a countable set V of variables. The meaning of $t_1 t_2$ is 'application': we evaluate t_1 (a function) in t_2. The meaning of $\lambda x.t$ is 'λ-abstraction': this function takes a value a and responds with $t[a/x]$, the term t in which x is substituted by a. A sound model of the λ-calculus can be obtained from every cpo D equipped with a split monomorphism

$$m\colon [D,D] \rightarrowtail D$$

with a splitting $e\colon D \twoheadrightarrow [D,D]$: given a valuation $v\colon V \to D$, the interpretation $[\![t]\!]_v$ of a λ-term t in D is defined as follows:

$$[\![x]\!]_v = v(x), \qquad [\![t_1 t_2]\!]_v = e([\![t_1]\!]_v)([\![t_2]\!]_v), \qquad [\![\lambda x.t]\!]_v = m(h),$$

where $h\colon D \to D$ is the function that maps $d \in D$ to $[\![t]\!]_{v[x:=d]}$ and $v[x := d]$ is the modificatication of v that maps x to d: $v[x := d](x) = d$ and $v[x := d](y) = v(y)$ for every $y \neq x$. It is not difficult to prove that h is continuous and that the above interpretation makes D a model of the λ-calculus (cf. [3, Thm. 3.2.12]).

Note that for a set D that is not a singleton, there is no injection $[D,D] \rightarrowtail D$, since $[D,D]$ has larger cardinality than D.

Scott [4] decided to use the cartesian closed category of continuous lattices to obtain a model of the λ-calculus. But Smyth and Plotkin [5] made it clear that working in $\mathsf{CPO}_\perp$ is sufficient (and simpler). In fact, consider the locally continuous functor

$$F\colon \mathsf{CPO}_\perp^{\mathsf{op}} \times \mathsf{CPO}_\perp \to \mathsf{CPO}_\perp \quad \text{with} \quad F(X,Y) = [X,Y]_\perp,$$

which adds a new least element to the cpo $[X,Y]$. If D is the initial algebra for F^E (Theorem 5.3.1), then

$$D \cong [D,D]_\perp$$

is a non-trivial model of λ-calculus (cf. Abramsky [1], Abramsky and Ong [2]). Indeed, we obtain a split monomorphism in CPO:

$$m = ([D,D] \hookrightarrow [D,D]_\perp \xrightarrow{\cong} D).$$

7 Terminal Coalgebras as Algebras, Initial Algebras as Coalgebras

Proof sketch for Prop. 7.3.5. In the first display the flat equation morphism is

$$e = F\mathsf{inr} + \mathsf{id}_A \colon FA + A \to F(FA + A) + A.$$

References

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[3] Hendrik P. Barendregt. Functional programming and lambda calculus. In Jan van Leeuwen, editor, *Handbook of Theoretical Computer Science, Volume B: Formal Models and Semantics*, pages 321–363. Elsevier and MIT Press, 1990.

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