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Problems for Chapter 21 of *Advanced Mathematics for Applications*

Linear Operators in Infinite-Dimensional Spaces

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Notation: In the statement of the following problems the word "operator" is to be taken to mean "linear operator" unless explicitly stated. The following notation is used in the problems that follow:

- $L^p(a, b)$ with a and b finite or infinite, denotes the space of functions $u(x)$ over the interval (a, b) such that
$$\|u\|^p = \int_a^b |u|^p \, dx < \infty .$$
In particular, L^2 denotes the space of square integrable functions.
- $C^k[a, b]$ is the set of functions continuous with their first k derivatives on the closed interval $a \le x \le b$. These spaces are normed with the sup norm.
- ℓ^p denotes the space of real or complex sequences of numbers $c = \{c_n\} = (c_0, c_1, c_2, , \ldots)$ such that
$$\|c\|^p = \sum_{n=0}^{\infty} |c_n|^p < \infty .$$

see Boccara example 1 p 273

1 General

1. Let S, S_1 and S_2 be linear vector spaces over the same scalar field, and let A_1 be a linear operator from S to S_1 and A_2 a linear operator from S_1 to S_2. Show that the composition $\mathsf{A}_2\mathsf{A}_1$ from S to S_2 is a linear operator.
2. Define an operator A transforming functions $u(x)$ belonging $L^2(-a, a)$ with $0 < a < \infty$ into functions $v(x) \in L^2(-a, a)$ according to the rule
$$v(x) = \mathsf{A}u(x) = \frac{1}{2}\left[u(x) + u(-x)\right] .$$
Find the domain and range of A.
3. Let A be an operator from a linear vector space S into a linear vector space S'. Given a subspace $S_1 \subset S$, show that $\mathsf{A}S_1$ is a subspace of S'.
4. Let A be a linear operator from a space S_1 to a space S_2. Prove that: (a) If M is a linear manifold of S_1, its image $\mathsf{A}M$ is a linear manifold of S_2; (b) If A is invertible and N is a linear manifold of S_2, then $\mathsf{A}^{-1}N$ is a linear manifold of S_1.
5. Let A be an arbitrary operator from a Hilbert space into itself and let a and b be two complex number such that $|a| = |b|$. Show that $a\mathsf{A} + b\mathsf{A}^*$ is normal.

6. Let A be an operator from a linear vector space S into a linear vector space S'. Let $S_1 \subset S$ be a subspace of S such that $S_1 \bigcap \mathcal{N}(\mathsf{A}) = \emptyset$. Show that A operating only on elements of S_1 is an injective operator. Show also that the spaces S_1 and $\|sfAS_1$ are isomorphic.

7. Let A be an operator from a Banach space S into itself with the property that $\sum_{n=0}^{\infty} \mathsf{A}^n u$ converges for each $u \in S$. Show that $\mathsf{I} - \mathsf{A}$ is injective and that its range coincides with S.

8. In the space ℓ^1 define the shift operator A by

$$\mathsf{A}c \equiv \mathsf{A}\,(c_0,\, c_1,\, \ldots,\, c_n,\, \ldots) = (c_1,\, c_2,\, \ldots,\, c_{n+1},\, \ldots)\,.$$

What is the norm of A so defined? What is its range? Is the range dense in ℓ^1?

9. In the space ℓ^1 define an operator A by

$$\mathsf{A}c \equiv \mathsf{A}\,(c_0,\, c_1,\, \ldots,\, c_n,\, \ldots) = (0,\, 1\,c_1,\, 2\,c_2,\, \ldots,\, n\,c_n,\, \ldots)\,.$$

What are the domain and the range of A so defined? Is the operator bonded? Is its domain dense in ℓ^1?

10. In the space $C[0,1]$ define the operator

$$\mathsf{A}u(x) = \int_0^x (x-y)u(y)\,\mathrm{d}y \qquad 0 \le x \le 1\,.$$

Find the norm and range of A. Is its range dense in $C[0,1]$?

11. Let the functions $\{u_n(x)\}$ be continuous and differentiable for $0 < x < \infty$ and, for every fixed $x > 0$, let $\{u_n(x)\} \in \ell^1$. Is the operator $\mathsf{D}\{u_n(x)\} = \{u'_n(x)\}$ bounded on ℓ^1?

12. Show that, if A commutes with $\mathsf{A}\mathsf{A}^*$, then A is normal.

2 Bounded operators

1. Prove that any operator from a finite-dimensional normed space into an arbitrary normed space is bounded.

2. What is the norm of the operator defined on $L^2(0,\infty)$ by

$$\mathsf{A}u(x) = \begin{cases} u(x-a) & x \ge a \\ 0 & x < a \end{cases}\,,$$

where $a > 0$?

3. Let A and B be two bounded linear operators on a Banach space. Suppose that A^{-1} exists and that $\|\mathsf{A}^{-1}\mathsf{B}\| < 1$. Show that $\mathsf{A} - \mathsf{B}$ is invertible and find the expression of its inverse in terms of a series.

4. Prove that, similarly to a matrix (see p. 493), a bounded operator A can be uniquely decomposed as $\mathsf{A} = \mathsf{A}_1 + i\mathsf{A}_2$.

5. Prove that the two operators A_1 and A_2 arising in the canonical decomposition $\mathsf{A} = \mathsf{A}_1 + i\mathsf{A}_2$ of an operator A commute if and only A is normal.

6. Let A and B bet two bounded, self-adjoint (in general non-commuting) operators on a Hilbert space. Show that $\mathsf{AB} + \mathsf{BA}$ and $i(\mathsf{AB} - \mathsf{BA})$ are self-adjoint bounded operators on the same space.

7. Consider the space of infinite numerical sequences $\{c\} = (c_1, c_1, c_2, \ldots)$ (real or complex) such that the series $|\sum_{n=1}^{\infty} n! c_n| < \infty$ equipped with the norm

$$\|c\| = \sum_{n=1}^{\infty} n! |c_n| .$$

Show that the operator from this space into ℓ^1 defined by

$$\mathsf{B}c = \left(\frac{c_1}{1!}, \frac{c_2}{2}!, \frac{c_3}{3!}, \ldots\right)$$

is bounded. Find its norm.

8. Given a function $f(x)$ defined in $L^2(0, 2\pi)$ with Fourier coefficients $\{f_n\}$, $-\infty < n < \infty$, define the action of an operator A acting on f as generating a function $g = \mathsf{A}f$ having Fourier coefficients $\lambda_n f_n$ where $\{\lambda_n\}$ are a given set of complex numbers. What condition on the λ_n will render A bounded?

9. Let B_1 and B_2 be two bounded operators from a space S to a space S'. Show that the set of elements $u \in S$ such that $\mathsf{B}_1 u = \mathsf{B}_2 u$ is a closed subset of S.

10. Let B_n be a sequence of bounded operators from a Hilbert space H to a Hilbert space H'. Show that, if $\mathsf{B}_n u$ is a Cauchy sequence for each $u \in H$, then there is a bounded operator B such that $\mathsf{B}_n \to \mathsf{B}$ strongly.

11. By appealing to the Gelfand-Beurling formula (21.2.47) p. 635 prove that, for a bounded normal operator N,

$$\|\mathsf{N}\| = \sup_{\lambda \in \sigma(\mathsf{N})} |\lambda| = r_\sigma(\mathsf{N}) .$$

where $\sigma(\mathsf{N})$ denotes the spectrum of N and r_σ its spectral radius.

12. Prove that, if the bounded operator B is Hermitian, then $\|\mathsf{B}^2\| = \|\mathsf{B}\|^2$.

13. Prove that, if B is a bounded operator in a Hilbert space, then

$$\mathcal{N}(\mathsf{B}^*) = \mathcal{N}(\mathsf{B}\mathsf{B}^*) , \qquad \overline{\mathcal{R}(\mathsf{B})} = \overline{\mathcal{R}(\mathsf{B}\mathsf{B}^*)} .$$

14. In the space ℓ^2 find the adjoint of the operator defined by

$$\mathsf{B}\{c_1, c_2, c_3, \ldots\} = \{c_1, \frac{1}{2}c_2, \frac{1}{3}c_3, \ldots\} .$$

15. Consider the space ℓ^2 of square-summable sequences $\mathbf{u} = \{u_j\}$ with

$$\|\mathbf{u}\|^2 = \sum_{j=1}^{\infty} |u_j|^2 < \infty$$

and the operator A acting on ℓ defined by

$$\mathsf{A}\mathbf{u} = \left\{\frac{1}{1}u_1, \frac{1}{2}u_2, \ldots, \frac{1}{n}u_n, \ldots\right\} .$$

(a) Find eigenvalues and eigenvectors of A.
(b) Show that $\lambda = 0$ belongs to the continuous spectrum by exhibiting a sequence $\{\mathbf{u}_k\}$ such that $\|\mathbf{u}_k\| = 1$, $\|\mathsf{A}\mathbf{u}_k\| \leq 1/k$.
(c) By solving explicitly the equation $\mathsf{A}\mathbf{u} = \mathbf{v}$, where $\mathbf{v} \in \ell^2$, find A^{-1} and verify that it is unbounded as was to be expected from the fact that $\lambda = 0$ belongs to the continuous spectrum.

16. Section 21.2.4 describes the Neumann series for the resolvent. Consider the more general problem

$$(\mathsf{L} - \epsilon \mathsf{M})\, u \;=\; f$$

where $|\epsilon| \ll 1$, L, M are bounded operators, and u and f are vectors in a Hilbert space. For given ϵ, L, M ad f, calculate u correct to order ϵ using the same general idea. As an application of this procedure, let L and M be the 2×2 matrices

$$\mathsf{L} = \begin{vmatrix} 1 & 0 \\ 0 & \text{-}1 \end{vmatrix} \qquad \mathsf{M} = \begin{vmatrix} 2 & \text{-}1 \\ \text{-}1 & 2 \end{vmatrix}$$

and $f \;=\; |f_1 \;\; f_2|^T$. Calculate the exact solution and verify that the approximate solution that you have found has an error of order ϵ^2.

2.1 Contractions

1. Show that every contraction operator is continuous.

2. Define on the real line the operator

$$\mathsf{A}x \;=\; \frac{1}{2}\,(x + \sin x)\;.$$

Show that A is a contraction. Are there fixed points?

3. Given a function $f(x)$ defined in $L^2(0, 2\pi)$ with Fourier coefficients $\{f_n\}$, $-\infty < n < \infty$, define the action of an operator A acting on f as generating a function $g = \mathsf{A}f$ having Fourier coefficients $\lambda_n f_n$ where $\{\lambda_n\}$ are a given set of complex numbers. What condition on the λ_n will render A a contraction mapping?

4. For $s > 0$ define the family of *Picard operators* defined on elements of $L^2(-\infty, \infty)$ by

$$(\mathsf{A}_s f)(x) \;=\; \frac{1}{2} s \int_{-\infty}^{\infty} e^{-s|x-y|} f(y)\, \mathrm{d}y\;.$$

Show that the A_s are contraction mappings of $L^2(-\infty, \infty)$ into itself.

5. For $s > 0$ define the family of *Poisson operators* defined on elements of $L^2(-\infty, \infty)$ by

$$(\mathsf{A}_s f)(x) \;=\; \frac{s}{\pi} \int_{-\infty}^{\infty} \frac{f(x+y)}{y^2 + x^2}\, \mathrm{d}y\;.$$

Show that the A_s are contraction mappings of $L^2(-\infty, \infty)$ into itself and have a unique fixed point.

2.2 Projection operators

1. Show that the operator defined in $L^2(-\pi, \pi)$ by

$$\mathsf{P}u(x) \;=\; \int_{-\pi}^{\pi} \left(\sum_{k=m}^{n} \frac{e^{ik(x-\xi)}}{2\pi} \right) u(\xi)\, \mathrm{d}\xi$$

is a projection for any pair of integers m, n.

2. Let P be the orthogonal projection on a finite-dimensional subspace M of a Hilbert space H. (a) Is P bounded? What is its norm? (b) Is P compact? (c) What is the adjoint of P? What are its domain and range? (d) Would P be compact if M were infinite-dimensional?

3. Let P_1 and P_2 be two projection operators on a Hilbert space. Under what conditions are (a) $\mathsf{P}_1 + \mathsf{P}_2$, and (b) $\mathsf{P}_1\mathsf{P}_2$ projection operators?

4. Is the following operator acting on $L^2(-\infty, \infty)$ a projection operator

$$\mathsf{A}u(x) \;=\; \left\{ \begin{array}{cc} u(x) & x \geq a \\ 0 & x < a \end{array} \right. \;?$$

5. Define an operator A transforming functions $u(x)$ belonging $L^2(-a,a)$ with $0 < a \leq \infty$ into functions $v(x) \in L^2(-a,a)$ according to the rule

$$v(x) \;=\; \mathsf{A}u(x) \;=\; \frac{1}{2}\,[u(x) + u(-x)]\ .$$

   Show that A is an orthogonal projection.

6. Show that, if A is a Hermitian idempotent operator on a Hilbert space and its nullspace N is nontrivial (i.e., neither the whole space nor the zero vector), then A is a projection operator onto $N^{\perp}$.

7. If $\emptyset \subset S_1 \subset S_2 \subset H$ and P_1 and P_2 are the projectors on S_1 and S_2, find $\mathsf{P}_1\mathsf{P}_2$ and $\mathsf{P}_2\mathsf{P}_1$.

8. Let P_1 and P_2 be two projection operators onto subspaces S_1 and S_2 of a Hilbert space. and suppose that they commute. Show that $\mathsf{I} - \mathsf{P}_1$, $\mathsf{I} - \mathsf{P}_2$, $\mathsf{P}_1\mathsf{P}_2$, $\mathsf{P}_1 + \mathsf{P}_2 - \mathsf{P}_1\mathsf{P}_2$ and $\mathsf{P}_1 + \mathsf{P}_2 - 2\mathsf{P}_1\mathsf{P}_2$ are all orthogonal projection operators. How are the ranges of these projection operators related to S_1 and S_2?

9. Let P be the orthogonal projection on a closed manifold M in the Hilbert space H. Find its eigenvalues and their multiplicities. What are the solvability conditions for the equations

$$\mathsf{P}u \;=\; f, \qquad \mathsf{P}u - u \;=\; f?$$

   Interpret your answer in the light of the theorem requiring orthogonality of f to the solutions of the homogeneous adjoint equation. Find the corresponding solutions if the solvability conditions are satisfied.

10. Let H_1 and H_2 be closed subspaces of a Hilbert space H, and let P_1 and P_2 be the corresponding orthogonal projectors. Show that $S_1 \subset S_2$ if and only if $\mathsf{P}_2\mathsf{P}_1 = \mathsf{P}_1$, in which case $\mathsf{P}_1\mathsf{P}_2 = \mathsf{P}_1$,

11. Let $\{\mathsf{P}_n\}$ be a family of orthogonal projectors in a Hilbert space H constituting a resolution of the identity operator so that $\sum_n \mathsf{P}_n = \mathsf{I}$. Define an operator A by

$$\mathsf{A}u \;=\; \left(\sum_n \lambda_n \mathsf{P}_n\right) u$$

   for every u in H, where $\{\lambda_n\}$ is abounded family os scalars. Sow that

   - If A is unitary, then $|\lambda_n| = 1$ for all n;
   - If A is self-adjoint, then all the λ_n are real and the so are the eigenvalues of A;
   - If A is positive, then so are all the λ_n.

12. Show that an orthogonal projector operator is compact if and only if its range is finite-dimensional.

2.3 Compact operators

1. Let C be a operator defined on the space ℓ^2 by

$$\mathsf{C}u_n = \lambda_n u_n$$

where $u_n \in \ell^2$ and $|\lambda_n| \to 0$. Show that C is compact.

2. In the space ℓ^p (with $p \geq 1$) define the operator C by

$$\mathsf{C}c = \left(\frac{c_1}{1}, \frac{c_2}{2}, \frac{c_3}{3}, \ldots\right).$$

Show that C is compacts.

3. Show that the set of all compact operators on a Hilbert space H is a linear subspace of the space of all bounded operators on H closed in the operator norm.

4. Show that a linear operator C from a Hilbert space H to a Hilbert space H' is compact if and only if $\mathsf{C}^*\mathsf{C}$ is compact.

5. Show that, if a linear operator C from a Hilbert space H to a Hilbert space H' is compact, also its adjoint C^* is. if and only if $\mathsf{C}^*\mathsf{C}$ is compact.

2.4 Unitary operators

1. Is the following operator acting on $L^2(0,\infty)$

$$\mathsf{A}u(x) = \begin{cases} u(x-a) & x \geq a \\ 0 & x < a \end{cases},$$

where $a > 0$, unitary?

2. Define an operator A transforming functions $u(x)$ belonging $L^2(-\infty,\infty)$ into functions $v(x)$ in the same space according to the rule

$$v(x) = \mathsf{A}u(x) = \begin{cases} u(x) & x \geq 0 \\ -u(x) & x < 0 \end{cases}.$$

Show that A is a unitary operator

3. Let B be a bounded operator and let i belong to its resolvent set. Show that $\mathsf{B} + i\mathsf{I}$ and $(\mathsf{B} - i\mathsf{I})^{-1}$ commute.

4. Let A be a self-adjoint operator mapping a subset of a Hilbert space H into H. Prove that the operator

$$\mathsf{U} = (\mathsf{A} - i\mathsf{I})(\mathsf{A} + i\mathsf{I})^{-1} = (\mathsf{A} + i\mathsf{I})^{-1}(\mathsf{A} - i\mathsf{I})$$

is unitary; U is called the *Cayley transform* of A.

5. Show that, if A is a self-adjoint operator on a Hilbert space, then $e^{i\mathsf{A}\xi}$ is strongly continuous, i.e.,

$$\lim_{\xi \to c} \|e^{i\mathsf{A}\xi} - e^{i\mathsf{A}c}\| = 0.$$

6. Show that if a unitary operator is positive-definite, then it is the identity operator.

2.5 Integral operators

1. Determine the eigenvalues and eigenfunctions of the Fredholm integral operator

$$\mathsf{L}\,u \equiv \int_0^1 (1 - 3xy)\, u(y)\, dy.$$

Find the general solution of the equation

$$u(x) \;=\; f(x) + \mu \mathsf{L} u,$$

where $f(x)$ is given, when $1/\mu$ is not an eigenvalue, When $1/\mu$ is an eigenvalue, determine the solvability conditions on f and write the solution for the class of functions f that satisfy these conditions.

2. Determine the eigenvalues and eigenvectors of the Fredholm integral operator

$$\mathsf{L} u \equiv \int_{-\pi}^{\pi} [\sin(x-y) + \sin(x+y)]\, u(y)\, dy,$$

in the range $-\pi < x < \pi$.

3. Determine the eigenvalues and eigenvectors of the Fredholm integral operator

$$\mathsf{L} u \equiv \frac{1}{2} \int_0^1 \exp(-|x-y|)\, u(y)\, dy,$$

in the range $0 < x < 1$.

4. Determine the eigenvalues and eigenvectors of the Fredholm integral operator

$$\mathsf{L} u \equiv \frac{1}{2} \int_0^1 \exp(-|x-y|)\, u(y)\, dy,$$

in the range $-\infty < x < \infty$.

5. Determine the eigenvalues and eigenvectors of the Fredholm integral operator

$$\mathsf{L} u \equiv \frac{1}{2} \int_0^1 \exp(-|x-y|)\, u(y)\, dy,$$

in the range $-1 < x < 1$.

6. Consider the integral equation

$$u(x) \;=\; f(x) + \lambda \int_0^{\infty} \cos(2xy)\, u(y)\, dy,$$

where f is a given continuous function.

   (*a*) Determine the solution. [Hint: Multiply by $\cos(2xz)$ and integrate. Assume that all the interchanges of integrations that you need are legitimate.]

   (*b*) From the answer to the previous problem you will find critical values of λ for which the solution may break down. Verify directly that these are eigenvalues of the integral operator. [Hint: Proceed as before. You do not need to find the eigenfunctions to answer this question.]

   (*c*) If you can find the eigenfunctions, so much the better. If you can't, state the conditions on f for a solution to exist when λ has one of the critical values. Is the solution unique in this case?

7. Given the integral equation

$$(\exp b^2)\ u(x) + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left[-(x-y)^2\right] u(y)\, dy \;=\; (\sinh b^2)\ \cos 2bx\,,$$

where b is a real positive constant, describe an approximate solution method based on a suitable Bubnov-Galerkin method, i.e., choose a suitable set of basis functions and a suitable scalar product. Describe how you would set up the calculation. Solve the equation exactly.

8. Consider an integral operator with kernel $K(x, y)$ in $\mathsf{L}^2(a, b)$. The operator is self-adjoint (i.e., $K(x, y) = \overline{K(y, x)}$) and Hilbert-Schmidt (i.e. $\int_a^b dx \int_a^b dy |K|^2 < \infty$). Let λ_j and $v_j(x)$ be its eigenvalues and eigenfunctions. Show that

$$K(x, y) \;=\; \sum_j \lambda_j v_j(x) \overline{v_j(y)}$$

(convergence being with respect to the norm in $\mathsf{L}^2(a, b) \times \mathsf{L}^2(a, b)$), and also that

$$\int_a^b dx \int_a^b dy |K(x, y)|^2 \;=\; \sum_j |\lambda_j|^2$$

9. In the space $L^2(-\infty, \infty)$ consider the operator defined by

$$\mathsf{L} \;=\; \frac{1}{2}\left[u(x-a) + u(x+a)\right]$$

for some real constant a. Is it bounded? Is it self-adjoint?

10. In the integral equation

$$u(x) \;=\; x^2 + \int_0^1 \sin(axy)\, u(y)\, dy,$$

assume $|a| \ll 1$, replace $\sin(axy)$ by the first two terms of its power series expansion and obtain an approximate solution. This technique, which in effect approximates a non-separable kernel by a separable one, is sometimes useful.

11. Find the value(s) of α for which the integral equation:

$$u(y) \;=\; -\alpha^2 \int_0^1 G(x, y) u(x)\, dx$$

has a solution and calculate this (these) solution(s). Here

$$G(x, y) \;=\; -\frac{\sin kx_< \ \sin k(1 - x_>)}{k \sin k}$$

with k a given real number and $x_> = \max(x, y)$, $x_> = \min(x, y)$.

12. (a) Consider the integral operator

$$\mathsf{K}v \;=\; \int_0^{\pi} K(x, y)\, v(y)\, dy\,,$$

where

$$K(x, y) \;=\; \begin{cases} x(y-\pi) & 0 \le x \le y \\ y(x-\pi) & y \le x \le \pi \end{cases}$$

acting on $C_0^2[0,\pi]$ functions, i.e., functions vanishing at $x=0$ and $x=\pi$ and possessing a continuous second derivative. Find the eigenvalues and the normalized eigenfunctions of K satisfying

$$\mathsf{K}v_n \;=\; \pi\lambda_n v_n\,.$$

(A good way to proceed is to express the kernel in terms of the Heaviside step function and differentiate twice.)
(b) Solve the integral equation

$$u(x) \;=\; f(x) + \frac{\mu^2}{\pi}\int_0^\pi K(x,y)\,u(y)\,dy$$

where $f(0)=f(\pi)=0$, by expanding u in a series of eigenfunctions of K.
(c) Solve this equation directly and indicate what you expect the relation between the two solutions to be.

13. Consider the operator

$$\mathsf{K}v \;=\; \int_0^\pi \sin(x-y)\,v(y)\,dy\,.$$

(a) Find eigenvalues and normalized eigenfunctions satisfying $\mathsf{K}v \;=\; \lambda v$. ($b$) Is the operator compact? (c) Is $\lambda=0$ an eigenvalue? If so, what is its degeneracy? (d) If $\lambda=0$ is an eigenvalue, show one of the many possible orthonormal bases in its eigenspace.

3 Unbounded operators

1. Let A be a symmetric operator on a scalar product space and B another operator such that $\mathsf{AB}=0$. Prove that this situation is only possible when the ranges of the two operators are orthogonal.

2. Consider the operator $\mathsf{A}f \equiv f'$ on the domain

$$\mathcal{D}_{\mathsf{A}} \;=\; \{f\in L^2(a,b) : f'\in L^2(a,b), f(a)=0\}.$$

Determine the adjoint A^* with its domain of definition. Is A symmetric? Is it self-adjoint?

3. Consider the operator $\mathsf{A}f \equiv f'$ on the domain

$$\mathcal{D}_{\mathsf{A}} \;=\; \{f\in L^2(a,b) : f'\in L^2(a,b), f(a)=f(b)\}.$$

Determine the adjoint A^* with its domain of definition. Is A symmetric? Is it self-adjoint?

4. Let (e_0, e_1, e_2 …, e_n, …) be an orthonormal basis in the Hilbert space ℓ^2. Define the *annihilation operator* by

$$\mathsf{A}e_0 \;=\; 0\,, \qquad \mathsf{A}e_n \;=\; \sqrt{n}\,e_{n-1}\,.$$

Show that the adjoint of this operator, called the *creation operator*, is given, for $n=0$, 1, …, by

$$\mathsf{A}^*e_n \;=\; \sqrt{n+1}\,e_{n+1}\,.$$

5. Determine the formal adjoint L^* of the operator L defined by

$$\mathsf{L}\,u \equiv [p(x)u'(x)]' + q(x)u(x), \tag{1}$$

with $p(x)>0$, $a<x<b$, acting on functions such that $u(a)=0$, $u'(a)=0$. Determine the conditions that the functions belonging to the domain of L^* must satisfy so that

$$(v,\mathsf{L}u) \;=\; (\mathsf{L}^*v,u), \tag{2}$$

with a vanishing conjunct.

6. Determine the formal adjoint L^* of the operator L defined by

$$\mathsf{L}u \equiv xu''(x) + (2-x)u'(x) - u(x), \tag{3}$$

with $0 < x < 1$. If L operates on functions such that

$$|u(0)| < \infty, \qquad |u'(0)| < \infty, \qquad u(0) = u'(1), \tag{4}$$

determine the conditions that the functions belonging to the domain of L^* must satisfy so that

$$(v, \mathsf{L}u) = (\mathsf{L}^* v, u), \tag{5}$$

with a vanishing conjunct.

7. Consider the Sturm-Liouville operator

$$\mathsf{L}u \equiv -\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right] + q(x)u$$

acting on functions $u \in L^2(a,b)$ which, in $a < x < b$, satisfy

$$u(b) - \alpha u(a) - \beta \left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=a} = 0\,, \qquad \left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=b} - \gamma u(a) - \delta \left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=a} = 0\,.$$

Determine the conditions satisfied by the (generally complex) numbers α, β, γ and δ which make the operator symmetric.

4 Inverses

1. Let A and B be two operators possessing inverses, A^{-1} and B^{-1}. Show that $(\mathsf{AB})^{-1} = \mathsf{B}^{-1}\mathsf{A}^{-1}$.

2. Show that the linear transformation defined on $L^2(-\infty,\infty)$ by

$$v(x) = \frac{1}{a}\int_{-\infty}^{x} e^{-a(x-\xi)}u(\xi)\,\mathrm{d}\xi$$

with a a given constant, is one-to-one.

3. Show that, when it exists, the inverse A^{-1} of a operator A from a normed space into another normed space is linear.

4. Prove that, when it exists, the inverse A^{-1} of an operator is unique.

5. Show that, if B is a bounded operator on a Banach space admitting an inverse B^{-1}, then $(\mathsf{B}^{-1})^n = (\mathsf{B}^n)^{-1}$.

6. Show that, if B_1 and B_2 are two bounded operators on a Banach space admitting inverses B_1^{-1} and B_2^{-1}, then $\mathsf{B}_1\mathsf{B}_2$ also admits an inverse given by $(\mathsf{B}_1\mathsf{B}_2)^{-1} = \mathsf{B}_2^{-1}\mathsf{B}_1^{-1}$.

7. Show that, if B_1 and B_2 are two bounded operators on a Banach space and $(\mathsf{B}_1\mathsf{B}_2)$ admits an inverse, then also B_1 and B_2 must have inverses.

8. Show that, if A is positive definite, and $\lambda < 0$, then $(\mathsf{A} - \lambda\mathsf{I})^{-1}$ exists.

9. Let A_n and A be positive self-adjoint operators and let $\mathsf{R}_{n,\lambda}$ and R_λ be their respective resolvents. Show that $\mathsf{R}_{n,\lambda} \to \mathsf{R}_\lambda$ for any non-real λ in the strong sense if and only if $(\mathsf{A}_n + \mathsf{I})^{-1} \to (\mathsf{A} + \mathsf{I})^{-1}$ in the strong sense.

5 Solvability conditions and the Fredholm alternative

1. Solve the following integral equation for all values of B

$$u(x) \;=\; B\int_0^{2\pi} \sin{(x+y)}\, u(y)\, dy \;+\; f(x),$$

$0 < x < 2\pi$. Give explicitly any solvability condition that need be imposed on f.

2. Solve the integral equation

$$u(x) \;=\; f(x) + \lambda \int_0^1 x\, t\, u(t)\, dt\,,$$

where f and λ are given. Does the solution exist for any λ? Is there a solvability condition? After studying the problem in general, consider in detail the particular case $f = b - x^2$, where b is a given parameter; discuss the possible cases that arise as b and λ are varied.

3. Let C be a compact non-normal operator. Show that, if 1 is not an eigenvalue of C, then there is one and only one solution of the equation $\mathsf{C}u - u = f$ for all $f \in H$. If, on the other hand, 1 is an eigenvalue, then the equation has a solution if and only if $f \perp \mathcal{N}(\mathsf{C}^* - \mathsf{I})$.

4. Solve the integral equation

$$u(x) - \lambda \int_0^{\pi/2} K(x,\xi) u(\xi)\, \mathrm{d}\xi \;=\; 1$$

where

$$K(x,\xi) \;=\; \left\{ \begin{array}{ll} \sin x\,, \cos\xi & \text{for} 0 \le x \le \xi \le \pi/2 \\ \sin\xi\,, \cos x & \text{for} 0 \le \xi \le x \le \pi/2 \end{array} \right. .$$

Are there special values of λ one should pay attention to?

5. Solve the integral equation

$$u(x) - \lambda \int_0^1 e^{-|x-\xi|} u(\xi)\, \mathrm{d}\xi \;=\; x\,.$$

Are there special values of λ one should pay attention to?

6. Find the general solution of the Fredholm integral equation

$$u(x) \;=\; f(x) + \lambda \int_0^1 e^x e^t u(t)\, dt$$

Are there value(s) of λ for which a solution for arbitrary f? does not exist? When λ equals one of these value(s), what is the condition on f for a solution to exist? Interpret in the ligh of the Fredholm alternative thorem.

7. Consider, over the interval $0 < x < 1$, the problem

$$-\frac{d}{dx}\left(x\frac{du}{dx}\right) + \frac{N^2}{x} u \;=\; \lambda^2 x u - F(x)\,,$$

where N is a given non-zero integer, λ^2 a given real positive number, and F a given function. The boundary conditions are $u(0)$ regular, $u(1) = 0$. Are there (λ-dependent) restrictions on F for the solution to exist? (Don't worry about whether the range of the differential operator is closed or not – proceed as if it were.) After answering this question, verify your answer by obtaining the explicit solution of the problem by means of the method of variation of parameters.

6 Resolvent and spectrum

1. By proceeding as in Example 21.2.5 p. 633 find the resolvent kernel (see p. 144) of the Fredholm equation
$$u(x) \;=\; f(x) + \mu \int_0^1 (x-y)\, u(y)\, \mathrm{d}y \;.$$
Show that it is an analytic function of μ and find its singularities.

2. Let S be Banach space and A, B operators on S. Show that, if $\lambda \in \rho(\mathsf{A}) \bigcap \rho(\mathsf{B})$, then the resolvents of A and B satisfy the so-called second resolvent equation
$$(\mathsf{A}-\lambda\mathsf{I})^{-1} - (\mathsf{B}-\lambda\mathsf{I})^{-1} \;=\; (\mathsf{B}-\lambda\mathsf{I})^{-1}(\mathsf{B}-\mathsf{A})(\mathsf{A}-\lambda\mathsf{I})^{-1}\,.$$

3. Define on $L^2(=\pi,\pi)$ an operator A acting on functions $u(x) = \sum_{n=-\infty}^{\infty} u_n e^{int}$ by
$$\mathsf{A}u(x) \;=\; \sum_{n=-\infty}^{\infty} u_{n+1} e^{int}\,.$$
Let $v_n(\lambda)$ be the Fourier coefficient of $\mathsf{R}_\lambda[u]$, when it exists, and show that
$$v_n \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{u(y)}{e^{-iy}-\lambda} e^{-iny}\, \mathrm{d}y \;.$$
Discuss the existence of v_n as a function of λ and determine the spectrum of A.

4. Let A be a non-negative self-adjioint operator on a Hilbert space and let $\lambda > 0$. Use the resolvent equation (21.10.1) p, 669 to show that $(\mathsf{A}+\lambda\mathsf{I})^{-1}$ is compact if and only if $(\mathsf{A}+\mathsf{I})^{-1}$ is compact.

5. Consider on the interval $0 < x < \pi$ the operator L acting on the vector $\mathbf{v}(x) = (v_1(x), v_2(x))$ according to
$$\mathsf{L}\mathbf{v} \;\equiv\; \left|\begin{array}{cc} \frac{d}{dx} & -1 \\ 1 & \frac{d}{dx} \end{array}\right| \left|\begin{array}{c} v_1 \\ v_2 \end{array}\right| \;=\; \left\{\begin{array}{l} \frac{dv_1}{dx} - v_2 \\ \frac{dv_2}{dx} + v_1 \end{array}\right. ,$$
with $v_1(0) = v_1(\pi) = 0$. Define the adjoint of this operator and consider the inhomogeneous equation
$$\mathsf{L}\mathbf{v} \;=\; \mathbf{f},$$
where $\mathbf{f}^T(x) \;=\; |f_1(x) \;\; f_2(x)|$ and obtain explicitly the solvability condition for this problem. Define the scalar product by
$$(\mathbf{w},\mathbf{v}) \;=\; \int_0^\pi |w_1^* \;\; w_2*| \left|\begin{array}{c} v_1 \\ v_2 \end{array}\right| dx \;=\; \int_0^\pi (w_1^* v_1 + w_2^* v_2)\, dx.$$

6. Consider the space ℓ^2 of square-summable sequences $\mathbf{u} = \{u_j\}$ with
$$\|\mathbf{u}\|^2 \;=\; \sum_{j=1}^{\infty} |u_j|^2 < \infty$$
and the operator A acting on ℓ^2 defined by
$$\mathsf{A}\mathbf{u} \;=\; \left\{\left(a+\frac{1}{1}\right)u_1, \left(a+\frac{1}{2}\right)u_2, \ldots, \left(a+\frac{1}{n}\right)u_n, \ldots\right\}.$$

where a is a real constant. (a) Find eigenvalues and eigenvectors of A.
(b) Is the equation $(\mathsf{A}-\lambda\mathsf{I})\mathbf{u}=\mathbf{v}$, where $\mathbf{v}\in\ell^2$, always solvable for any λ and $\mathbf{v}$?
(c) Give the explicit form of $(\mathsf{A}-\lambda\mathsf{I})^{-1}$. For what value(s) of λ is this operator unbounded? By definition, these value(s) constitute the continuous spectrum of A.

7. Show that a number λ belongs to the approximate point spectrum of an operator A (see p. **) if and only if $\mathsf{A}-\lambda\mathsf{I}$ is not bounded below. Furthermore, show that the approximate point spectrum of an operator is a subset of its spectrum.

8. Sjow that the spectrum of a unitary operator lies on the unit circle.

9. If C is a compact operator, show that, among its eigenvalues λ, there is one, λ_M, such that $|\lambda_M| = \max|\lambda|$. Furthermore $|\lambda_M| = \|\mathsf{C}\|$ and

$$\|\mathsf{C}\| = \sup_{\|v\|=1} |(v, Cv)| \,.$$

10. Let C be a compact self-adjoint operator with the spectral decomposition $\mathsf{C} = \sum_n \lambda_n \mathsf{P}_n$. For $-\infty < \lambda < \infty$ define the spectral family of operators

$$\mathsf{E}_\lambda u = \sum_{\lambda_n \le \lambda} \mathsf{P}_n u$$

for all $u \in H$. Show that E_λ is a projector for any λ and that $\mathsf{E}_\lambda \le \mathsf{E}_\mu$ if $\lambda \le \mu$.