Essential Mathematics for the Political and Social Research

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Cambridge University Press

Lecture Slides, Chapter 2: Analytic Geometry

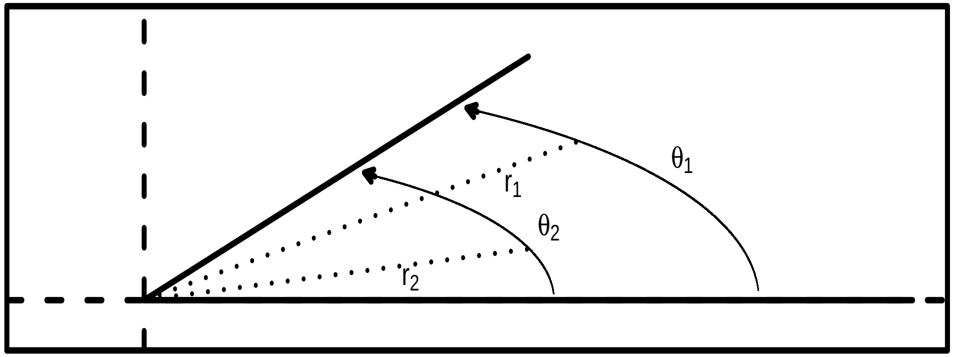
Chapter 2 Objectives

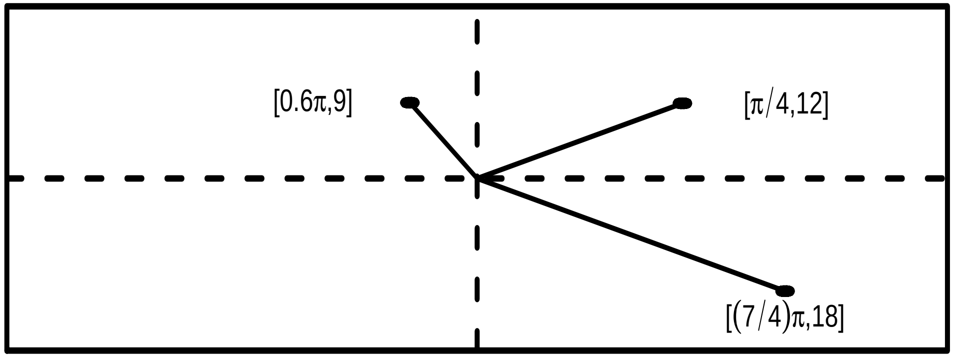
- This chapter introduces the basic principles of analytic geometry and trigonometry.
- Even if one is not studying some spatial phenomenon, such functions and rules can still be relevant.
- Trigonometric functions occur in formal models, game theory, spatial analysis of political and social phenomena, and elsewhere.
- We will also expand beyond Cartesian coordinates and look at polar coordinate systems.

Radian Measurement and Polar Coordinates

- There is a second system that can be employed when it is convenient to think in terms of a movement around a circle.
- *Radian Measurement* treats the angular distance around the center of a circle (also called the pole or origin for obvious reasons) in the *counterclockwise* direction as a proportion of 2π.
- Radian measurement is based on the formula for the circumference of a circle: $c = 2\pi r$, where r is the radius.
- So it is convenient to describe angles around the circle with the range of numbers from 0 to 2π.
- If we assume a unit radius ($r = 1$), then the linkage is obvious: from a starting point, moving 2π around the circle (a complete revolution) returns us to the radial point where we began.

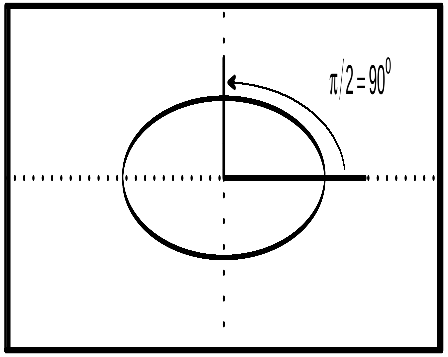
Polar Coordinates

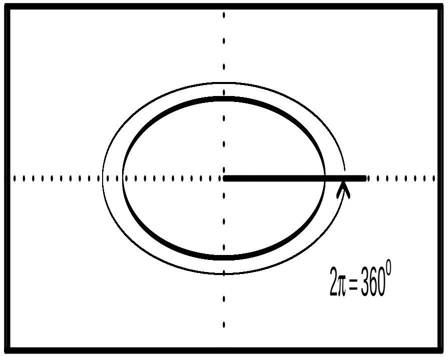




Degrees as an Alternative Measurement of Angles

- Most are more comfortable with a different measure of angles, degrees, which are measured from 0 to 360.
- So 2π is equal to 360^o in this context: we translate between radians and degrees for angles simply by multiplying a radian measure by $360/2\pi$ or a degree measure by $2\pi/360$.
- Degrees is more arbitrary and less convenient mathematically.





Cycling and Direction

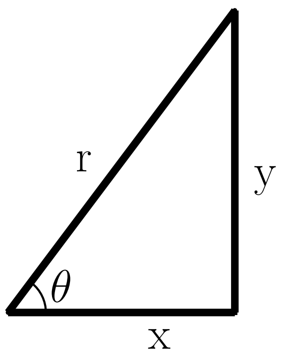
- Negative radian measurement also makes sense because sometimes it will be convenient to move in the negative in a clockwise direction.
- The system also "restarts" at 2π in this direction as well, meaning the function value becomes zero.
- Positive and negative angular distances have interesting equalities, such as $-\frac{3}{2}\pi = \frac{1}{2}\pi$.
- Table converting between polar and cartesian:

Polar to Cartesian	Cartesian to Polar
$x = r\cos(\theta)$	$\theta = \arctan(\frac{y}{x})$
$y = r\sin(\theta)$	$r = \sqrt{x^2 + y^2}$

What Is Trigonometry?

- The topic of trigonometry started as the study of triangles.
- Initial interest is in the angle θ of a right triangle as shown in the figure at the right.
- The Greeks were interested in the ratios of the sizes of the sides (r, y, and x) of this triangle, and they noticed that these could be related to the angle of θ.

Right Triangle



Right Triangles

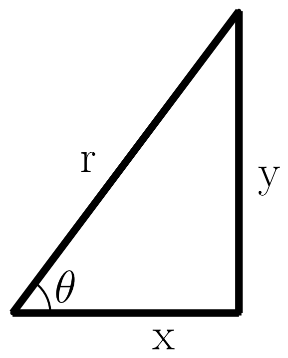
- The Greeks were very interested in the properties of *right* triangles in particular and considered them to be important cases.
- The Pythagorean Theorem is a the relevant mathematical assertion here, but there are additional relations of interest.
- The basic relations are ratios of sides related to acute angles.
- These have technical sounding names but are basic relations.

Basic Trigonometric Functions

- Consider θ, x, y, and r from the figure.
- There are six core definitions for the even functions cosine and secant, as well as the odd functions sine, cosecant, tangent, and cotangent, given by:

$$\sin(\theta) = \frac{y}{r} \qquad \csc(\theta) = \frac{r}{y}$$
$$\cos(\theta) = \frac{x}{r} \qquad \sec(\theta) = \frac{r}{x}$$
$$\tan(\theta) = \frac{y}{x} \qquad \cot(\theta) = \frac{x}{y}.$$

Right Triangle



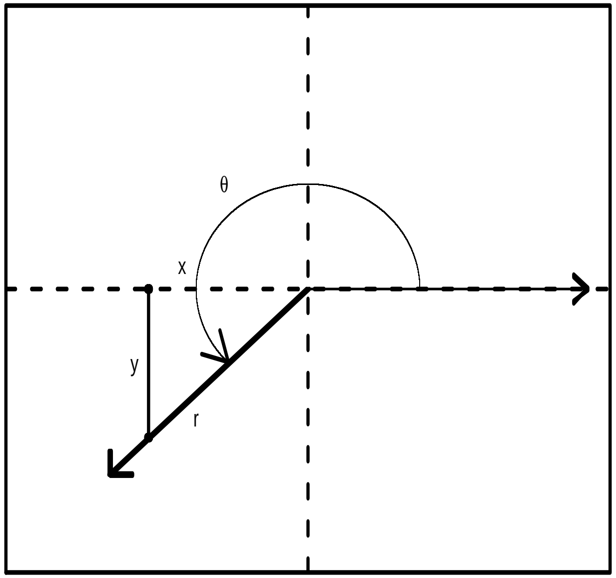
Reciprical Functions

- There are also reciprocal relations:
  - $\sin(\theta) = \csc(\theta)^{-1}$,
  - $\cos(\theta) = \sec(\theta)^{-1}$,
  - $\tan(\theta) = \cot(\theta)^{-1}$.
- Also from these inverse properties we get:
  - $\sin(\theta)\csc(\theta) = 1$,
  - $\cos(\theta)\sec(\theta) = 1$,
  - $\tan(\theta)\cot(\theta) = 1$.
- This implies that, of the six basic trigonometric functions, sine, cosine, and tangent are the more fundamental.

Generalization

- Defining with a triangle necessarily restricts us to acute angles.
- To expand the scope to a full 360^o we need to be broad and use a full Cartesian coordinate system.
- The next figure shows the more general setup where the triangle defined by sweeping the angle θ counterclockwise from the positive side of the x-axis along with an r value gives the x and y distances.
- Now the trigonometric values are defined exactly in the way that they were in the table above except that x and y can now take on negative values.

General Trigonometric Setup



Common Angular Value

θ^o	0	45	90	135	180	225	270	315	360
$\sin(\theta)$	0	$\sqrt{2}/2$	1	$\sqrt{2}/2$	0	$-\sqrt{2}/2$	-1	$-\sqrt{2}/2$	0
$\csc(\theta)$	–	$\sqrt{2}$	1	$\sqrt{2}$	–	$-\sqrt{2}$	-1	$-\sqrt{2}$	–
$\cos(\theta)$	1	$\sqrt{2}/2$	0	$-\sqrt{2}/2$	-1	$-\sqrt{2}/2$	0	$\sqrt{2}/2$	1
$\sec(\theta)$	1	$\sqrt{2}$	–	$-\sqrt{2}$	-1	$-\sqrt{2}$	–	$\sqrt{2}$	1
$\tan(\theta)$	0	1	–	-1	0	1	–	-1	0
$\cot(\theta)$	–	1	0	-1	–	1	0	-1	–

(Here the notation "–" denotes that the value is undefined and comes from a division by zero operation.)

Relations

- The Pythagorean Theorem gives some very useful and well-known relations between the basic trigonometric functions:

$$\sin(\theta)^2 + \cos(\theta)^2 = 1 \qquad \csc(\theta)^2 - \cot(\theta)^2 = 1 \qquad \sec(\theta)^2 - \tan(\theta)^2 = 1.$$

- Addition or subtraction of angle values provides the following rules:

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$

$$\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b)$$

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$$

$$\tan(a+b) = \frac{\tan(a) + \tan(b)}{1 - \tan(a)\tan(b)}$$

$$\tan(a-b) = \frac{\tan(a) - \tan(b)}{1 + \tan(a)\tan(b)}.$$

Reverse Relationships

- Consider:

$$
\begin{aligned}
a &= \arcsin(b), && \text{if } b = \sin(a), && \text{provided that } -\frac{\pi}{2} \leq a \leq \frac{\pi}{2} \\
a &= \arccos(b), && \text{if } b = \cos(a), && \text{provided that } \quad 0 \leq a \leq \pi \\
a &= \arctan(b), && \text{if } b = \tan(a), && \text{provided that } -\frac{\pi}{2} \leq a \leq \frac{\pi}{2}.
\end{aligned}
$$

- Confusingly, these also have alternate terminology: $\arcsin(b) = \sin^{-1}(b)$, $\arccos(b) = \cos^{-1}(b)$, and $\arctan(b) = \tan^{-1}(b)$, which can easily be confused with the inverse relationships.

Radian Measures for Trigonometric Functions

- These functions "reset" at 360^o, but from what we have seen of the circle it makes sense to do this at 2π for sine and cosine or at π for tangent:

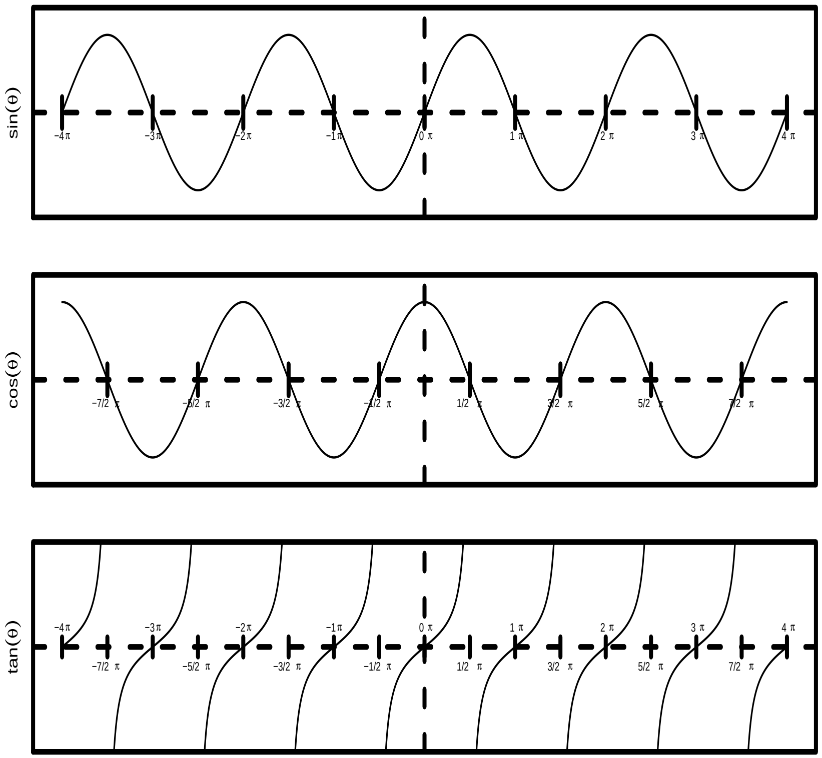
$$\sin(\theta + 2\pi) = \sin(\theta) \qquad \csc(\theta + 2\pi) = \csc(\theta)$$

$$\cos(\theta + 2\pi) = \cos(\theta) \qquad \sec(\theta + 2\pi) = \sec(\theta)$$

$$\tan(\theta + \pi) = \tan(\theta) \qquad \cot(\theta + \pi) = \cot(\theta).$$

- This property means that these functions "cycle" for increasing theta values. This is illustrated in the following figure.

Illustrating Basic Trigonometric Function in Radians



Complements

- We say that two angles are complementary if they sum to 90^o, that is, $\pi/2$.
- This gives the following interesting relationships:

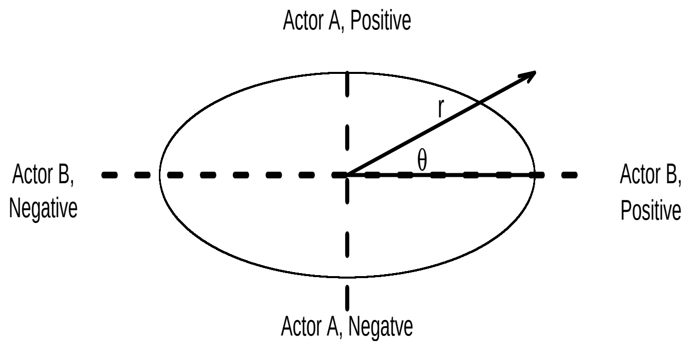
$$\sin\left(\frac{\pi}{2}-\theta\right)=\cos(\theta) \qquad \cos\left(\frac{\pi}{2}-\theta\right)=\sin(\theta)$$

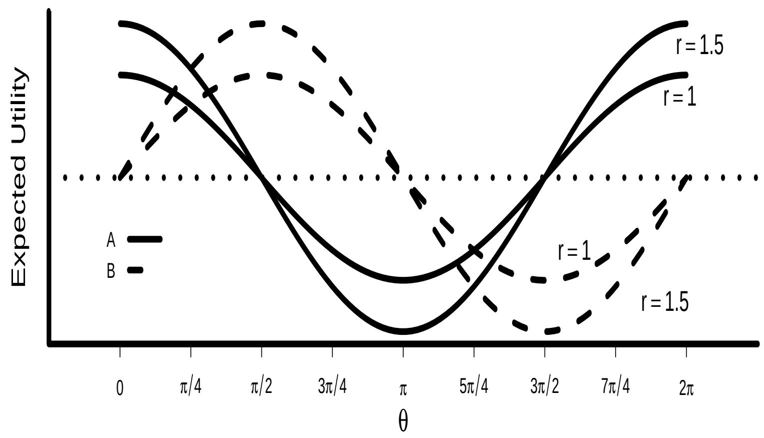
$$\tan\left(\frac{\pi}{2}-\theta\right)=\cot(\theta) \qquad \cot\left(\frac{\pi}{2}-\theta\right)=\tan(\theta).$$

An Expected Utility Model of Conflict Escalation

- A model used by Bueno De Mesquita and Lalman (1986), and more recently Stam (1996), specifies the expected utility for conflict escalation radiating outward from the origin on a two-dimensional polar metric whereby θ specifies an angular direction and r specifies the intensity or stakes of the conflict.
- The y-axis gives Actor A's utility, where the positive side of the origin gives expected utility increasing outcomes (A continuing to fight) and the negative side of the origin gives expected utility decreasing outcomes (A withdrawing).
- Likewise, the x-axis gives Actor B's expected utility, where the positive side of the origin gives expected utility increasing outcomes (B continuing to fight) and the negative side of the origin gives utility decreasing outcomes (B withdrawing).
- The value of θ determines "who wins" and "who loses," where it is possible to have both actors receive positive (or negative) expected utility.
- The model thus asserts that nations make conflict escalation decisions based on these expected utilities for themselves as well as those assessed for their adversary.

Views of Conflict Escalation





An Expected Utility Model of Conflict Escalation

- Even though the circle depiction is a very neat mnemonic about trade-offs, it does not show very well the consequences of θ to an individual actor.
- If we transform to Cartesian coordinates, then we get the illustration in the bottom panel of the figure.
- We can directly see the difference in expected utility between Actor A (solid line) and Actor B (dashed line) at some value of θ by taking the vertical distance between the curves at that x-axis point.
- Since r is the parameter that controls the value of the conflict (i.e., what is at stake here may be tiny for Ecuador/Peru but huge for U.S./U.S.S.R.), then the circle in the upper panel of the figure gives the universe of possible outcomes for one scenario.

An Expected Utility Model of Conflict Escalation

- The points where the circle intersects with the axes provide absolute outcomes: Somebody wins or loses completely and somebody has a zero outcome.
- We see in the lower panel that going from $r = 1$ to $r = 1.5$ magnifies the scope of the positive or negative expected utility to A and B.
- Increasing r has different effects on the expected utility difference for differing values of θ, something that was not very apparent from the polar depiction because the circle obviously just increased in every direction.

Conic Sections and Some Analytical Geometry

- Start with a reminder of the definition of a circle in analytical geometry terms.
- Intuitively we know that for a center c and a radius r, a circle is the set of points that are exactly r distance away from c in a two-dimensional Cartesian coordinate system.
- More formally:

  A circle with radius $r > 0$ and center at point $[x_c, y_c]$ is defined by the quadratic, multi-variable expression

  $$r^2 = (x - x_c)^2 + (y - y_c)^2.$$

- This is "multivalued" because x and y both determine the function value.

The Unit Ciricle

- The most common form is the unit circle which has radius $r = 1$ and is centered at the origin $(x_c = 0, y_c = 0)$.
- This simplifies the more general form down to merely
$$1 = x^2 + y^2.$$
- Suppose we break this quadratic form out and rearrange according to
$$\begin{aligned} r^2 &= (x - x_c)^2 + (y - y_c)^2 \\ &= x^2 - 2x_c x + x_c^2 + y^2 - 2y_c y + y_c^2 \end{aligned}$$
and then make the variable changes $a = -2x_c$ and $b = -2y_c$. Then
$$\begin{aligned} &= x^2 + ax + \frac{a^2}{4} + y^2 + by + \frac{b^2}{4} \\ &= x^2 + ax + y^2 + by + c, \end{aligned}$$
defining in the last step that $c = \frac{a^2}{4} + \frac{b^2}{4} - r^2$.

The Unit Ciricle

- Note here that r, c, x_c, and y_c are all fixed, known constants.
- r can be backed out of c above as $r = \frac{1}{2}\sqrt{a^2 + b^2 - 4c}$.
- Recall that r is the radius of the circle, so we can look at the size of the radius that results from choices of the constants.
- First note that if $a^2 + b^2 = 4c$, then the circle has no size and condenses down to a single point.
- If $a^2 + b^2 < 4c$, then the square root is a complex number (and not defined on the Cartesian system we have been working with), and therefore the shape is undefined.
- So the only condition that provides a circle is $a^2 + b^2 > 4c$, and thus we have a simple test.

The Unit Ciricle

- Testing for a Circular Form: Does this quadratic equation define a circle?

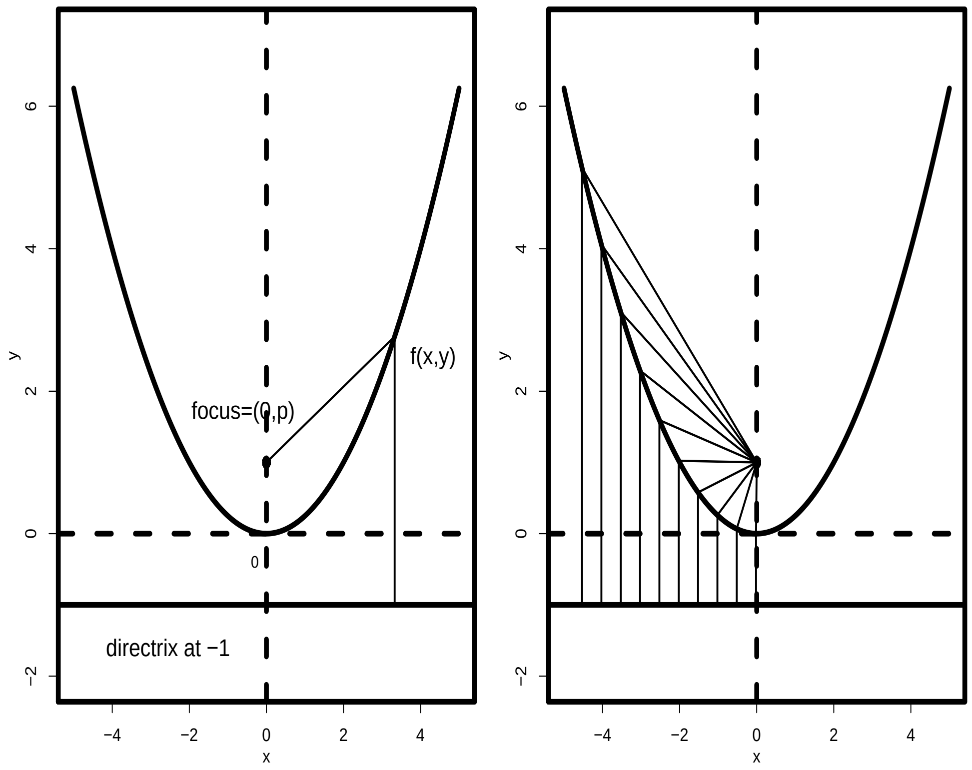
$$x^2 + y^2 - 12x + 12y + 18 = 0$$

- Here $a = -12$, $b = 12$, and $c = 18$. So $a^2 + b^2 = 144 + 144 = 288$ is greater than $4c = 4 \cdot 18 = 72$, and this is therefore a circle.

The Parabola

- The parabola is produced by slicing through a three-dimensional cone with a two-dimensional plane such that the plane goes through the flat bottom of the cone.
- A parabola in two dimensions is the set of points that are equidistant from a given (fixed) point and a given (fixed) line.
- The point is called the focus of the parabola, and the line is called the directrix of the line.
- The next figure shows the same parabola in two frames for a focus at $[0, p]$ and a directrix at $y = -p$, for $p = 1$.
- The first panel shows the points labeled and a single equidistant principle illustrated for a point on the parabola.
- The second frame shows a series of points on the left-hand side of the access with their corresponding equal line segments to p and d.

Characteristics of the Parabola



More Mathematical Details on the Parabola

- There is a formula that dictates the relationship between the y-axis points and the x-axis points for the parabola:
$$y = \frac{x^2}{4p}.$$
- This is a parabola facing upward and continuing toward greater positive values of y for positive values of p.
- Making p negative, then the focus will be below the x-axis and the parabola will face downward.
- Swapping x and y in the definitional formula, then the parabola will face left or right (depending on the sign of p).
- A more general form of a parabola is for the parabola whose focus is not on one of the two axes:
$$(y - y') = \frac{(x - x')^2}{4p},$$
which will have a focus at $[x', p + y']$ and a directrix at $y = y' - p$.

Presidential Support as a Parabola

- Stimson (1976) found that early post-war presidential support scores fit a parametric curve.
- Presidents begin their 4 year term with large measures of popularity, which declines and then recovers before the end of their term, all in the characteristic parabolic form.
- Stimson took Gallup poll ratings of full presidential terms from Truman's first term to Nixon's first term and "fits" (estimates) parabolic forms that best followed the data values.
- Fortunately, Gallup uses the same question wording for over 30 years, allowing comparison across presidents.

Presidential Support as a Parabola

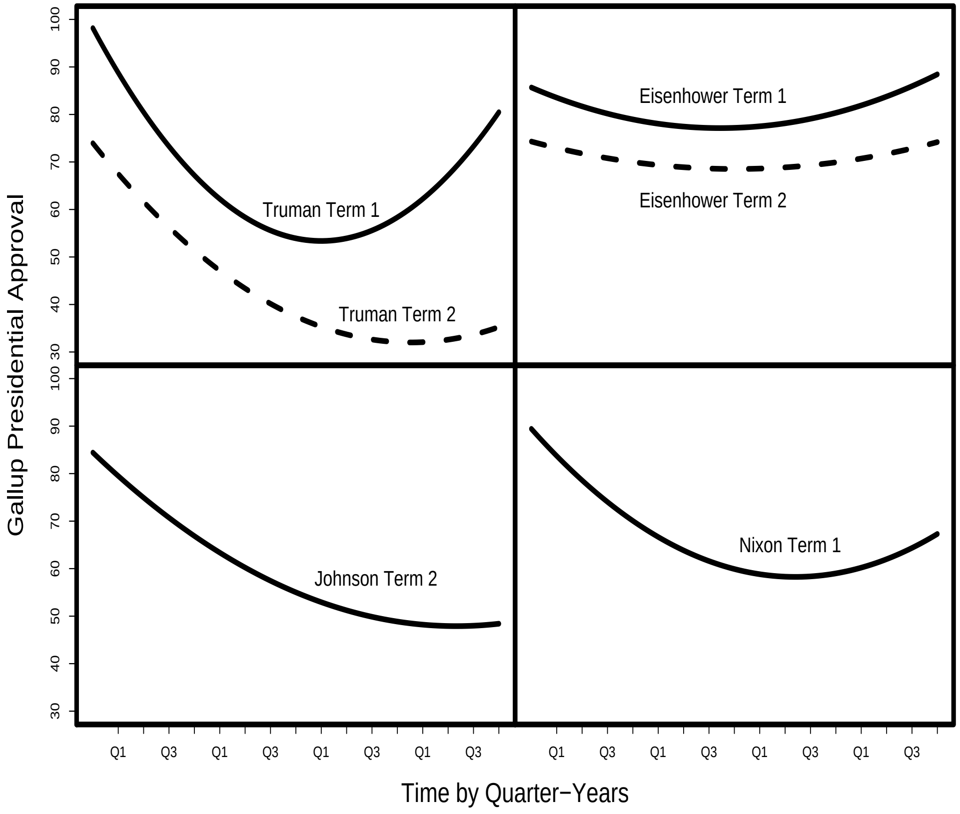
- Using the general parabolic notation above, the estimated parameters are:

	y'	x'	$1/4p$
Truman Term 1	53.37	2.25	8.85
Truman Term 2	32.01	3.13	4.28
Eisenhower Term 1	77.13	1.86	2.47
Eisenhower Term 2	68.52	2.01	1.43
Johnson Term 2	47.90	3.58	2.85
Nixon Term 1	58.26	2.60	4.61

Presidential Support as a Parabola

- Since y' is the lowest point in the upward facing parabola, we can use these values to readily see low ebbs in a given presidency and compare them as well.
- Not surprisingly, Eisenhower did the best in this regard.
- It is interesting to note that, while the forms differ considerably, the hypothesized shape bears out (Stimson gave a justification of the underlying statistical work in his article).
- Regardless of political party, wartime status, economic conditions, and so on, there was a persistent parabolic phenomenon for approval ratings of presidents across the four years.

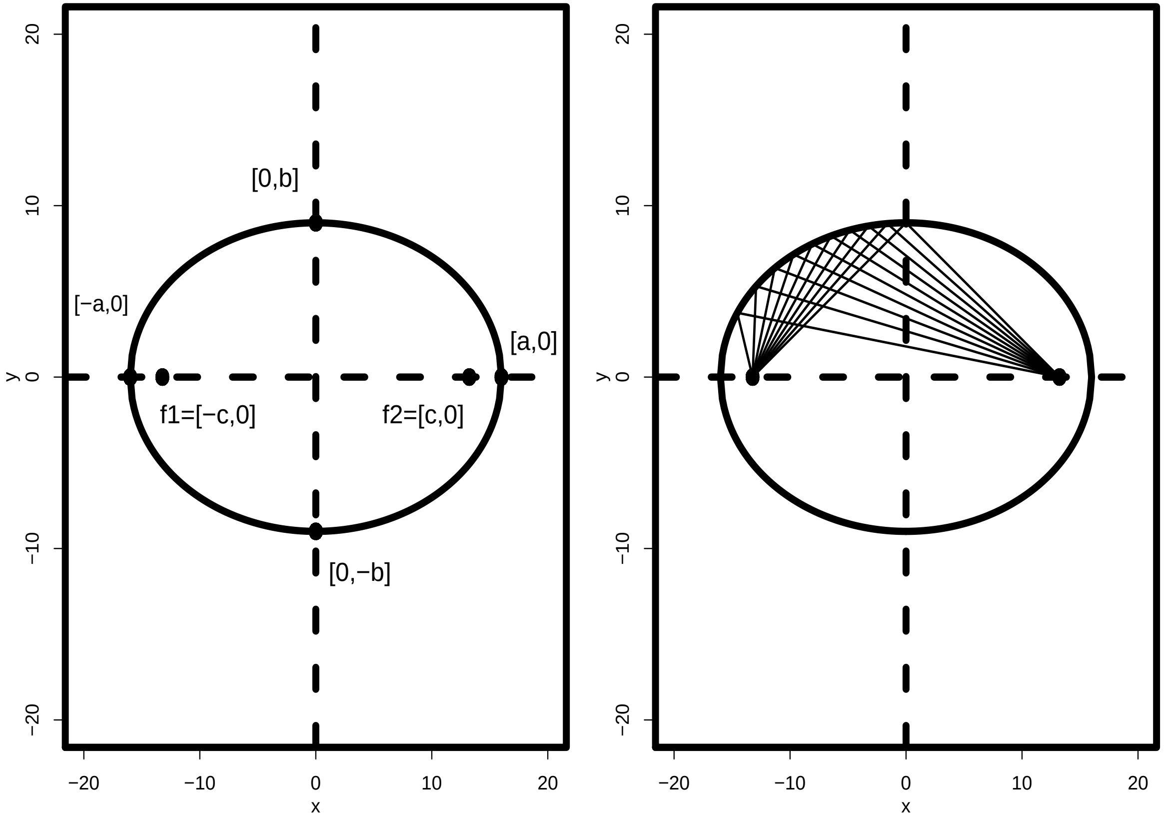
Parabolic Presidential Popularity Figure



The Ellipse

- Another fundamental conic section is produced by slicing a two-dimensional plane through a three-dimensional cone such that the plane *does not* cut through the flat base of the cone.
- More precisely, an ellipse in two dimensions is the set of points whose summed distance to two points is a constant.
- For two foci f_1 and f_2, each point p_i on the ellipse has $|p_i - f_1| + |p_i - f_2| = k$.
- Consider the figure. . .

Characteristics of an Ellipse



The Ellipse

- Things are much easier if the ellipse is set up to rest on the axes, but this is not a technical necessity.
- Suppose that we define the foci as resting on the x-axis at the points $[-c, 0]$ and $[c, 0]$.
- Then with the assumption $a > b$, we get the standard form of the ellipse from:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ where } \ c = \sqrt{a^2 - b^2}.$$

.

- This form is pictured in the two panels of the figure for $a = 16$, $b = 9$.
- The first panel shows the two foci as well as the four vertices of the ellipse where the ellipse reaches its extremum in the two dimensions at $x = \pm 16$ and $y = \pm 9$.
- The second panel shows for selected points in the upper left quadrant the line segments to the two foci.

The Ellipse

- For any one of these points along the ellipse where the line segments meet, the designated summed distance of the two segments must be a constant.
- Can we determine this sum easily?
- Each of the four vertices must also satisfy this condition.
- Use the Pythagorean Theorem and pick one of them:

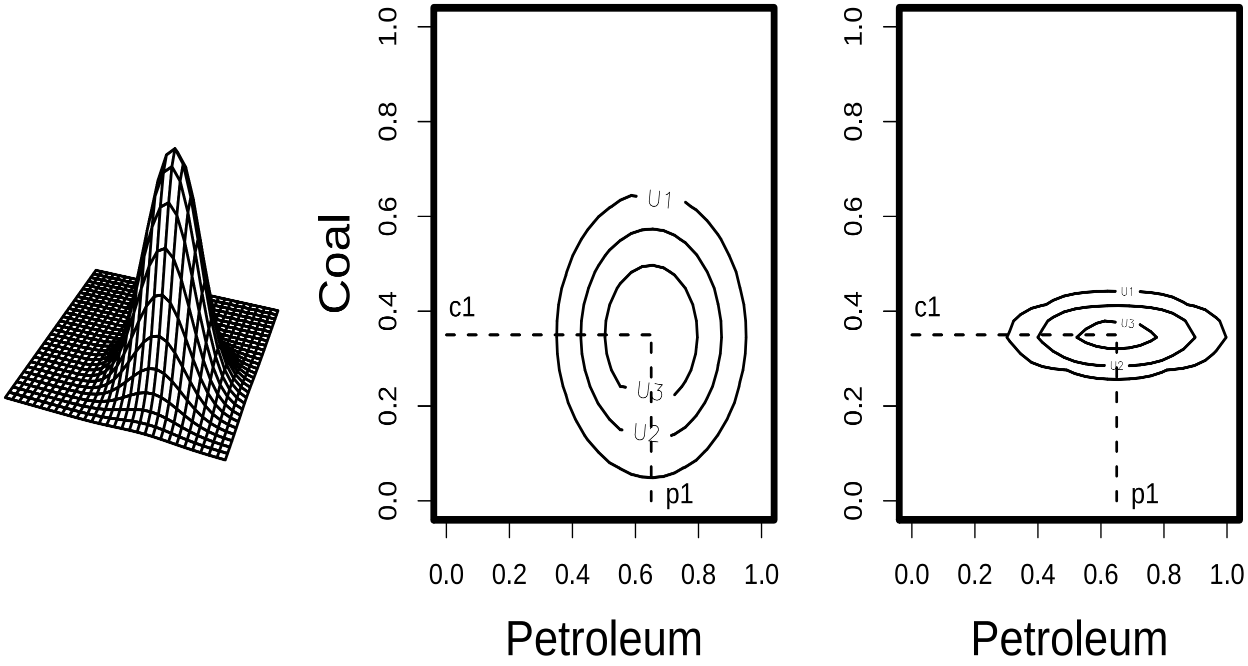
$$\frac{k}{2} = \sqrt{c^2 + b^2} = \sqrt{(\sqrt{a^2 - b^2})^2 + b^2} = a.$$

- Because this hypotenuse is only one-half of the required distance, we know that $k = 2a$, where a is greater than b.
- If one picks any of the paired line segments in the second panel of the figure and flattens them out down on the x-axis below, they will exactly join the two x-axis vertices.
- This is called the major axis of the ellipse the other one is called the minor axis of the ellipse.

Elliptical Voting Preferences

- Suppose that Congress needs to vote on a budget for research and development spending that divides total possible spending between coal and petroleum.
- A hypothetical member of Congress is assumed to have an *ideal* spending level for each of the two projects that trades off spending in one dimension against spending in the other.
- As an example, the highest altitude ideal point, and therefore the mode of the preference structure, is located at $[petroleum = 0.65,\ coal = 0.35]$ on a standardized metric (i.e., dollars removed).
- The figure here shows the example representative's utility preference in two ways: a three-dimensional wire-frame drawing that reveals the dimensionality now present over the petroleum/coal grid (the first panel), and a contour plot that illustrates the declining levels of utility preference, U_1, U_2, U_3 (the second panel).
- The further away from $[0.65,\ 0.35]$, the less happy the individual is with any proposed bill: She has a lower returned utility of a spending level at $[0.2, 0.8]$ than at $[0.6, 0.4]$.
- By this means, any point outside a given contour provides less utility than all the points inside this contour, no matter what direction from the ideal point.

Multidimensional Issue Preference



Elliptical Voting Preferences

- Suppose our example legislator was adamant about her preference on coal and somewhat flexible about petroleum.
- Then the resulting generalization of circular preferences in two dimensions would be an ellipse (third panel) that is more elongated in the petroleum direction.
- The fact that the ellipse is a more generalized circle can be seen by comparing the equation of a circle centered at the origin with radius 1 ($1 = x^2 + y^2$) to the standard form of the ellipses above.
- If $a = b$, then $c = 0$ and there is only one focus, which would collapse the ellipse to a circle.

The Hyperbola

- The third fundamental conic section comes from a different visualization.
- Suppose that instead of one cone we had two that were joined at the tip such that the flat planes at opposite ends were parallel.
- We could then take this three-dimensional setup of two cones "looking at each other" and slice through it with a plane that cuts equally through each of the flat planes for the two cones.
- This will produce a hyperbola, which is the set of points such that the *difference* between two foci f_1 and f_2 is a constant.

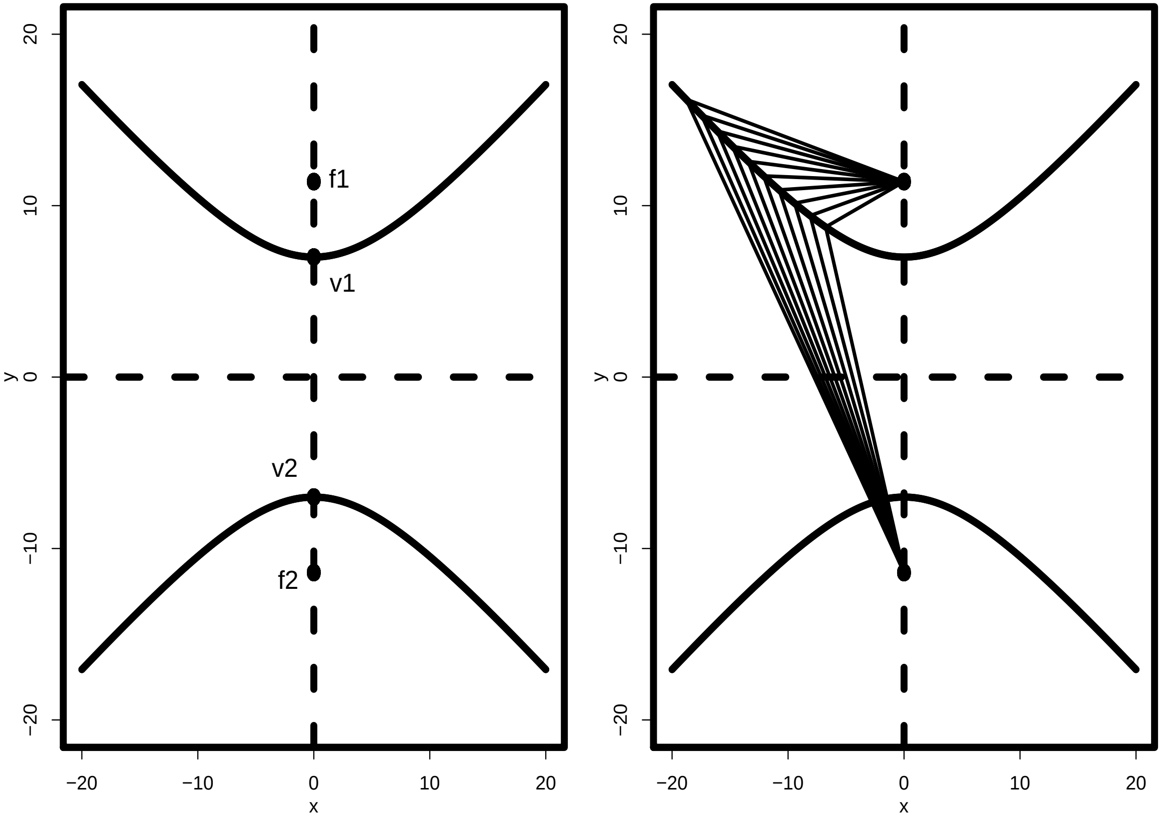
The Hyperbola

- We saw with the ellipse that $|p_i - f_1| + |p_i - f_2| = 2a$, but now we assert that $|p_i - f_1| - |p_i - f_2| = 2a$ for each point on the hyperbola.
- If the hyperbola is symmetric around the origin with open ends facing vertically, then $f_1 = c$, $f_2 = -c$, and the standard form of the hyperbola is given by:

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1, \text{ where } c = \sqrt{a^2 + b^2}.$$

- Notice that this is similar to the equation for the ellipse except that the signs are different.

Characteristics of a Hyperbola



The Hyperbola

- A hyperbola is actually two separate curves in 2-space, shown in the figure for $a = 9$ and $b = 8$ in the standard form.
- As with the previous two figures, the first panel shows the individual points of interest, the foci and vertexes, and the second panel shows a subset of the segments that define the hyperbola.
- Unfortunately these segments are not quite as visually intuitive as they are with the ellipse, because the hyperbola is defined by the difference of the two connected segments rather than the sum.

Hyperbolic Discounting in Evolutionary Psychology and Behavioral Economics

- There is a long literature that seeks to explain why people make decisions about immediate versus deferred rewards.

- The classic definition of this phenomenon is that of Böm-Bawerk (1959): "Present goods have in general greater subjective value than future (and intermediate) goods of equal quantity and quality."

- While it seems clear that everybody prefers $100 today rather than in one year, it is not completely clear what larger value is required to induce someone to wait a year for the payment.

- That is, what is an appropriate discounting schedule that reflects general human preferences and perhaps accounts for differing preferences across individuals or groups.

- This is a question of human and social cognitive and emotional affects on economic decisions.

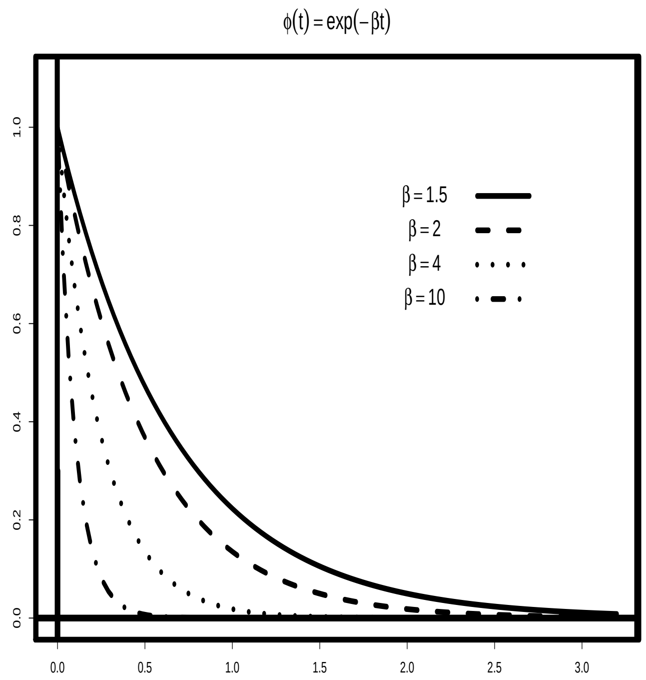
Hyperbolic Discounting in Evolutionary Psychology and Behavioral Economics

- One way to mathematically model these preferences is with a declining curve that reflects discounted value as time increases.
- A person might choose to wait one week to get $120 instead of $100 but not make such a choice if the wait was one year.
- The basic model is attributed to Samuelson (1937), who framed these choices in discounted utility terms by specifying (roughly) a positive constant followed by a positive discount rate declining over time.
- This can be linear or curvilinear, depending on the theory concerned.
- The first mathematical model for such discounted utility is the exponential function:

$$\phi(t) = \exp(-\beta t),$$

  where t is time and β is a parameter that alters the rate of decay.
- This is illustrated in the following figure.

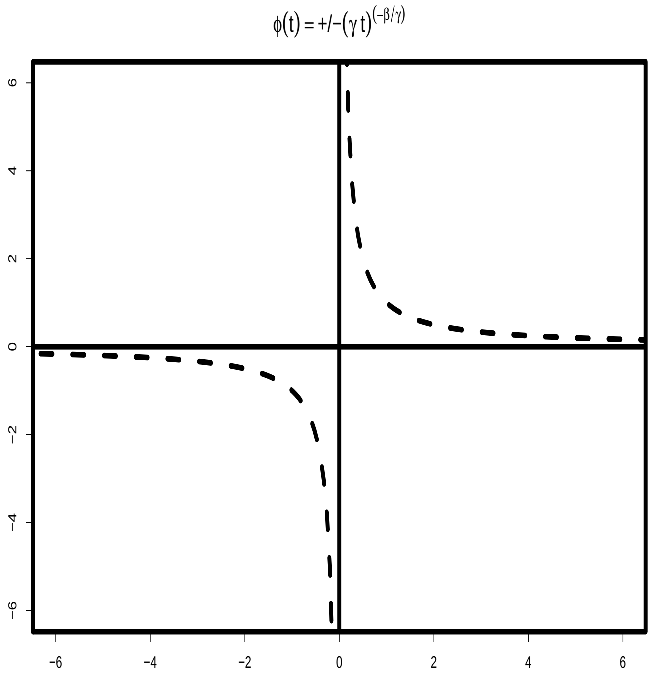
Exponential Curves



Hyperbolic Discounting in Evolutionary Psychology and Behavioral Economics

- The exponential function does not explain discounting empirically because with only one parameter to control the shape it is not flexible enough to model observed human behavior.
- Recall that the exponentiation here is just raising 2.718282 to the $-\beta t$ power.
- Because there is nothing special about the constant, why not parameterize this as well?
- We can also add flexibility to the exponent by moving t down and adding a second parameter to control the power; the result is
$$\phi(t) = \pm(\gamma t)^{-\beta/\gamma}.$$

Derived Hyperbola Forma



Hyperbolic Discounting in Evolutionary Psychology and Behavioral Economics

- This is actually a hyperbolic functional form for $t = x$ and $\phi(t) = y$, in a more general form than we saw above.
- To illustrate this as simply as possible, set $\gamma = 1$, $\beta = 1$, and obtain the plot.
- So this is just like the hyperbola that we saw before, except that it is aligned with axes such that the two curves are confined to the first and third quadrants.
- The phenomenon that we are interested in (discounting) is confined to $\mathfrak{R}^+$, so we only need the form in the upper right quadrant.
- Also, by additively placing a 1 inside the parentheses we can standardize the expression such that discounting at time zero is 1 (i.e., people do not have a reduction in utility for immediate payoff) for every pair of parameter values.
- Loewenstein and Prelec (1992) used the constant to make sure that the proposed hyperbolic curves all intersect at the point (0,0.3).
- The constant is arbitrary, and the change makes it very easy to compare different hyperbolic functions.

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- Interestingly, as γ gets very small and approaches zero, the hyperbolic function above becomes the previous exponential form.