Invitation to Linear Programming and Game Theory

David Vella, Skidmore College

Solutions Manual

This manual, to complement the text ***Invitation to Linear Programming and Game Theory,*** contains hints, partial, or complete solutions to every exercise in the text.

In the Preface to the text, I stated that *In addition to mathematical influences, music of various genres has always been important to me, and like-minded readers may enjoy the fact that this book is riddled with references in homage to songs and artists*. Many of those music references and allusions will be obvious, but a few are more obscure. For the benefit of those readers who may have enjoyed finding those references as a sort of 'Easter Egg', I have spelled them out here and provided, when possible, a link to a relevant website for the band or artist.

Chapter 1. Preliminaries

Section 1.1 Mathematical Models

This section contained no exercises or musical references.

Section 1.2 Systems of Linear Equations

Music References in the Text:

In Example 1.1, Peter, Paul & Mary alludes to the folk trio popular in the 1960s. Three of the songs they released were 'Lemon Tree', 'If I Had a Hammer', and 'Leavin' on a Jet Plane.' (Website: http://www.peterpaulandmary.com)

Example 1.2 pays musical homage to the Steely Dan song 'Kid Charlemagne' from their 1976 album 'the Royal Scam'. 'Walter' is a reference to the late Walter Becker, who, along with Donald Fagen, formed the songwriting team and creative core of the band. (Website: https://www.steelydan.com)

Example 1.3 is a pun on the 1977 song 'Two Tickets to Paradise' by singer-songwriter Eddie Money. (Website: https://www.eddiemoney.com)

Solutions to Exercises (and more music references):

1. Solve the system of equations we derived for example 1.2.

   Solution: The system is easy to solve using substitution or elimination, especially if one multiplies the second equation by a factor of 10 to clear the decimals. The result is $(x, y) = (4,2)$, so $x = 4$ liters of the 40% solution and $y = 2$ liters of the 10% solution.

2. Solve the system of equations we derived for example 1.3, and interpret the answer obtained in terms of Eddie's goals.
   Solution: The system is

$$\begin{gathered} x + y + z = 3260 \\ 2x + y - z = 0 \\ 7x - y - z = 100 \end{gathered}$$

   Although the system can be solved by elimination, we will illustrate a solution which is a hybrid of both methods. We begin with substitution.

   Choosing to solve for z in the second equation we obtain

$$z = 2x + y$$

Substitute this expression for z into the other two equations and simplify. The result is the system of two equations:

$$3x + 2y = 3260$$
$$5x - 2y = 100$$

There is nothing to prevent us from switching to elimination now to take advantage of the fact that it is easy to eliminate y by just adding these two equations. We obtain

$$8x = 3360$$

Dividing by 8 yields $x = 420$. Substitute this into the equation

$$3x + 2y = 3260$$

And simplify to obtain

$$3(420) + 2y = 3260$$
$$1260 + 2y = 3260$$
$$2y = 2000$$
$$y = 1000$$

Finally, from our expression for z above, we obtain

$$z = 2x + y = 2(420) + 1000 = 1840$$

Thus, the mathematical solution is

$$(x, y, z) = (420{,}1000{,}1840)$$

For the interpretation stage of the model, we need to find the total amount of money this represents. Since x represents dollar coins, y represents quarters, and z represents dimes, the total is

$$\$420 + 1000(\$0.25) + 1840(\$0.10) = \$854$$

This is enough money for the tickets with $4 to spare.

3. Johnny Vishnu leads a yoga class at the *Industrial Nirvana Studio and Gym*. Five times the number of men in the class is four more than three times the number of women. If there are 36 students altogether, how many more women than men are there?

   Music tip of the hat to jazz fusion guitar pioneer John McLaughlin. Over the years he has released many albums as a solo artist and with bands, including the renowned early 1970s jazz fusion group the Mahavishnu Orchestra. The name of the gym in this exercise is an allusion to McLaughlin's 2006 album Industrial Zen.
   (Websites: https://www.johnmclaughlin.com/ and https://en.wikipedia.org/wiki/Mahavishnu_Orchestra)

   a) Set up a system of linear equations for this problem.

   Solution: Let x be the number of men in the class and y the number of women. The equations in standard form are:

$$5x - 3y = 4$$

$$x + y = 36$$

b) Solve the system by substitution and answer the question.

Solution: To avoid working with fractions, it is easiest to use the second equation to solve for one of the variables in terms of the other. For example, $y = 36 - x$. Substitute this into the first equation. The reader can finish and obtain the solution $(x, y) = (14,22)$, so there are 14 men and 22 women. Thus, there are 8 more women than men.

c) Solve the system by elimination.
Solution: Left for the reader. One approach is to multiply through the second equation by 3, then add it to the first equation. This leads to the same answer as part b, of course.

4. Mr. Gilmour invested a total of $\$10{,}000$ in two mutual funds, the Windpillow Fund earns 6% interest, and the Crazy Diamond Fund earns 8% interest. If he earned \$730 interest in one year, how much did he invest in each fund?

Music Homage to David Gilmour, the guitarist for the band Pink Floyd. Two songs are alluded to in this example. The first is 'A Pillow of Winds' from their 1971 album Meddle, and the second is 'Shine On You Crazy Diamond' from their 1975 album Wish You Were Here. (Website: https://www.pinkfloyd.com)

a) Set up a system of linear equations for this problem.

Solution: Let x be the amount invested in the Windpillow fund and y the amount invested in the Crazy Diamond fund. The two equations come from knowing the total amount invested and the total interest for one year:

$$x + y = 10{,}000$$

$$.06x + .08y = 730$$

b) Solve the system by substitution and thereby answer the question.

Solution: Details left for the reader. The solution is $(x, y) = (3500{,}6500)$, so \$3,500 was invested in the Windpillow fund and \$6500 in the Crazy Diamond fund.

c) Solve the system by elimination.

Solution: To start, one approach would be to multiply through the second equation by 100 to clear the decimals. Ultimately, one obtains the same answer as in part b.

5. The *Astral Moon Dance Troupe* is taking their show on the road. The troupe has 26 members and 1300 cubic feet of props, supplies and luggage to transport. They call *Morrison Vans and Trucks,* a company that rents panel trucks and 10-foot moving vans. Each panel truck can seat 4 people and carry 80 cubic feet of luggage. Each moving van can seat 2 people and carry 300 cubic feet of luggage. How many of each type of truck should the dancers rent?

Music Reference to singer-songwriter Van Morrison, who has had a long career as a solo artist after leaving his band Them in the 1960s. His 1968 debut solo album was called 'Astral Weeks', and his 1970 album was named for its title track 'Moondance'. (Website: https://www.vanmorrison.com)

a) Set up a system of linear equations for this problem.

Solution: Let x be the number of panel trucks and y the number of moving vans. One equation comes from seating all the troupe members, the other from having the space to store all their luggage and equipment.

$$4x + 2y = 26$$

$$80x + 300y = 1300$$

b) Solve the system by substitution and answer the question.

Solution: The final solution is $(x, y) = (5,3)$, so they should rent 5 panel trucks and 3 moving vans.

c) Solve the system by elimination.

Solution: One approach is to divide the second equation by 20, and subtract the first equation to eliminate x. Ultimately, one obtains the same solution as in part b.

6. The painter Peter Black has made fourteen paintings for an exhibit in the Gabriel Gallery in Solsbury. The paintings are all exactly the same size and have the same size border around the paintings. If the length of his paintings (without the border) are 8 inches less than twice the width, and if the 2-inch border adds exactly 224 square inches to the canvas, what are the dimensions of the painted area (without the border)? Set up the system and solve it by whatever method you prefer.

Music Homage to Peter Gabriel, who has enjoyed a long solo career after leaving the progressive rock band Genesis in 1975. On his first solo album in 1977 he included the song 'Solsbury Hill', which allegedly addresses his departure from Genesis. On his 1992 album 'Us' he included a song entitled 'Fourteen Black Paintings', which is itself an homage to the famous 'Black Paintings' of artist Francisco Goya. (Website: https://petergabriel.com)

Solution: The equations in this problem come from comparing the without the border to the area with the border. At first glance, it appears that the equations might not be linear since they deal with area – however, all the terms cancel except the linear terms! So, let x stand for the length of the paintings (without the border) and let y stand for the width of the paintings (without the border). The first equation is $x = 2y - 8$, or in standard form, $-x + 2y = 8$. For the second equation, note that the area of the painted region is xy, but when the 2-inch border is included (on all four sides), the area of the canvas is $(x + 4)(y + 4)$. Thus we know

$$(x + 4)(y + 4) = xy + 224$$

$$xy + 4x + 4y + 16 = xy + 224$$

$$4x + 4y = 208$$

It follows that the system is:

$$-x + 2y = 8$$

$$4x + 4y = 208$$

To solve this system, it is convenient to divide the third equation by 4. Then, using either substitution or elimination, arrive at the solution $(x, y) = (32,20)$, so the paintings (without the borders) are 32 inches by 20 inches.

7. Leo, a mechanic at *Freeway Star Automotive*, needs to winterize a fleet of race cars. He has a vat containing 80% concentration antifreeze, and another vat containing 30% concentration antifreeze. How many liters of each should he mix together in order to obtain 50 liters of antifreeze at a 60% concentration? Set up the system and solve by your preferred method.

Music allusion to the rock band Deep Purple. On their 1972 album Machine Head, they opened with the song 'Highway Star', and followed with 'Maybe I'm a Leo'. (Website: https://deep-purple.com)

Solution: Let x be the number of liters of the 80% concentration and y the number of liters of the 30% concentration to be mixed. As noted in the text with concentration mixing problems, one equation comes from the total volume (50 liters), the other from the volume of pure antifreeze, which should be $.6(50) = 30$ liters. We obtain

$$x + y = 50$$

$$.8x + .3y = 30$$

The solution is $(x, y) = (30,20)$.

8. Leo's friend Ritchie has a vehicle with an 8-liter radiator, which is currently filled with antifreeze at a 50% concentration. How many liters should Leo drain from the radiator

and replace with an 80% concentration in order to leave the radiator full with a 60% concentration? Set Up the system and solve by your preferred method.

The music reference to Deep Purple continues here. Ritchie alludes to Ritchie Blackmore, the lead guitarist for Deep Purple for a time, including on the album Machine Head.

Solution: Let x be the number of liters to be drained from the radiator, and let y be the number of liters of 80% antifreeze to be added. In this problem, of course, $y = x$ (or in standard form $x - y = 0$), so it is actually possible to set this up as a single equation with a single variable rather than as a system. However, we prefer to set it up this way because it is a more general model. What if we started with a radiator that wasn't completely full, but wanted it full after we added the more concentrated antifreeze? In that case, x and y would not be the same, but we could still solve the problem the same way.

In this problem, the system becomes:

$$x - y = 0$$

$$.5(8 - x) + .8y = 4.8$$

Written in standard form, this is

$$x - y = 0$$

$$-.5x + .8y = .8$$

In this form, the first equation represents the net change in total volume, while the second equation represents the net change in volume of pure antifreeze. Solving the system leads to $(x, y) = \left(\frac{8}{3}, \frac{8}{3}\right)$. That is, we must drain $\frac{8}{3}$ liters from the radiator and replace it with an equal amount of 80% concentration antifreeze.

9. Mr. Nelson travels a lot. In one pocket he has American dollar coins, in another he has Euro coins, and in a third pocket he keeps Hong Kong Dollar coins. He has five more Hong Kong dollars than Euros, and two-thirds as many Hong Kong dollars as American dollars. If he has 58 coins altogether, how many of each type of coin does he have? Set up the system and solve with your preferred method.

Music Reference to the 1950s/60s teen idol, rock star, and actor Ricky Nelson. In 1961, he had a hit with the song 'Travelin' Man', in which the protagonist travels the world and has 'a girl in every port', including the USA, Europe, and Hong Kong. (Website: https://en.wikipedia.org/wiki/Ricky_Nelson)

Solution: Let x be the number of US dollar coins, y the number of Euro coins, and z the number of Hong Kong dollar coins. The system is

$$-y + z = 5$$

$$\frac{2}{3}x - z = 0$$

$$x + y + z = 58$$

Starting off with the method of substitution, the second equation gives us $z = \frac{2}{3}x$. Substitute into the other two equations and simplify to obtain the following system

$$\frac{5}{3}x + y = 58$$

$$\frac{2}{3}x - y = 5$$

Although we could continue with the substitution method, why not to switch to elimination, since adding these two equations will eliminate the y? From there it is easy to finish and obtain the solution $(x, y, z) = (27,13,18)$.

10. *Squeeze from a Stranger Coffee Shop* sells hot coffee for $3.00 per cup, iced coffee for $4.00 per cup, and latte for $5.00 per cup. One day they sold 495 cups altogether and made $2,035 in revenue. If they sold 25 more iced coffees than lattes, how many of each type of cup did they sell? Set up the system and solve with your preferred method.

Music tip of the hat to the British rock band Squeeze. Their 1982 album Sweets from a Stranger contained the song 'Black Coffee in Bed.' (Website: http://www.squeezeofficial.com)

Solution: Let x be the number of cups of hot coffee sold, y the number of cups of iced coffee sold and z the number of lattes sold. The system is

$$x + y + z = 495$$

$$3x + 4y + 5z = 2035$$

$$y - z = 25$$

To solve, use the third equation to solve for y in terms of z, then plug in this expression for y in the remaining equations. The result is a system of two equations and two unknowns, which can easily be solved by substitution or elimination. The final answer is $(x, y, z) = (120,200,175)$.

11. The music department at Tippett College sponsored a lecture/concert featuring free jazz legend Julie Keith. Tickets for students were $15, tickets for faculty were $25, and tickets for the general public were $40. A total of 970 tickets were sold, for a total of $31,100. If the total number of general public tickets sold was 30 more than twice the number of student tickets plus the number of faculty tickets, how many or each type of ticket were sold? Set up the system and solve with your preferred method.

Music homage to British free jazz pianist Keith Tippett. Before he passed away in June 2020, he released many albums spanning a career of over 50 years. Some were solo performances, some with collaborators and bands, some with his partner and wife, vocalist Julie Tippetts. (Website: https://en.wikipedia.org/wiki/Keith_Tippett)

Solution: Let x be the number of student tickets, y the number of faculty tickets, and z the number of general public tickets sold. The system is

$$x + y + z = 970$$

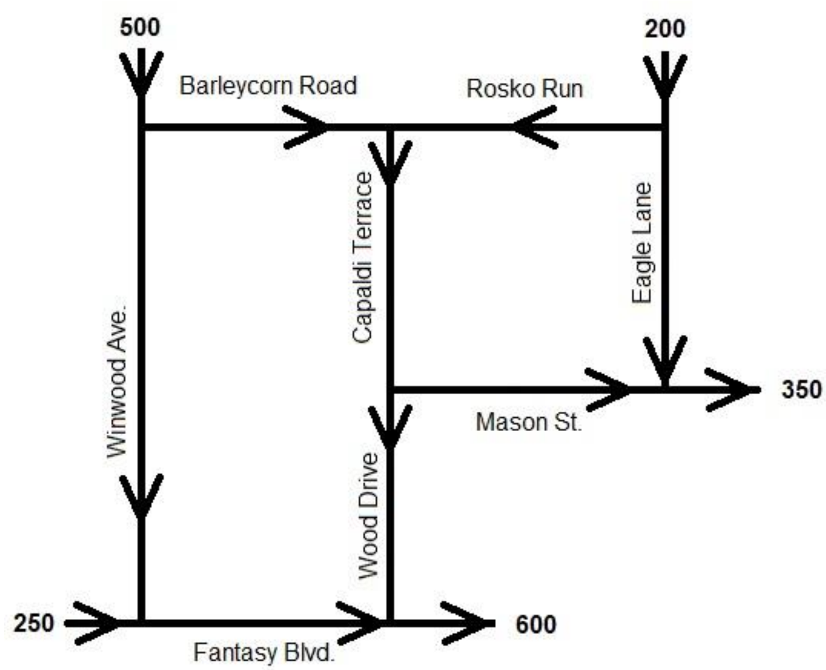
$$15x + 25y + 40z = 31{,}100$$

$$-2x - y + z = 30$$

One approach to solving the system is to solve for z in terms of the other variables using the third equation, then substitute this expression into the remaining two equations. The resulting system of two equations and two unknowns is easily solved by substitution or elimination. The answer is $(x, y, z) = (260, 80, 630)$.

Remark: If you are not experienced in using elimination on systems of three equations and three unknowns, you are encouraged to try elimination in this problem instead of the approach outlined above. If you have difficulties, they should be resolved in the next sections of the text where we develop a more streamlined approach to elimination.

12. Consider the following network of one-way streets:



The Berkshire city planners want to study the traffic patterns through this network. The numbers are the number of vehicles per day that enter or leave the network. Set up the system of linear equations for this network. Clearly indicate to which street each variable is associated. Do not solve!

Music homage to the rock band Traffic. Their 1967 debut album contained the song 'Dear Mr. Fantasy', as well as the song 'Berkshire Poppies'. The founding members were Steve Winwood, Dave Mason, Jim Capaldi, and Chris Wood. Later, touring members included Rosko Gee and others. They also released albums entitled John Barleycorn Must Die (1970) and When the Eagle Flies (1974). It will also not be lost on the reader that the problem is about ….traffic flow! (Website: https://en.wikipedia.org/wiki/Traffic)

Solution: We need a variable for each street segment, so there are eight variables. The following table names the variables:

Street Name	Variable
Barleycorn Road	s
Rosko Run	t
Winwood Ave.	u
Capaldi Terrace	v
Eagle Lane	w
Mason St.	x
Wood Dr.	y
Fantasy Blvd.	z

In each case, the variable stands for the number of vehicles per day that traverse the associated street. We have an equation for each intersection, so there are seven equations. We denote each intersection by a letter as follows (reading across the diagram horizontally):

Intersection	Letter
Barleycorn Rd. & Winwood Ave.	A
Barleycorn Rd., Rosko Run, & Capaldi Terr.	B
Rosko Run & Eagle Lane	C
Capaldi Terr., Mason St., & Wood Dr.	D
Mason St. & Eagle Lane	E
Winwood Ave. & Fantasy Blvd.	F
Fantasy Blvd. & Wood Dr.	G

The equations below have been converted to standard form, one for each intersection A-G, in order:

$$s + u = 500$$

$$s + t - v = 0$$

$$t + w = 200$$

$$v - x - y = 0$$

$$w + x = 350$$

$$-u + z = 250$$

$$y + z = 600$$

Section 1.3 Elimination and Matrices

1.3.1 Non-degenerate Examples

Solutions to Exercises:

1-11. Solve exercises 1-11 from Section 1.2 by pivoting the augmented coefficient matrix.

1. **Solution**: In the textbook we obtained the system

$$x + y = 6$$
$$0.4x + 0.1y = 1.8$$

So, pivoting the augmented coefficient matrix, we obtain:

$$\begin{bmatrix} 1 & 1 & 6 \\ 0.4 & 0.1 & 1.8 \end{bmatrix} R_2 \to R_2 - .4R_1$$

$$\begin{bmatrix} 1 & 1 & 6 \\ 0 & -0.3 & -0.6 \end{bmatrix} R_2 \to \left(-\frac{1}{0.3}\right) R_2$$

$$\begin{bmatrix} 1 & 1 & 6 \\ 0 & 1 & 2 \end{bmatrix} R_1 \to R_1 - R_2$$

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \end{bmatrix}$$

This is the desired form, so the answer appears in the last column: $(x, y) = (4,2)$, which agrees with the solution obtained in Section 1.2.

2. **Solution**: In the textbook, we obtained the system:

$$x + y + z = 3260$$
$$2x + y - z = 0$$
$$7x - y - z = 100$$

In the first pivot we take care of the first column:

$$\begin{bmatrix} 1 & 1 & 1 & 3260 \\ 2 & 1 & -1 & 0 \\ 7 & -1 & -1 & 100 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 7R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 3260 \\ 0 & -1 & -3 & -6520 \\ 0 & -8 & -8 & -22720 \end{bmatrix}$$

We then negate the second row in order to put a 1 in the 22 position, so the second row can be used for the pivot row next:

$$\begin{bmatrix} 1 & 1 & 1 & 3260 \\ 0 & 1 & 3 & 6520 \\ 7 & -8 & -8 & -22720 \end{bmatrix} \begin{matrix} R_1 \to R_1 - R_2 \\ \\ R_3 \to R_3 + 8R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & -2 & -3260 \\ 0 & 1 & 3 & 6520 \\ 0 & 0 & 16 & 29440 \end{bmatrix}$$

That completes the second pivot (Note the first two columns are unit columns). Now divide the third row by 16 to put a 1 in the 33 position, and use the third row as the pivot row:

$$\begin{bmatrix} 1 & 0 & -2 & -3260 \\ 0 & 1 & 3 & 6520 \\ 0 & 0 & 1 & 1840 \end{bmatrix} \begin{matrix} R_1 \to R_1 + 2R_3 \\ R_2 \to R_2 - 3R_3 \\ \\ \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 420 \\ 0 & 1 & 0 & 1000 \\ 0 & 0 & 1 & 1840 \end{bmatrix}$$

Thus, the solution agrees with what we obtained in section 1.2:

$(x, y, z) = (420{,}1000{,}1840)$.

3. **Solution**: The system is

$$5x - 3y = 4$$
$$x + y = 36$$

We could put a 1 in the 11 position by dividing the first row by 5. However, if we simply interchange the equations we can avoid fractions:

$$\begin{bmatrix} 1 & 1 & 36 \\ 5 & -3 & 4 \end{bmatrix} R_2 \to R_2 - 5R_1$$

$$\begin{bmatrix} 1 & 1 & 36 \\ 0 & -8 & -176 \end{bmatrix} R_2 \to \left(-\frac{1}{8}\right) R_2$$

$$\begin{bmatrix} 1 & 1 & 36 \\ 0 & 1 & 22 \end{bmatrix} \begin{matrix} R_1 \to R_1 - R_2 \\ \\ \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 14 \\ 0 & 1 & 22 \end{bmatrix}$$

Thus, $(x, y) = (14{,}22)$.

4. Solution: The system is

$$\mathrm{x}+\mathrm{y}=10000$$
$$.06x+.08y=730$$

Thus, the augmented coefficient matrix is

$$\begin{bmatrix} 1 & 1 & 10000 \\ .06 & .08 & 730 \end{bmatrix}$$

We leave it to the reader to complete the pivoting. (Suggestion: start with $R_2 \to 100R_2$.

5. **Solution**: The system is

$$4x+2y=26$$
$$80x+300y=1300$$

One way to pivot:

$$\begin{bmatrix} 4 & 2 & 26 \\ 80 & 300 & 1300 \end{bmatrix} R_1 \to \left(\frac{1}{4}\right) R_1$$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{13}{2} \\ 80 & 300 & 1300 \end{bmatrix} R_2 \to R_2 - 80R_1$$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{13}{2} \\ 0 & 260 & 780 \end{bmatrix} R_2 \to \left(\frac{1}{260}\right) R_2$$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{13}{2} \\ 0 & 1 & 3 \end{bmatrix} R_1 \to R_1 - \left(\frac{1}{2}\right) R_2$$

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \end{bmatrix}$$

Thus, the solution is $(x,y)=(5,3)$.

6. **Solution**: The system is

$$-x+2y=8$$
$$4x+4y=208$$

Start by negating the first row to put a 1 in the 11 position, and pivot with the first row and column:

$$\begin{bmatrix} 1 & -2 & -8 \\ 4 & 4 & 208 \end{bmatrix} R_2 \to R_2 - 4R_1$$

$$\begin{bmatrix} 1 & -2 & -8 \\ 0 & 12 & 240 \end{bmatrix} R_2 \to \left(\frac{1}{12}\right) R_2$$

$$\begin{bmatrix} 1 & -2 & -8 \\ 0 & 1 & 20 \end{bmatrix} R_1 \to R_1 + 2R_2$$

$$\begin{bmatrix} 1 & 0 & 32 \\ 0 & 1 & 20 \end{bmatrix}$$

Thus, the solution is $(x,y)=(32,20)$.

7. **Solution**: The augmented matrix is

$$\begin{bmatrix} 1 & 1 & 50 \\ .8 & .3 & 30 \end{bmatrix}$$

Pivoting left for the reader. (Suggestion: start with $R_2 \to 10R_2$.)

8. **Solution**: The system is

$$x - y - 0$$
$$-.5x + .8y = .8$$

Thus,

$$\begin{bmatrix} 1 & -1 & 0 \\ -.5 & .8 & .8 \end{bmatrix} R_2 \to R_2 + .5R_1$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & .3 & .8 \end{bmatrix} R_2 \to \left(\frac{10}{3}\right) R_2$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & \frac{8}{3} \end{bmatrix} R_1 \to R_1 + R_2$$

$$\begin{bmatrix} 1 & 0 & \frac{8}{3} \\ 0 & 1 & \frac{8}{3} \end{bmatrix}$$

Thus. The solution is $(x, y) = \left(\frac{8}{3}, \frac{8}{3}\right)$.

9. **Solution**: The system is

$$-y + z = 5$$
$$\tfrac{2}{3}x - z = 0$$
$$x + y + z = 58$$

One way to pivot is to begin by switching the first and last rows:

$$\begin{bmatrix} 0 & -1 & 1 & 5 \\ \frac{2}{3} & 0 & -1 & 0 \\ 1 & 1 & 1 & 58 \end{bmatrix} R_1 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & 1 & 1 & 58 \\ \frac{2}{3} & 0 & -1 & 0 \\ 0 & -1 & 1 & 5 \end{bmatrix} R_2 \to R_2 - \frac{2}{3}R_1$$

$$\begin{bmatrix} 1 & 1 & 1 & 58 \\ 0 & -\frac{2}{3} & -\frac{5}{3} & -\frac{116}{3} \\ 0 & -1 & 1 & 5 \end{bmatrix}$$

That completes the first pivot. Now we take care of the second column, beginning by multiplying through the second row by $-\frac{3}{2}$, then:

$$\begin{bmatrix} 1 & 1 & 1 & 58 \\ 0 & 1 & \frac{5}{2} & 58 \\ 0 & -1 & 1 & 5 \end{bmatrix} \begin{matrix} R_1 \to R_1 - R_2 \\ \\ R_3 \to R_3 + R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & -\frac{3}{2} & 0 \\ 0 & 1 & \frac{5}{2} & 58 \\ 0 & 0 & \frac{7}{2} & 63 \end{bmatrix}$$

That completes the second pivot. We now take care of the third column, beginning with replacing the third row by $\frac{2}{7}$ times itself to put a 1 in the 33 position, and then:

$$\begin{bmatrix} 1 & 0 & -\frac{3}{2} & 0 \\ 0 & 1 & \frac{5}{2} & 58 \\ 0 & 0 & 1 & 18 \end{bmatrix} \begin{matrix} R_1 \to R_1 + \frac{3}{2}R_3 \\ R_2 \to R_2 - \frac{5}{2}R_3 \\ \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 27 \\ 0 & 1 & 0 & 13 \\ 0 & 0 & 1 & 18 \end{bmatrix}$$

Thus, the solution is $(x, y, z) = (27,13,18)$.

10. **Solution**: The system is

$$x + y + z = 495$$
$$3x + 4y + 5z = 2035$$
$$y - z = 25$$

Thus,

$$\begin{bmatrix} 1 & 1 & 1 & 495 \\ 3 & 4 & 5 & 2035 \\ 0 & 1 & -1 & 25 \end{bmatrix} R_2 \to R_2 - 3R_1$$

$$\begin{bmatrix} 1 & 1 & 1 & 495 \\ 0 & 1 & 2 & 550 \\ 0 & 1 & -1 - & 25 \end{bmatrix} \begin{matrix} R_1 \to R_1 - R_2 \\ \\ R_3 \to R_3 - R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & 55 \\ 0 & 1 & 2 & 550 \\ 0 & 0 & -3 & -525 \end{bmatrix} R_3 \to -\frac{1}{3}R_3$$

$$\begin{bmatrix} 1 & 0 & -1 & 55 \\ 0 & 1 & 2 & 550 \\ 0 & 0 & 1 & 175 \end{bmatrix} \begin{matrix} R_1 \to R_1 + R_3 \\ R_2 \to R_2 - 2R_3 \\ \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 120 \\ 0 & 1 & 0 & 200 \\ 0 & 0 & 1 & 175 \end{bmatrix}$$

Thus, the solution is $(x, y, z) = (120,200,175)$.

11. **Solution:** The system is

$$x + y + z = 970$$
$$15x + 25y + 40z = 31100$$
$$-2x - y + z = 30$$

Thus,

$$\begin{bmatrix} 1 & 1 & 1 & 970 \\ 15 & 25 & 40 & 31100 \\ -2 & -1 & 1 & 30 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 15R_1 \\ R_3 \to R_3 + 2R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 970 \\ 0 & 10 & 25 & 16550 \\ 0 & 1 & 3 & 1970 \end{bmatrix} R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & 1 & 1 & 970 \\ 0 & 1 & 3 & 1970 \\ 0 & 10 & 25 & 16550 \end{bmatrix} \begin{matrix} R_1 \to R_1 - R_2 \\ \\ R_3 \to R_3 - 10R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & -2 & -1000 \\ 0 & 1 & 3 & 1970 \\ 0 & 0 & -5 & -3150 \end{bmatrix} R_3 \to \left(-\frac{1}{5}\right) R_3$$

$$\begin{bmatrix} 1 & 0 & -2 & -1000 \\ 0 & 1 & 3 & 1970 \\ 0 & 0 & 1 & 630 \end{bmatrix} \begin{matrix} R_1 \to R_1 + 2R_3 \\ R_2 \to R_2 - 3R_3 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 260 \\ 0 & 1 & 0 & 80 \\ 0 & 0 & 1 & 630 \end{bmatrix}$$

Thus, the solution is $(x, y, z) = (260,80,630)$.

1.3.2 Degenerate Examples and the General Gauss-Jordan Elimination Algorithm

Solutions to Exercises:

1. In Example 8, we suggested that there were several ways to start the elimination process to put a leading 1 in the first row and column. One of the suggestions, which we did not illustrate, was the operation $R_1 \to R_1 + R_2$. Finish the example using this suggestion and verify that your answer is the same as the one we obtained in the text.

   Solution: Pivot as follows:

$$\begin{bmatrix} 5 & -1 & 2 & 21 \\ -4 & 3 & 1 & 8 \\ 1 & 1 & 3 & 26 \end{bmatrix} R_1 \to R_1 + R_2$$

$$\begin{bmatrix} 1 & 2 & 3 & 29 \\ -4 & 3 & 1 & 8 \\ 1 & 1 & 3 & 26 \end{bmatrix} \begin{matrix} R_2 \to R_2 + 4R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 29 \\ 0 & 11 & 13 & 124 \\ 0 & -1 & 0 & -3 \end{bmatrix} R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & 2 & 3 & 29 \\ 0 & -1 & 0 & -3 \\ 0 & 11 & 13 & 124 \end{bmatrix} R_2 \to (-1)R_2$$

$$\begin{bmatrix} 1 & 2 & 3 & 29 \\ 0 & 1 & 0 & 3 \\ 0 & 11 & 13 & 124 \end{bmatrix} \begin{matrix} R_1 \to R_1 - 2R_2 \\ \\ R_3 \to R_3 - 11R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & 23 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 13 & 91 \end{bmatrix} R_3 \to \left(\frac{1}{13}\right) R_3$$

$$\begin{bmatrix} 1 & 0 & 3 & 23 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 7 \end{bmatrix} R_1 \to R_1 - 3R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

   Thus, the solution is $(x, y, z) = (2,3,7)$, which agrees with the solution found in the text.

In exercises 2-9, solve the system using the Gauss-Jordan algorithm. If the solution is infinite, write the solution in parametric form. Note that the steps used in pivoting are not unique. We present one possible set of steps. In the case of infinitely many solutions, different steps used for pivoting might result in solutions whose parametric form looks quite different.

2.

$$10x + 3y = 20$$
$$2x + y = 2$$

Solution: Pivoting left for the reader. The (unique) solution is $(x, y) = \left(\frac{7}{2}, -5\right)$.

3.

$$u + 2v + 3w = 11$$
$$2u + 4v - 4w = 2$$
$$3u + v + 5w = 25$$

Solution:

$$\begin{bmatrix} 1 & 2 & 3 & 11 \\ 2 & 4 & -4 & 2 \\ 3 & 1 & 5 & 25 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 11 \\ 0 & 0 & -10 & -20 \\ 0 & -5 & -4 & -8 \end{bmatrix} R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & 2 & 3 & 11 \\ 0 & -5 & -4 & -8 \\ 0 & 0 & -10 & -20 \end{bmatrix} \begin{matrix} R_2 \to \left(-\frac{1}{5}\right) R_2 \\ R_3 \to \left(-\frac{1}{10}\right) R_3 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 11 \\ 0 & 1 & \frac{4}{5} & \frac{8}{5} \\ 0 & 0 & 1 & 2 \end{bmatrix} R_1 \to R_1 - 2R_2$$

$$\begin{bmatrix} 1 & 0 & \frac{7}{5} & \frac{39}{5} \\ 0 & 1 & \frac{4}{5} & \frac{8}{5} \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{matrix} R_1 \to R_1 - \frac{7}{5} R_3 \\ R_1 \to R_2 - \frac{4}{5} R_3 \\ \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Thus, the solution is $(x, y, z) = (5,0,2)$.

4. a)

$$x + 2y - z = 70$$
$$2x - y + 8z = 150$$

Solution:

$$\begin{bmatrix} 1 & 2 & -1 & 70 \\ 2 & -1 & 8 & 150 \end{bmatrix} R_2 \to R_2 - 2R_1$$

$$\begin{bmatrix} 1 & 2 & -1 & 70 \\ 0 & -5 & 10 & 10 \end{bmatrix} R_2 \to \left(-\frac{1}{5}\right) R_2$$

$$\begin{bmatrix} 1 & 2 & -1 & 70 \\ 0 & 1 & -2 & -2 \end{bmatrix} R_1 \to R_1 - 2R_2$$

$$\begin{bmatrix} 1 & 0 & 3 & 74 \\ 0 & 1 & -2 & -2 \end{bmatrix}$$

This is reduced echelon form, so we are finished pivoting. The variable z is a free variable, and we can move it to the other side of the equations, obtaining

$$x = 74 - 3z$$
$$y = -2 + 2z$$
$$z \text{ is free}$$

Replacing z by the parameter t, the solution in parametric form is:

$$x = 74 - 3t$$
$$y = -2 + 2t$$
$$z = t$$

We can also write the solution as a set of ordered triples:

$$(x, y, z) = (74 - 3t, -2 + 2t, t),$$

where t is an arbitrary real number.

b) Observe that, thinking geometrically, each of these equations in part (a) is a plane in three-dimensional space, and since there are only two planes, their intersection cannot be a single point. The intersection of two planes is either empty (if they are parallel), or a line, or a plane (if they are coincident.) Thus, even before starting the elimination process, we know there cannot be a unique solution.

More generally, whenever there are more variables than there are equations, the system must be degenerate, with infinitely many (or no) solutions. Explain why, but do not use a geometric argument. Instead, appeal to the augmented coefficient matrix. [**Hint**: Think about free variables.]

Solution: The maximum number of leading 1's possible is the number of rows. Since (without counting the rightmost column) there are more columns than rows, there will be columns without leading 1's. If the system is consistent, these columns correspond to free variables, leading to infinitely many solutions. If it is inconsistent, there are no solutions. Either way, the system is degenerate.

5.

$$6x + 3y + 4z = 41$$
$$x - y + 2z = 17$$
$$2x + 5y + z = 22$$
$$7x + 7y + 3z = 46$$

Solution:

$$\begin{bmatrix} 6 & 3 & 4 & 41 \\ 1 & -1 & 2 & 17 \\ 2 & 5 & 1 & 22 \\ 7 & 7 & 3 & 46 \end{bmatrix} R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} 1 & -1 & 2 & 17 \\ 6 & 3 & 4 & 41 \\ 2 & 5 & 1 & 22 \\ 7 & 7 & 3 & 46 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - 6R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - 7R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & -1 & 2 & 17 \\ 0 & 9 & -8 & -61 \\ 0 & 7 & -3 & -12 \\ 0 & 14 & -11 & -73 \end{bmatrix} R_2 \to \left(\frac{1}{9}\right) R_2$$

$$\begin{bmatrix} 1 & -1 & 2 & 17 \\ 0 & 1 & -\frac{8}{9} & -\frac{61}{9} \\ 0 & 7 & -3 & -12 \\ 0 & 14 & -11 & -73 \end{bmatrix} \begin{matrix} R_1 \to R_1 + R_2 \\ \\ R_3 \to R_3 - 7R_2 \\ R_4 \to R_4 - 14R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & -1 & \frac{10}{9} & \frac{92}{9} \\ 0 & 1 & -\frac{8}{9} & -\frac{61}{9} \\ 0 & 0 & \frac{29}{9} & \frac{319}{9} \\ 0 & 0 & \frac{13}{9} & \frac{13}{9} \end{bmatrix} \begin{matrix} \\ \\ R_3 \to \left(\frac{9}{29}\right) R_3 \\ R_4 \to \left(\frac{9}{13}\right) R_4 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & \frac{10}{9} & \frac{92}{9} \\ 0 & 1 & -\frac{8}{9} & -\frac{61}{9} \\ 0 & 0 & 1 & 11 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

It should be clear at this point that there is no solution since the third equation gives $z = 11$, while the fourth gives $z = 1$. But to confirm, we apply one more operation: $R_4 \to R_4 - R_3$ to obtain:

$$\begin{bmatrix} 1 & -1 & \frac{10}{9} & \frac{92}{9} \\ 0 & 1 & -\frac{8}{9} & -\frac{61}{9} \\ 0 & 0 & 1 & 11 \\ 0 & 0 & 0 & -10 \end{bmatrix}$$

From the last equation we see the contradiction $0 = -10$, so indeed there is no solution, and we may stop pivoting.

6.

$$3w + 2x - y - 2z = 12$$
$$2w - x + 5y + 4z = 10$$
$$8w + 3x + 3y = 34$$

$$2w + 6x - 12y - 12z = 4$$

Solution:

$$\begin{bmatrix} 3 & 2 & -1 & -2 & 12 \\ 2 & -1 & 5 & 4 & 10 \\ 8 & 3 & 3 & 0 & 34 \\ 2 & 6 & -12 & -12 & 4 \end{bmatrix} R_1 \to R_1 - R_2$$

$$\begin{bmatrix} 1 & 3 & -6 & -6 & 2 \\ 2 & -1 & 5 & 4 & 10 \\ 8 & 3 & 3 & 0 & 34 \\ 2 & 6 & -12 & -12 & 4 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 8R_1 \\ R_4 \to R_4 - 2R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 3 & -6 & -6 & 2 \\ 0 & -7 & 17 & 16 & 6 \\ 0 & -21 & 51 & 48 & 18 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} R_2 \to \left(-\frac{1}{7}\right) R_2$$

$$\begin{bmatrix} 1 & 3 & -6 & -6 & 2 \\ 0 & 1 & -\frac{17}{7} & -\frac{16}{7} & -\frac{6}{7} \\ 0 & -21 & 51 & 48 & 18 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 \to R_1 - 3R_1 \\ R_3 \to R_3 + 21R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & \frac{9}{7} & \frac{6}{7} & \frac{32}{7} \\ 0 & 1 & -\frac{17}{7} & -\frac{16}{7} & -\frac{6}{7} \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} R_3 \to \left(-\frac{1}{3}\right) R_3$$

$$\begin{bmatrix} 1 & 0 & \frac{9}{7} & \frac{6}{7} & \frac{32}{7} \\ 0 & 1 & -\frac{17}{7} & -\frac{16}{7} & -\frac{6}{7} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 \to R_1 - \frac{6}{7} R_3 \\ R_2 \to R_2 + \frac{16}{7} R_3 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & \frac{9}{7} & 0 & \frac{32}{7} \\ 0 & 1 & \frac{17}{7} & 0 & -\frac{6}{7} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix is now in reduced echelon form. Note there is no leading 1 in the third column, so y is a free variable. The solution is:

$$w = \frac{32}{7} - \frac{9}{7} y$$

$$x = -\frac{6}{7} + \frac{17}{7} y$$

y is free

$$z = 0$$

Replacing y by the parameter t, the solution in parametric form is:

$$w = \frac{32}{7} - \frac{9}{7}t$$
$$x = -\frac{6}{7} + \frac{17}{7}t$$
$$y = t$$
$$z = 0$$

We leave it to the reader to also rewrite this in terms of ordered quadruples if desired.

7.

$$2x + 3y + 5z = 15$$
$$6x - 3y + 10z = 18$$
$$20x - 5z = 23$$

Solution:

$$\begin{bmatrix} 2 & 3 & 5 & 15 \\ 6 & -3 & 10 & 18 \\ 20 & 0 & -5 & 23 \end{bmatrix} R_1 \to \frac{1}{2}R_1$$

$$\begin{bmatrix} 1 & \frac{3}{2} & \frac{5}{2} & \frac{15}{2} \\ 6 & -3 & 10 & 18 \\ 20 & 0 & -5 & 23 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 6R_1 \\ R_3 \to R_3 - 20R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & \frac{3}{2} & \frac{5}{2} & \frac{15}{2} \\ 0 & -12 & -5 & -27 \\ 0 & -30 & -55 & -127 \end{bmatrix} R_2 \to \left(-\frac{1}{12}\right)R_2$$

$$\begin{bmatrix} 1 & \frac{3}{2} & \frac{5}{2} & \frac{15}{2} \\ 0 & 1 & \frac{5}{12} & \frac{9}{4} \\ 0 & -30 & -55 & -127 \end{bmatrix} \begin{matrix} R_1 \to R_1 - \frac{3}{2}R_2 \\ R_3 \to R_3 + 30R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & \frac{15}{8} & \frac{33}{8} \\ 0 & 1 & \frac{5}{12} & \frac{9}{4} \\ 0 & 0 & -\frac{85}{2} & -\frac{119}{2} \end{bmatrix} R_3 \to \left(-\frac{2}{85}\right)R_3$$

$$\begin{bmatrix} 1 & 0 & \frac{15}{8} & \frac{33}{8} \\ 0 & 1 & \frac{5}{12} & \frac{9}{4} \\ 0 & 0 & 1 & \frac{7}{5} \end{bmatrix} \begin{matrix} R_1 \to R_1 - \frac{15}{8}R_3 \\ R_2 \to R_2 - \frac{5}{12}R_3 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{3}{2} \\ 0 & 1 & 0 & \frac{5}{3} \\ 0 & 0 & 1 & \frac{7}{5} \end{bmatrix}$$

Thus, $(x, y, z) = \left(\frac{3}{2}, \frac{5}{3}, \frac{7}{5}\right)$.

8.

$$5w + 8x - y + z = 100$$
$$w + 2x + y + 2z = 60$$
$$2w + 4y - z = 50$$
$$4w + 5x + 2y + z = 100$$

Solution:

$$\begin{bmatrix} 5 & 8 & -1 & 1 & 100 \\ 1 & 2 & 1 & 2 & 60 \\ 2 & 0 & 4 & -1 & 50 \\ 4 & 5 & 2 & 1 & 100 \end{bmatrix} R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} 1 & 2 & 1 & 2 & 60 \\ 5 & 8 & -1 & 1 & 100 \\ 2 & 0 & 4 & -1 & 50 \\ 4 & 5 & 2 & 1 & 100 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - 5R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - 4R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & 2 & 60 \\ 0 & -2 & -6 & -9 & -200 \\ 0 & -4 & 2 & -5 & -70 \\ 0 & -3 & -2 & -7 & -140 \end{bmatrix} R_2 \to R_2 - R_4$$

$$\begin{bmatrix} 1 & 2 & 1 & 2 & 60 \\ 0 & 1 & -4 & -2 & -60 \\ 0 & -4 & 2 & -5 & -70 \\ 0 & -3 & -2 & -7 & -140 \end{bmatrix} \begin{matrix} R_1 \to R_1 - 2R_2 \\ \\ R_3 \to R_3 + 4R_2 \\ R_4 \to R_4 + 3R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 9 & 6 & 180 \\ 0 & 1 & -4 & -2 & -60 \\ 0 & 0 & -14 & -13 & -310 \\ 0 & 0 & -14 & -13 & -320 \end{bmatrix} R_3 \to R_3 - R_4$$

$$\begin{bmatrix} 1 & 0 & 9 & 6 & 180 \\ 0 & 1 & -4 & -2 & -60 \\ 0 & 0 & 0 & 0 & 10 \\ 0 & 0 & -14 & -13 & -320 \end{bmatrix}$$

The third row reads $0 = 10$, a contradiction. So, there are no solutions.

9.

$$-2p + 3q + 6r + s - 2t + 6u = 0$$
$$p + 4q + 4s + 2t - u = 60$$
$$q + 3r + 2s - 3u = 30$$

$$2p-3q+9r+2t+3u=50$$
$$4p+6q+10r-12s+8t-6u=20$$

Solution:

$$\begin{bmatrix} -2 & 3 & 6 & 1 & -2 & 6 & 0 \\ 1 & 4 & 0 & 4 & 2 & -1 & 60 \\ 0 & 1 & 3 & 2 & 0 & -3 & 30 \\ 2 & -3 & 9 & 0 & 2 & 3 & 50 \\ 4 & 6 & 10 & -12 & 8 & -6 & 20 \end{bmatrix} R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} 1 & 4 & 0 & 4 & 2 & -1 & 60 \\ -2 & 3 & 6 & 1 & -2 & 6 & 0 \\ 0 & 1 & 3 & 2 & 0 & -3 & 30 \\ 2 & -3 & 9 & 0 & 2 & 3 & 50 \\ 4 & 6 & 10 & -12 & 8 & -6 & 20 \end{bmatrix} \begin{matrix} R_2 \to R_2 + 2R_1 \\ \\ R_4 \to R_4 - 2R_1 \\ R_5 \to R_5 - 4R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 4 & 0 & 4 & 2 & -1 & 60 \\ 0 & 11 & 6 & 9 & 2 & 4 & 120 \\ 0 & 1 & 3 & 2 & 0 & -3 & 30 \\ 0 & -11 & 9 & -8 & -2 & 5 & -70 \\ 0 & -10 & 10 & -28 & 0 & -2 & -220 \end{bmatrix} R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & 4 & 0 & 4 & 2 & -1 & 60 \\ 0 & 1 & 3 & 2 & 0 & -3 & 30 \\ 0 & 11 & 6 & 9 & 2 & 4 & 120 \\ 0 & -11 & 9 & -8 & -2 & 5 & -70 \\ 0 & -10 & 10 & -28 & 0 & -2 & -220 \end{bmatrix} \begin{matrix} R_1 \to R_1 - 4R_2 \\ \\ R_3 \to R_3 - 11R_2 \\ R_4 \to R_4 + 11R_2 \\ R_5 \to R_5 + 10R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & -12 & -4 & 2 & 11 & -60 \\ 0 & 1 & 3 & 2 & 0 & -3 & 30 \\ 0 & 0 & -27 & -13 & 2 & 37 & -210 \\ 0 & 0 & 42 & 14 & -2 & -28 & 260 \\ 0 & 0 & 40 & -8 & 0 & -32 & 80 \end{bmatrix} R_3 \to \left(-\frac{1}{27}\right) R_3$$

$$\begin{bmatrix} 1 & 0 & -12 & -4 & 2 & 11 & -60 \\ 0 & 1 & 3 & 2 & 0 & -3 & 30 \\ 0 & 0 & 1 & \frac{13}{27} & -\frac{2}{27} & -\frac{37}{27} & \frac{70}{9} \\ 0 & 0 & 42 & 14 & -2 & -28 & 260 \\ 0 & 0 & 40 & -8 & 0 & -32 & 80 \end{bmatrix} \begin{matrix} R_1 \to R_1 + 12R_3 \\ R_2 \to R_2 - 3R_3 \\ \\ R_4 \to R_4 - 42R_3 \\ R_5 \to R_5 - 40R_3 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{16}{9} & \frac{10}{9} & -\frac{49}{9} & \frac{300}{9} \\ 0 & 1 & 0 & \frac{5}{9} & \frac{2}{9} & \frac{10}{9} & \frac{60}{9} \\ 0 & 0 & 1 & \frac{13}{27} & -\frac{2}{27} & -\frac{37}{27} & \frac{70}{9} \\ 0 & 0 & 0 & -\frac{56}{9} & \frac{10}{9} & \frac{266}{9} & -\frac{600}{9} \\ 0 & 0 & 0 & -\frac{736}{9} & \frac{80}{27} & \frac{616}{27} & -\frac{2080}{9} \end{bmatrix} R_4 \to \left(-\frac{9}{56}\right) R_4$$

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 0 & \frac{16}{9} & \frac{10}{9} & -\frac{49}{9} & \frac{300}{9} \\ 0 & 1 & 0 & \frac{5}{9} & \frac{2}{9} & \frac{10}{9} & \frac{60}{9} \\ 0 & 0 & 1 & \frac{13}{27} & -\frac{2}{27} & -\frac{37}{27} & \frac{70}{9} \\ 0 & 0 & 0 & 1 & -\frac{5}{28} & -\frac{19}{4} & \frac{75}{7} \\ 0 & 0 & 0 & -\frac{736}{27} & \frac{80}{27} & \frac{616}{27} & -\frac{2080}{9} \end{array}\right] \begin{array}{l} R_1 \to R_1 - \frac{16}{9}R_4 \\ R_2 \to R_2 - \frac{5}{9}R_4 \\ R_3 \to R_3 - \frac{13}{27}R_4 \\ \\ R_5 \to R_5 + \frac{736}{27}R_4 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{10}{7} & 3 & \frac{100}{7} \\ 0 & 1 & 0 & 0 & \frac{9}{28} & \frac{15}{4} & \frac{5}{7} \\ 0 & 0 & 1 & 0 & \frac{1}{84} & \frac{11}{12} & \frac{55}{21} \\ 0 & 0 & 0 & 1 & -\frac{5}{28} & -\frac{19}{4} & \frac{75}{7} \\ 0 & 0 & 0 & 0 & -\frac{40}{21} & -\frac{320}{3} & \frac{1280}{21} \end{bmatrix} R_5 \to \left(-\frac{21}{40}\right) R_5$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{10}{7} & 3 & \frac{100}{7} \\ 0 & 1 & 0 & 0 & \frac{9}{28} & \frac{15}{4} & \frac{5}{7} \\ 0 & 0 & 1 & 0 & \frac{1}{84} & \frac{11}{12} & \frac{55}{21} \\ 0 & 0 & 0 & 1 & -\frac{5}{28} & -\frac{19}{4} & \frac{75}{7} \\ 0 & 0 & 0 & 0 & 1 & 56 & -32 \end{bmatrix} \begin{array}{l} R_1 \to R_1 - \frac{10}{7}R_5 \\ R_2 \to R_2 - \frac{9}{28}R_5 \\ R_3 \to R_3 - \frac{1}{84}R_5 \\ R_4 \to R_4 + \frac{5}{28}R_5 \\ \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -77 & 60 \\ 0 & 1 & 0 & 0 & 0 & -\frac{57}{4} & 11 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{4} & 3 \\ 0 & 0 & 0 & 1 & 0 & \frac{21}{4} & 5 \\ 0 & 0 & 0 & 0 & 1 & 56 & -32 \end{bmatrix}$$

Replacing u with a parameter z, the solution in parametric form is:

$$p = 60 + 77z$$

$$q = 11 + \frac{57}{4}z$$

$$r = 3 - \frac{1}{4}z$$

$$s = 5 - \frac{21}{4}z$$

$$t = -32 - 56z$$

$$u = z$$

Notice that if we replace the parameter z by $4x$, where x is also a parameter, we can write the solution in parametric form without fractions:

$$p = 60 + 308x$$
$$q = 11 + 57x$$
$$r = 3 - x$$
$$s = 5 - 21x$$
$$t = -32 - 224x$$
$$u = 4x$$

10. Consider the linear system:

$$p + 20q + r - 2s + t = 10$$
$$q + 10r - 6s + t = 216$$
$$r + 8s - t = 18$$
$$s + 2t = 257$$
$$t = 121$$

a) Notice that the last equation tells us that $t = 121$. Substitute this into the previous equation, which involves only s and t, and solve it for s. Then substitute the values of s and t into the third equation and solve for r. Keep going, substituting in the values of the solved variables in the previous equation at each step until you obtain the complete solution. This method of solution is called *back substitution*, and it works because the augmented coefficient matrix has all 0 entries below each leading 1.

Solution: Note that $t = 121$ gives:

$$s + 2(121) = 257$$
$$s = 257 - 242 = 15$$

Now we obtain:

$$r + 8(15) - 121 = 18$$
$$r = 18 - 120 + 121 = 19$$

Continuing:

$$q + 10(19) - 6(15) + 121 = 216$$
$$q = 216 - 190 + 90 - 121 = -5$$

Finally:

$$p + 20(-5) + 19 - 2(15) + 121 = 10$$
$$p = 10 + 100 - 19 + 30 - 121 = 0$$

Thus, the solution is:

$$p = 0$$
$$q = -5$$
$$r = 19$$
$$s = 15$$
$$t = 121$$

b) Solve the system as usual using Gauss-Jordan elimination. You should obtain the same solution as in part a.

Solution:

$$\begin{bmatrix} 1 & 20 & 1 & -2 & 1 & 10 \\ 0 & 1 & 10 & -6 & 1 & 216 \\ 0 & 0 & 1 & 8 & -1 & 18 \\ 0 & 0 & 0 & 1 & 2 & 257 \\ 0 & 0 & 0 & 0 & 1 & 121 \end{bmatrix} R_1 \to R_1 - 20R_2$$

$$\begin{bmatrix} 1 & 0 & -199 & 118 & -19 & -4310 \\ 0 & 1 & 10 & -6 & 1 & 216 \\ 0 & 0 & 1 & 8 & -1 & 18 \\ 0 & 0 & 0 & 1 & 2 & 257 \\ 0 & 0 & 0 & 0 & 1 & 121 \end{bmatrix} \begin{matrix} R_1 \to R_1 + 199R_3 \\ R_2 \to R_2 - 10R_3 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1710 & -218 & -728 \\ 0 & 1 & 0 & -86 & 11 & 36 \\ 0 & 0 & 1 & 8 & -1 & 18 \\ 0 & 0 & 0 & 1 & 2 & 257 \\ 0 & 0 & 0 & 0 & 1 & 121 \end{bmatrix} \begin{matrix} R_1 \to R_1 - 1710R_4 \\ R_2 \to R_2 + 86R_4 \\ R_3 \to R_3 - 8R_4 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -3638 & -440{,}198 \\ 0 & 1 & 0 & 0 & 183 & 22{,}138 \\ 0 & 0 & 1 & 0 & -17 & -2{,}038 \\ 0 & 0 & 0 & 1 & 2 & 257 \\ 0 & 0 & 0 & 0 & 1 & 121 \end{bmatrix} \begin{matrix} R_1 \to R_1 + 3638R_5 \\ R_2 \to R_2 - 183R_5 \\ R_3 \to R_3 + 17R_5 \\ R_4 \to R_4 - 2R_5 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -5 \\ 0 & 0 & 1 & 0 & 0 & 19 \\ 0 & 0 & 0 & 1 & 0 & 15 \\ 0 & 0 & 0 & 0 & 1 & 121 \end{bmatrix}$$

This agrees with the answer in part a.

Remark: If we relax the definition of reduced row echelon form so that condition 3 reads that in every column with a leading 1, every entry *below* the leading 1 is a 0, the resulting matrix is said to be in ***row echelon form*** (as opposed to reduced row echelon form.) A variation of the Gauss-Jordan elimination algorithm to solve a system is to use row operations to put the augmented matrix in row echelon form, then use back substitution as is part (a) of exercise 10 above.

11. a) Solve the following system via Gauss-Jordan elimination:

$$x + 2y + 5z = 30$$
$$3x + 6y - 10z = 65$$
$$5x + 10y = 125$$

Solution:

$$\begin{bmatrix} 1 & 2 & 5 & 30 \\ 3 & 6 & -10 & 65 \\ 5 & 10 & 0 & 125 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 5R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 5 & 30 \\ 0 & 0 & -25 & -25 \\ 0 & 0 & -25 & -25 \end{bmatrix} R_2 \to \left(-\frac{1}{25}\right) R_2$$

$$\begin{bmatrix} 1 & 2 & 5 & 30 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -25 & -26 \end{bmatrix} \begin{matrix} R_1 \to R_1 - 5R_2 \\ \\ R_3 \to R_3 + 25R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 25 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is in reduced echelon form, and y is a free variable, which we set to a parameter t. The solution is:

$$x = 25 - 2t$$
$$y = t$$
$$z = 1$$

b) Instead, solve it using back substitution as indicated in the above remark.

Solution: The first two steps are exactly the same as in part a, until we end up with a matrix in row echelon form (but not reduced form):

$$\begin{bmatrix} 1 & 2 & 5 & 30 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -25 & -25 \end{bmatrix} R_3 \to R_3 + 25R_2$$

$$\begin{bmatrix} 1 & 2 & 5 & 30 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since this matrix is now in row echelon form, we may begin the back substitution. The second row yields $z = 1$, so the first row becomes, via back substitution:

$$x + 2y + 5(1) = 30$$

$$x + 2y = 25$$

Since y is a free variable (there is no leading 1 in the column for y), we are done, and obtain the same final solution as above:

$$x = 25 - 2t$$
$$y = t$$
$$z = 1$$

12. Solve the system arising from the traffic flow problem in exercise 12 of Section 1.2. What other conditions on the solutions are there arising from the restriction that all the variables must be nonnegative integers?

Solution: The system of equations is

$$\begin{aligned} s+u &= 500 \\ s+t-v &= 0 \\ t+w &= 200 \\ v-x-y &= 0 \\ w+x &= 350 \\ -u+z &= 250 \\ y+z &= 600 \end{aligned}$$

Which leads to the augmented matrix and pivoting as follows:

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 500 \\ 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 200 \\ 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 350 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 250 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 600 \end{bmatrix} R_2 \to R_2 - R_1$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 500 \\ 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & -500 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 200 \\ 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 350 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 250 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 600 \end{bmatrix} R_3 \to R_3 - R_2$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 500 \\ 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & -500 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 700 \\ 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 350 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 250 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 600 \end{bmatrix} \begin{matrix} R_1 \to R_1 - R_3 \\ R_2 \to R_2 + R_3 \\ \\ \\ \\ R_6 \to R_6 + R_3 \\ \\ \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & -200 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 200 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 700 \\ 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 350 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 950 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 600 \end{bmatrix} \begin{matrix} R_1 \to R_1 + R_4 \\ \\ R_3 \to R_3 - R_4 \\ \\ \\ R_6 \to R_6 - R_4 \\ \\ \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & -200 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 200 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 700 \\ 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 350 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 950 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 600 \end{bmatrix} \begin{matrix} R_1 \to R_1 + R_5 \\ R_2 \to R_2 - R_5 \\ R_3 \to R_3 - R_5 \\ \\ \\ R_6 \to R_6 - R_5 \\ \\ \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 150 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & -150 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 350 \\ 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 350 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 600 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 600 \end{bmatrix} \begin{matrix} R_1 \to R_1 + R_6 \\ \\ R_3 \to R_3 - R_6 \\ R_4 \to R_4 + R_6 \\ \\ \\ R_7 \to R_7 - R_6 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 750 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & -150 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -250 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 600 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 350 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 600 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This is in reduced echelon form. There are two free variables, x and z, corresponding to the columns without leading 1's. Denote them by the parameters $p = x$ and $q = z$. Then the solution in parametric form is:

$$\begin{gathered} s = 750 - q \\ t = p - 150 \\ u = q - 250 \\ v = 600 + p - q \\ w = 350 - p \\ x = p \\ y = 600 - q \\ z = q \end{gathered}$$

Like the traffic example in the text, there are extra conditions which must be satisfied when we interpret this solution in terms of the real-life traffic problem. The conditions are that all the variables must be integers, since they count numbers of vehicles, and furthermore, they must all be nonnegative since the network cannot support 'backwards' flow. This puts conditions on p and q, namely:

Variable which is nonnegative	Conditions on $\boldsymbol{p}$ and $\boldsymbol{q}$
$\boldsymbol{s \geq 0}$	$q \leq 750$
$\boldsymbol{t \geq 0}$	$p \geq 150$
$\boldsymbol{u \geq 0}$	$q \geq 250$

$\boldsymbol{v \geq 0}$	$q - p \leq 600$
$\boldsymbol{w \geq 0}$	$p \leq 350$
$\boldsymbol{x \geq 0}$	$p \geq 0$
$\boldsymbol{y \geq 0}$	$q \leq 600$
$\boldsymbol{z \geq 0}$	$q \geq 0$

The inequalities which involve only p give this:

$$0 \leq 150 \leq p \leq 350$$

Similarly, the inequalities which involve only q give this:

$$0 \leq 250 \leq q \leq 600 \leq 750$$

The one inequality which involves both p and q can be written

$$q \leq p + 600$$

However, since p is an integer between 150 and 350, we know $p + 600$ is an integer between 750 and 950, and since q is already known to be less than 600, it will automatically be less than $p + 600$, whence this inequality is superfluous.

Hence, the solutions to the traffic flow problem consist of all the solutions in the above parametric form, subject to the extra conditions that p is an integer satisfying $150 \leq p \leq 350$, and q is an integer satisfying $250 \leq q \leq 600$. (In particular, there are a finite number of solutions.)

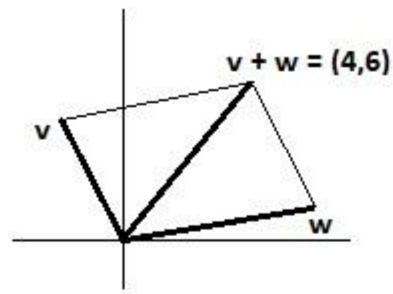
4. Vectors, Linear Combinations, Bases

Solutions to exercise:

1. Let $v = \binom{-2}{5}$ *and* $w = \binom{6}{1}$. Plot v, w and the following vectors in the plane:

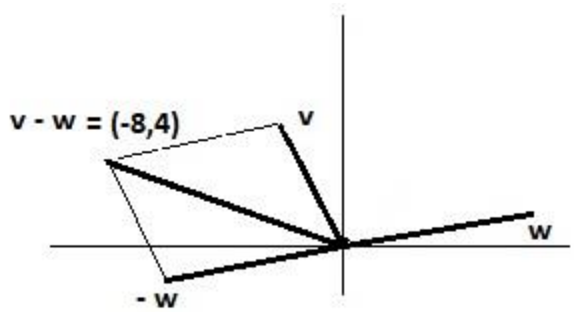
   a) $v + w$ (draw the parallelogram illustrating the parallelogram addition law.)

   Solution:



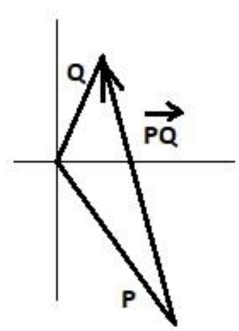
   b) $v - w$ (draw the parallelogram illustrating the parallelogram law. **HINT**: $v - w = v + (-w)$.)

   Solution:



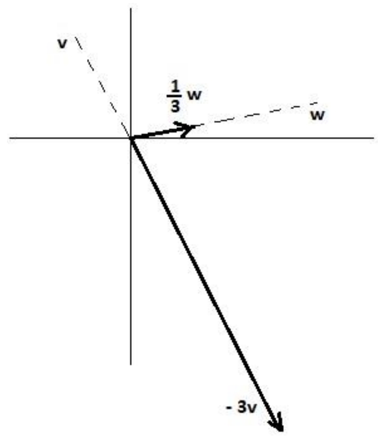
   c) The vector $\overrightarrow{PQ}$, where $P = (3, -4)$ and $Q = (1,3)$.

   Solution:



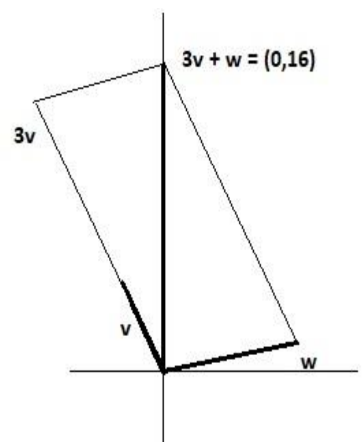
d) $z = -3v$ and $z = \frac{1}{3}w$.

Solution:



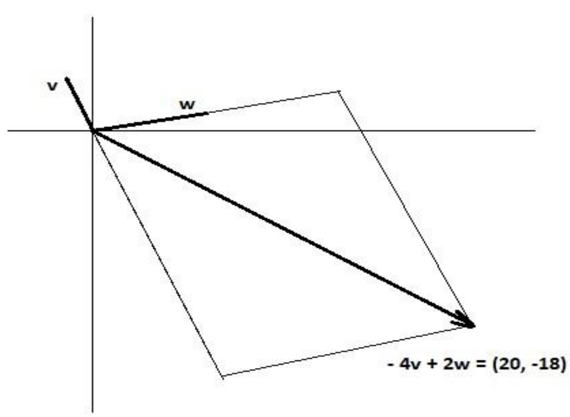
e) $z = 3v + w$

Solution:



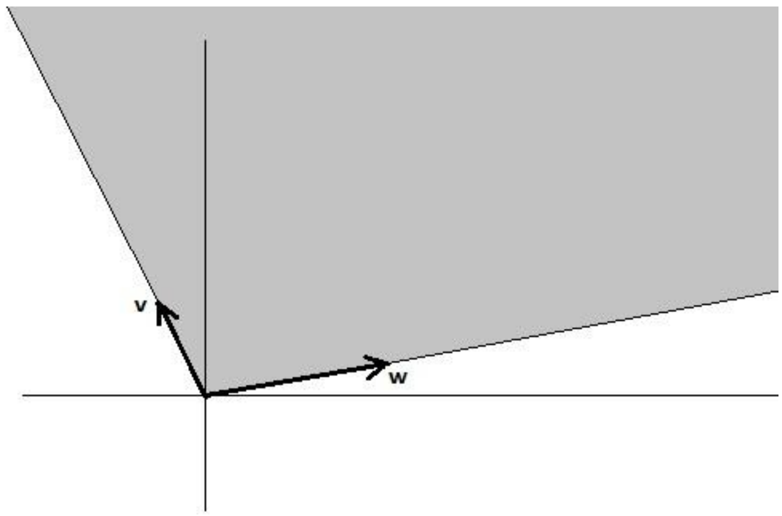
f) $z = -4v + 2w$

Solution:



g) Shade in the region of the plane that contains all the nonnegative linear combinations of v and w.

Solution:



2. In this exercise, we illustrate how theorems in Euclidean geometry may be prove using vectors. Let $ABCD$ be an arbitrary quadrilateral in the plane. Suppose the coordinates are $A = (x_1, y_1)$, $B = (x_2, y_2)$, $C = (x_3, y_3)$, and $D = (x_4, y_4)$. Let P be the midpoint of side AB, Q the midpoint of side BC, R the midpoint of side CD and S the midpoint of side DA.
   a) Find the coordinates of $P, Q, R,$ and S (using the midpoint formula for points in the plane you learned in your high school algebra class.)

Solution:

$$P = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$Q = \left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right)$$

$$R = \left(\frac{x_3 + x_4}{2}, \frac{y_3 + y_4}{2}\right)$$

$$S = \left(\frac{x_4+x_1}{2}, \frac{y_4+y_1}{2}\right).$$

   b) Compute the coordinates of the vector $\overrightarrow{PQ}$ and the coordinates of the vector $\overrightarrow{RS}$. Show that when you add these vectors, you obtain $\overrightarrow{PQ}+\overrightarrow{RS} = 0$.

Solution:

$$\overrightarrow{PQ} = Q - P = \left(\frac{x_3 - x_1}{2}, \frac{y_3 - y_1}{2}\right)$$

$$\overrightarrow{RS} = \left(\frac{x_1 - x_3}{2}, \frac{y_1 - y_3}{2}\right) = -\overrightarrow{PQ}$$

$$\overrightarrow{PQ} + \overrightarrow{RS} = (0,0).$$

c) Similarly, show that $\overrightarrow{QR} + \overrightarrow{SP} = 0$.

Solution:

$$\overrightarrow{QR} = R - Q = \left(\frac{x_4 - x_2}{2}, \frac{y_4 - y_2}{2}\right)$$

$$\overrightarrow{SP} = \left(\frac{x_2 - x_4}{2}, \frac{y_2 - y_4}{2}\right) = -\overrightarrow{QR}$$

$$\overrightarrow{QR} + \overrightarrow{SP} = (0,0).$$

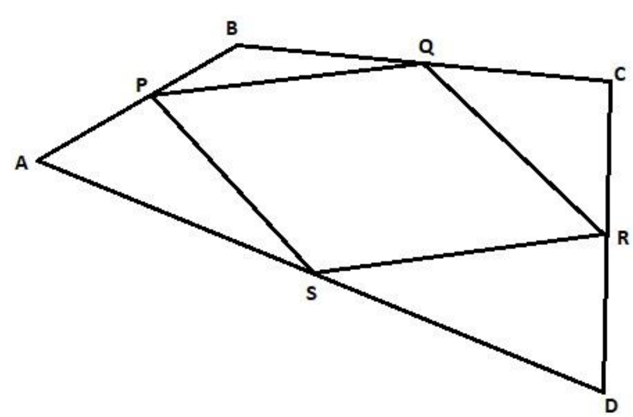
d) Explain why the results of parts (b) and (c) imply that the quadrilateral $PQRS$ is a parallelogram.

Solution: Since $\overrightarrow{PQ}+\overrightarrow{RS} = 0$, this means the vectors $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ must be parallel and the same length (and opposite orientation). Similarly, $\overrightarrow{QR}$ and $\overrightarrow{SP}$ are parallel and the same length. But these vectors are pairs of opposite sides of the quadrilateral $PQRS$. When opposite sides of a quadrilateral are parallel (and congruent), that is the definition of a parallelogram.

You have just proved the following

Theorem The quadrilateral $PQRS$ obtained by joining the midpoints of the sides of an arbitrary quadrilateral $ABCD$ is always a parallelogram.

The picture illustrates the theorem:



3. We mentioned that some of the properties of theorem 1.4 are easy to verify. For example, we verify the commutative law for addition of vectors. Let

$$u = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \text{ and } v = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Then

$$u + v = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix}$$

by definition of vector addition. But we now look at each coordinate – these are real numbers and we already know for real numbers that the commutative law holds. Thus, for each $i = 1,2, \dots, n$, we know $a_i + b_i = b_i + a_i$. Thus, the vector $u + v$ is the same as the vector

$$\begin{bmatrix} b_1 + a_1 \\ b_2 + a_2 \\ \vdots \\ b_n + a_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} + \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = v + u,$$

again, by using the definition of vector addition. Thus, the commutative law for addition of vectors follows from the commutative law for addition of real numbers.
Using similar ideas, working with coordinates, prove the following parts of theorem 1.4:

a) Part b of the theorem

Solution: This is the associative law. Let

$$u = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, v = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, w = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Then

$$(u + v) + w = \left(\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \right) + \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$$= \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} (a_1 + b_1) + c_1 \\ (a_2 + b_2) + c_2 \\ \vdots \\ (a_n + b_n) + c_n \end{bmatrix}$$

$$= \begin{bmatrix} a_1 + (b_1 + c_1) \\ a_2 + (b_2 + c_2) \\ \vdots \\ a_n + (b_n + c_n) \end{bmatrix}$$

(By the associative law for the addition of real numbers)

$$= \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 + c_1 \\ b_2 + c_2 \\ \vdots \\ b_n + c_n \end{bmatrix}$$

$$= \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \left(\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \right) = u + (v + w)$$

b) Part d of the theorem.

Solution: The existence of additive inverses. Let

$$v = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

be an arbitrary vector. Let w be the vector defined by negating each coordinate:

$$w = \begin{bmatrix} -b_1 \\ -b_2 \\ \vdots \\ -b_n \end{bmatrix}.$$

Then observe that

$$v + w = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} + \begin{bmatrix} -b_1 \\ -b_2 \\ \vdots \\ -b_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

This shows that w plays the role of $-v$, that is $v + (-v) = 0$, where $-v = w$.

c) Part e of the theorem.

Solution: The distributive law for scalar multiplication over vector addition. Let r be a real number and let

$$v = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \quad w = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

Then

$$r(v+w) = r\left(\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}\right)$$

$$= r\begin{bmatrix} b_1 + c_1 \\ b_2 + c_2 \\ \vdots \\ b_n + c_n \end{bmatrix} = \begin{bmatrix} r(b_1 + c_1) \\ r(b_2 + c_2) \\ \vdots \\ r(b_n + c_n) \end{bmatrix}$$

$$= \begin{bmatrix} rb_1 + rc_1 \\ rb_2 + rc_2 \\ \vdots \\ rb_n + rc_n \end{bmatrix}$$

$$= \begin{bmatrix} rb_1 \\ rb_2 \\ \vdots \\ rb_n \end{bmatrix} + \begin{bmatrix} rc_1 \\ rc_2 \\ \vdots \\ rc_n \end{bmatrix} = r\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} + r\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$$= rv + rw,$$

as claimed.

d) Part g of the theorem.

Solution: Associative law of scalar multiplication. Let r and s be real numbers and

$$v = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Then

$$r(sv) = r\begin{bmatrix} sb_1 \\ sb_2 \\ \vdots \\ sb_n \end{bmatrix} = \begin{bmatrix} r(sb_1) \\ r(sb_2) \\ \vdots \\ r(sb_n) \end{bmatrix}$$

$$= \begin{bmatrix} (rs)b_1 \\ (rs)b_2 \\ \vdots \\ (rs)b_n \end{bmatrix} = (rs)\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = (rs)v,$$

as claimed.

e) Part h of the theorem.

Solution: We must show that $1v = v$ for any vector v. Let

$$v = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Then

$$1v = 1\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} 1b_1 \\ 1b_2 \\ \vdots \\ 1b_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = v,$$

as claimed.

f) Prove that the additive inverse $-v$ of v is the same vector as the scalar multiple $(-1)v$. [**Hint**: The additive inverse $-v$ of v is defined by the property in part d of the theorem.]

Solution: Observe that

$$v + (-1)v$$

$$= 1v + (-1)v \text{ by part e}$$

$$= \big(1 + (-1)\big)v \text{ by part f}$$

$$= 0v = 0$$

The last equality because

$$0\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} 0b_1 \\ 0b_2 \\ \vdots \\ 0b_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

We have shown that $v + (-1)v = 0$, but by part d of the theorem, this means $(-1)v$ is the additive inverse of v: $(-1)v = -v$.

4. Let $S = \{u, v, w\} = \left\{\begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \\ 2 \\ -1 \end{bmatrix}\right\}$ be a set of vectors in $\mathbb{R}^4$. Find the following elements of $Span(S)$:

a) $2u + v + w$

Solution:

$$2u + v + w = \begin{bmatrix} 0 \\ 7 \\ 4 \\ 6 \end{bmatrix}$$

b) $5u + 2w$

Solution:

$$5u + 2w = \begin{bmatrix} -3 \\ 16 \\ 9 \\ 13 \end{bmatrix}$$

c) $7u - 2v - 3w$

Solution:

$$7u - 2v - 3w = \begin{bmatrix} 15 \\ 5 \\ 1 \\ 22 \end{bmatrix}$$

5. Let $v = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ and $w = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$. In this section of the text, we claimed that $z = \begin{bmatrix} -9 \\ 13 \\ 11 \end{bmatrix}$ is an element of $Span\{v, w\}$. To check this, we were led to solving the following system:

$$x + 3y = -9$$
$$3x - y = 13$$
$$5x + y = 11$$

Solve the system, and verify that the unique solution is $(x, y) = (3, -4)$ as we claimed.

Solution:

$$\begin{bmatrix} 1 & 3 & -9 \\ 3 & -1 & 13 \\ 5 & 1 & 11 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 5R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 3 & -9 \\ 0 & -10 & 40 \\ 0 & -14 & 56 \end{bmatrix} R_2 \to \left(-\frac{1}{10}\right) R_2$$

$$\begin{bmatrix} 1 & 3 & -9 \\ 0 & 1 & -4 \\ 0 & -14 & 56 \end{bmatrix} \begin{matrix} R_1 \to R_1 - 3R_2 \\ \\ R_3 \to R_3 + 14R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, $(x, y) = (3, -4)$ as claimed.

6. In this section of the text, we claimed that $z = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \notin Span\{v, w\}$, where v and w are the vectors from exercise 5. Verify this claim by showing that the system below has no solution:

$$\begin{gathered} x + 3y = 1 \\ 3x - y = 2 \\ 5x + y = 1 \end{gathered}$$

Solution:

$$\begin{bmatrix} 1 & 3 & 1 \\ 3 & -1 & 2 \\ 5 & 1 & 1 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 5R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & -10 & -1 \\ 0 & -14 & -4 \end{bmatrix} R_2 \to \left(-\frac{1}{10}\right) R_2$$

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & \frac{1}{10} \\ 0 & -14 & -4 \end{bmatrix} \begin{matrix} R_1 \to R_1 - 3R_2 \\ \\ R_3 \to R_3 + 14R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & \frac{7}{10} \\ 0 & 1 & \frac{1}{10} \\ 0 & 0 & -\frac{26}{10} \end{bmatrix}$$

The last equation reads $0 = -\frac{26}{10}$, which has no solution. Thus, the entire system has no solution, and so z is not in the span of $\{v, w\}$.

7. In this section of the text, we considered the example:

$$S = \left\{ \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 7 \\ 4 \end{bmatrix} \right\}$$

And we claimed that the vector $z = \begin{bmatrix} 14 \\ 16 \\ 22 \end{bmatrix}$ was an element of $Span(S)$, but that the expression of z as a linear combination of the elements of S was not unique. Verify this by finding the complete solution to the system we derived:

$$\begin{gathered} 4r + s - 2t = 14 \\ -r + 3s + 7t = 16 \\ 2r + 3s + 4t = 22 \end{gathered}$$

Write the solution in parametric form, and verify that the two specific solutions we gave in the text are elements of this parametric family.

Solution:

$$\begin{bmatrix} 4 & 1 & -2 & 14 \\ -1 & 3 & 7 & 16 \\ 2 & 3 & 4 & 22 \end{bmatrix} R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} -1 & 3 & 7 & 16 \\ 4 & 1 & -2 & 14 \\ 2 & 3 & 4 & 22 \end{bmatrix} R_1 \to -R_1$$

$$\begin{bmatrix} 1 & -3 & -7 & -16 \\ 4 & 1 & -2 & 14 \\ 2 & 3 & 4 & 22 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - 4R_1 \\ R_2 \to R_2 - 2R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & -3 & -7 & -16 \\ 0 & 13 & 26 & 78 \\ 0 & 9 & 18 & 54 \end{bmatrix} R_2 \to \frac{1}{13} R_2$$

$$\begin{bmatrix} 1 & -3 & -7 & -16 \\ 0 & 1 & 2 & 6 \\ 0 & 9 & 18 & 54 \end{bmatrix} \begin{matrix} R_1 \to R_1 + 3R_2 \\ \\ R_3 \to R_3 - 9R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is in reduced echelon form, and t is a free variable (we use the letter u for the parameter). The solution in parametric form is

$$r = 2 + u$$
$$s = 6 - 2u$$
$$t = u$$

When $u = 1$, we obtain $(r, s, t) = (3,4,1)$, and when $u = 0$, we obtain $(r, s, t) = (2,6,0)$, which were the two specific solutions given in the text.

8. In this section of the text, we considered the example:

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

We made two claims about this set:

i) Any vector $z = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in $\mathbb{R}^3$ is an element of $Span(S)$ (that is, $Span(S) = \mathbb{R}^3$), and

ii) For any vector z, the expression of z as a linear combination of the elements of S is unique (that is, S is a linearly independent set.)

In order to verify these claims, we would have to show that for any vector z, the following system has a unique solution:

$$x + z = a$$
$$x + y + z = b$$
$$y + z = c$$

Verify these claims by solving the system, and show that the unique solution is the one we gave in the text.

Solution:

$$\begin{bmatrix} 1 & 0 & 1 & a \\ 1 & 1 & 1 & b \\ 0 & 1 & 1 & c \end{bmatrix} R_2 \to R_2 - R_1$$

$$\begin{bmatrix} 1 & 0 & 1 & a \\ 0 & 1 & 0 & b-a \\ 0 & 1 & 1 & c \end{bmatrix} R_3 \to R_3 - R_2$$

$$\begin{bmatrix} 1 & 0 & 1 & a \\ 0 & 1 & 0 & b-a \\ 0 & 0 & 1 & a-b+c \end{bmatrix} R_1 \to R_1 - R_3$$

$$\begin{bmatrix} 1 & 0 & 0 & b-c \\ 0 & 1 & 0 & b-a \\ 0 & 0 & 1 & a-b+c \end{bmatrix}$$

Thus, the unique solution is $(x, y, z) = (b-c, b-a, a-b+c)$ as claimed in the text.

9. a) Consider the following set:

$$S = \{u, v, w\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \right\}$$

We mentioned in this section that $Span(S)$ must always contain the vector $\mathbf{0}$. That is because we always have the following linear combination, called the *trivial* linear combination:

$$0u + 0v + 0w = 0$$

In this exercise, show that the trivial linear combination is the ONLY linear combination of the 0 vector. That is, show that the only solution to the following system is the trivial solution $x = y = z = 0$:

$$x + 2y - z = 0$$
$$x + 2z = 0$$
$$x + y + 3z = 0$$

Remark: A system of linear equations with all 0's on the right side is said to be a *homogeneous* system.

Solution:

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 1 & 3 & 0 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -2 & 3 & 0 \\ 0 & -1 & 4 & 0 \end{bmatrix} R_2 \to \left(-\frac{1}{2}\right) R_2$$

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -\frac{3}{2} & 0 \\ 0 & -1 & 4 & 0 \end{bmatrix} \begin{matrix} R_1 \to R_1 - 2R_2 \\ \\ R_3 \to R_3 + R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & \frac{5}{2} & 0 \end{bmatrix} R_3 \to \frac{2}{5} R_3$$

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{matrix} R_1 \to R_1 - 2R_3 \\ R_2 \to R_2 + \frac{3}{2} R_3 \\ \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Thus, the unique solution is $(x, y, z) = (0,0,0)$ as claimed.

b) By contrast, consider the following set:

$$T = \{u, v, w\} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 11 \\ -4 \\ 13 \end{bmatrix} \right\}$$

Show that, in addition to the trivial linear combination, there are nontrivial linear combinations of the 0 vector. Do this by solving the system and writing the solution in parametric form:

$$\begin{gathered} x + 4y + 11z = 0 \\ -2x + y - 4z = 0 \\ 3x + 2y + 13z = 0 \end{gathered}$$

Solution:

$$\begin{bmatrix} 1 & 4 & 11 & 0 \\ -2 & 1 & -4 & 0 \\ 3 & 2 & 13 & 0 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 4 & 11 & 0 \\ 0 & 9 & 18 & 0 \\ 0 & -10 & -20 & 0 \end{bmatrix} R_2 \to \frac{1}{9} R_2$$

$$\begin{bmatrix} 1 & 4 & 11 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -10 & -20 & 0 \end{bmatrix} \begin{matrix} R_1 \to R_1 - 4R_2 \\ \\ R_3 \to R_3 + 10R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is reduced echelon form, with one free variable z. Setting $z = t$, the solution in parametric form is

$$\begin{gathered} x = -3t \\ y = 2t \\ z = t \end{gathered}$$

Thus, there are nontrivial (when $t \neq 0$) linear combinations of the zero vector.

c) An alternate way to show that there are nontrivial linear combinations of the vectors in T giving the zero vector is the following. Show that the third vector w is in fact in the span of the first two. That is, show that you can find scalars s, t such that

$$su + tv = w$$

Do this by solving the appropriate system. Once you have the solution, put all the vectors on one side, writing

$$su + tv - w = 0,$$

and simply note that this is a nontrivial (because the coefficient of w is non zero) linear combination of the elements of T equal to 0.

Solution: Using the same row operations as in part (b):

$$\begin{bmatrix} 1 & 4 & 11 \\ -2 & 1 & -4 \\ 3 & 2 & 13 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 4 & 11 \\ 0 & 9 & 18 \\ 0 & -10 & -20 \end{bmatrix} R_2 \to \frac{1}{9}R_2$$

$$\begin{bmatrix} 1 & 4 & 11 \\ 0 & 1 & 2 \\ 0 & -10 & -20 \end{bmatrix} \begin{matrix} R_1 \to R_1 - 4R_2 \\ \\ R_3 \to R_3 + 10R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, $(x, y) = (3,2)$. Indeed, we see directly that

$$3\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 11 \\ -4 \\ 13 \end{bmatrix}$$

That is,

$$3u + 2v = w,$$

which, in turn, gives a nontrivial linear combination of the zero vector:

$$3u + 2v - w = 0$$

Remark: Generalizing this example, it is not hard to prove the following:

a) A set S of vectors is linearly independent if and only if the only linear combination of elements of S yielding the zero vector is the trivial one.
(This is the alternate definition of linear independence given in most linear algebra textbooks.)

b) A set T of vectors is linearly dependent if and only if one of the vectors in T can be written as a linear combination of the other vectors in T.

10. Let $S = \{u, v\}$ be a set of two vectors. Show that S is a linearly dependent set if and only if u and v are parallel (that is, if and only if v is a nonzero scalar multiple of u.)

Solution: If S is linearly dependent, then there are nonzero scalars a and b such that

$$au + bv = 0$$

$$au = -bv$$

$$u = -\frac{b}{a}v,$$

The last step because $a \neq 0$.

Conversely, if u and v are parallel, then there is a nonzero scalar c with $u = cv$. But then $u - cv = 0$, with $c \neq 0$, so there is a nontrivial linear combination of the zero vector. Thus, by the remark (a) after the previous exercise, the set $\{u, v\}$ is linearly dependent.

5. Three Points of View

Solutions to exercises:

1. Convert the following system of equations to a vector equation; that is, into a linear combination. Do not solve, but write down explicitly what the set S of vectors is and what the linear combination on the right side (often denoted z in the text) is:

$$\begin{gathered} 19x + 5y - 12z = 109 \\ 24x + 11y + z = 168 \\ 16x - 2y + 30z = 150 \end{gathered}$$

Solution:

$$S = \left\{ \begin{bmatrix} 19 \\ 24 \\ 16 \end{bmatrix}, \begin{bmatrix} 5 \\ 11 \\ -2 \end{bmatrix}, \begin{bmatrix} -12 \\ 1 \\ 30 \end{bmatrix} \right\} = \{u, v, w\}$$

$$B = \begin{bmatrix} 109 \\ 168 \\ 150 \end{bmatrix}$$

The vector equation is:

$$x \begin{bmatrix} 19 \\ 24 \\ 16 \end{bmatrix} + y \begin{bmatrix} 5 \\ 11 \\ -2 \end{bmatrix} + z \begin{bmatrix} -12 \\ 1 \\ 30 \end{bmatrix} = \begin{bmatrix} 109 \\ 168 \\ 150 \end{bmatrix}$$

Or simply:

$$xu + yv + zw = B$$

2. Same directions as problem 1 for this system:

$$\begin{gathered} 2p + q + 5r - 3s + 10t = 800 \\ p + r + t = 90 \\ -p + 2q - 3r + 4s - 5t = 0 \end{gathered}$$

$$10p + 15q + 6r - 7s + 3t = 450$$
$$4p + 10q + r - 12t = 227$$

Solution: The vector equation is

$$p\begin{bmatrix} 2 \\ 1 \\ -1 \\ 10 \\ 4 \end{bmatrix} + q\begin{bmatrix} 1 \\ 0 \\ 2 \\ 15 \\ 10 \end{bmatrix} + r\begin{bmatrix} 5 \\ 1 \\ -3 \\ 6 \\ 1 \end{bmatrix} + s\begin{bmatrix} -3 \\ 0 \\ 4 \\ -7 \\ 0 \end{bmatrix} + t\begin{bmatrix} 10 \\ 1 \\ -5 \\ 3 \\ -12 \end{bmatrix} = \begin{bmatrix} 800 \\ 90 \\ 0 \\ 450 \\ 227 \end{bmatrix}.$$

3. The following matrix is the augmented matrix for a linear system:

$$\begin{bmatrix} 1 & 0 & 4 & -2 & 3 & 99 \\ 0 & 1 & 6 & 2 & 0 & 50 \\ 5 & -2 & 3 & 3 & 3 & 39 \\ 4 & 1 & 4 & 1 & 4 & 41 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{bmatrix}$$

a) Write down explicitly the system it represents. Do not solve it.
Solution: Using variables v, w, x, y, z, the system is:

$$v + 4x - 2y + 3z = 99$$
$$w + 6x + 3y = 50$$
$$5v - 2w + 3x + 3y + 3z = 39$$
$$4v + v + 4x + y + 4z = 41$$
$$6v + 5w + 4x + 3y + 2z = 1$$

b) Write down explicitly the corresponding vector equation (linear combination).
Solution:

$$v\begin{bmatrix} 1 \\ 0 \\ 5 \\ 4 \\ 6 \end{bmatrix} + w\begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \\ 5 \end{bmatrix} + x\begin{bmatrix} 4 \\ 6 \\ 3 \\ 4 \\ 4 \end{bmatrix} + y\begin{bmatrix} -2 \\ 2 \\ 3 \\ 1 \\ 3 \end{bmatrix} + z\begin{bmatrix} 3 \\ 0 \\ 3 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 99 \\ 50 \\ 39 \\ 41 \\ 1 \end{bmatrix}$$

4. In the chart above, in the case when there is a unique solution, explain why there must be rows of all 0's at the bottom of the augmented matrix in reduced echelon form in the case when $m > n$. [**Hint**: Think about free variables and leading 1's.]

Solution: There cannot be any free variables in a system with a unique solution. Therefore, any nonzero column must contain a leading 1. Since $m > n$, there cannot be a leading 1 in each row, so there are rows which are either all 0's or all 0's except for a nonzero entry in the rightmost column. In the latter case, there would be no solution, a contradiction to the assumption of a unique solution. Therefore, there are rows with all 0's.

5. In case $m < n$, and every row has a leading 1, explain why there cannot be a unique solution. Recall that the *rank* of a matrix is the number of leading 1's once the matrix has been put into reduced echelon form. More generally, explain why if $rank([A|z] < n$, there cannot be a unique solution.

   Solution: Since $n > m$, and every leading 1 is in a distinct row and column, there must be columns without leading ones. These columns represent free variables, so there must be infinitely many solutions. The same argument works in the more general case – if the rank is less than n, there will be columns without leading 1's and hence free variables.

6. In the third row of the chart above, the case of a unique solution, we are suggesting that if there is a unique solution for this one particular z, then S must be a linearly independent set. However, the alert reader will remember that the definition of linearly independence is that there must be a unique solution for <u>every</u> z in the span of S, not just for one particular z, Explain why the discrepancy doesn't matter. That is, explain why if the solution is unique for one z in the Span of S, it must be unique for every z in the span.

   a) Use the perspective of vector equations to answer this. [**Hint**: if you have two distinct solutions for a particular z, subtract them to obtain a nontrivial linear combination equal to the zero vector. Once you have that, you can add it to ANY solution for any z to produce a different solution for the same z.]

   Solution: Following the hint, suppose we have two distinct solutions for a particular z:

$$a_1v_1 + a_2v_2 + \cdots + a_nv_n = z$$

$$b_1v_1 + b_2v_2 + \cdots + b_nv_n = z$$

   Subtracting, we obtain a nontrivial (because at least one $a_i \neq b_i$) linear combination of the zero vector:

$$(a_1 - b_1)v_1 + (a_2 - b_2)v_2 + \cdots + (a_n - b_n)v_n = 0$$

   Then if w is any vector in the span of S, and we have a solution:

$$c_1v_1 + c_2v_2 + \cdots + c_nv_n = w,$$

   we can add this to the above linear combination of 0 above to obtain:

$$(c_1 + a_1 - b_1)v_1 + (c_2 + a_2 - b_2)v_2 + \cdots + (c_n + a_n - b_n)v_n = w$$

This shows that if one particular z has a non-unique solution, so does any w in the span of S. Conversely, if that particular z has a unique solution, then of a different z had a non-unique solution, so would z, a contradiction. So one particular z in the span of S has a unique linear combination in terms of S if and only if every vector in the span does so.

b) Instead, answer this from the perspective of the augmented coefficient matrix. [**Hint**: If you have a unique solution of ra particular z, can there be a leading 1 in the last column? If not, then what happens when you change z (i.e., change the last column)? Where do the leading 1's end up?]

Solution: Let $[A|z]$ be the augmented coefficient matrix. If there is a unique solution, the every leading 1 belongs to A (so is not in the last column), yet there are no free variables either, so every column (except the last column) has a leading 1. In particular, that means that, except for a possible number of rows of all 0's at the bottom, the reduced matrix looks like the identity matrix I_n. It follows that if we replace z to any other vector w in the span of S, the row operations we use to solve the system $[A|w]$ are exactly the same row operations as the ones we used to solve $[A|z]$, and therefore, the reduced form for $[A|w]$ also consists of the identity matrix I_n, possibly with rows of all 0's below it. In particular, there are no free variables, so the solution is unique.

7. Consider the system:

$$12w + 4x + 2y - 3z = 14$$
$$4w + x + y + 3z = 6$$

a) Verify that $(w, x, y, z) = (2, -3, 1, 0)$ is a solution. Determine whether or not it is a basic solution.

Solution: It is easy to check that it is a solution, which we leave to the reader. To see if it is a basic solution, rewrite the system as a vector equation:

$$w\begin{bmatrix}12\\4\end{bmatrix} + x\begin{bmatrix}4\\1\end{bmatrix} + y\begin{bmatrix}2\\1\end{bmatrix} + z\begin{bmatrix}-3\\3\end{bmatrix} = \begin{bmatrix}14\\6\end{bmatrix}$$

Since the given solution has only $z = 0$, the set S in question is

$$S = \left\{\begin{bmatrix}12\\4\end{bmatrix}, \begin{bmatrix}4\\1\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}\right\}$$

This set is NOT a basis of its span (students of linear algebra will recognize that a basis must have exactly two (linearly independent) vectors since the span is a two-dimensional plane.) To see directly that it is not a basis, one may proceed in several ways. For example, one can show that S is linearly dependent. To do this, one approach is to solve the homogeneous system:

$$12w + 4x + 2y = 0$$

$$4w + x + y = 0,$$

and verify that there are infinitely many solutions (the reduced echelon form will show that some columns correspond to free variables.) Again, we leave it to the reader to check this.

Instead, we could use Remark 23b at the end of the exercises in the last section. If a set S has the property that one vector in S is a combination of the remaining vectors in S, then it must be a linearly dependent set. In this example, observe that

$$\begin{bmatrix} 12 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 4 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Thus, the set S is not a basis so the solution is not a basic solution.

b) Verify that $(w, x, y, z) = (0,1,5,0)$ is a solution. Determine whether or not it is a basic solution.

Solution: Again, it is trivial to verify that this is a solution. To see if it is a basic solution, again rewrite the system as a vector equation:

$$w\begin{bmatrix} 12 \\ 4 \end{bmatrix} + x\begin{bmatrix} 4 \\ 1 \end{bmatrix} + y\begin{bmatrix} 2 \\ 1 \end{bmatrix} + z\begin{bmatrix} -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \\ 6 \end{bmatrix}$$

This time, $w = z = 0$, so the set in question is

$$S = \left\{\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$$

This time, we claim that S is a basis of its span, so the solution is a basic solution. Again, there are various ways to proceed to verify this. One way is to use exercise 10 from the last section. Since S has 2 vectors, it is a linearly independent set as long as the two vectors in it are not parallel. But they are visibly not parallel. Or just note that there is no scalar multiple s which yields

$$s\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

Once we know S is linearly independent, it is automatically a basis of its span. Another way to approach it is to show that for any vector $\begin{bmatrix} a \\ b \end{bmatrix}$, there is a unique solution to the equation:

$$x\begin{bmatrix} 4 \\ 1 \end{bmatrix} + y\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}.$$

This will establish that the span of S is the entire $\mathbb{R}^2$ plane (because there is a solution for any vector), and that S is linearly independent (because the solution is unique.) This can be accomplished by Gaussian elimination:

$$\begin{bmatrix} 4 & 2 & a \\ 1 & 1 & b \end{bmatrix} R_1 \to \frac{1}{4} R_1$$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{a}{4} \\ 1 & 1 & b \end{bmatrix} R_2 \to R_2 - R_1$$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{a}{4} \\ 0 & \frac{1}{2} & b - \frac{a}{4} \end{bmatrix} R_2 \to 2R_2$$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{a}{4} \\ 0 & 1 & 2b - \frac{a}{2} \end{bmatrix} R_1 \to R_1 - \frac{1}{2} R_2$$

$$\begin{bmatrix} 1 & 0 & \frac{a}{2} - b \\ 0 & 1 & 2b - \frac{a}{2} \end{bmatrix}$$

The solution is unique (which should be clear as soon as ne sees that there will be two leading 1's so that the matrix has rank 2!) Thus, S is a basis and the solution is a basic solution.

c) Given that $S = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$, find the corresponding basic solution.

Solution: Given that S is a basis, the corresponding basic solution is when we set $w = x = 0$ in the original equation:

$$0 \begin{bmatrix} 12 \\ 4 \end{bmatrix} + 0 \begin{bmatrix} 4 \\ 1 \end{bmatrix} + y \begin{bmatrix} 2 \\ 1 \end{bmatrix} + z \begin{bmatrix} -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \\ 6 \end{bmatrix}$$

$$y \begin{bmatrix} 2 \\ 1 \end{bmatrix} + z \begin{bmatrix} -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \\ 6 \end{bmatrix}$$

We solve this as usual:

$$\begin{bmatrix} 2 & -3 & 14 \\ 1 & 3 & 6 \end{bmatrix} R_1 \to \frac{1}{2} R_1$$

$$\begin{bmatrix} 1 & -\frac{3}{2} & 7 \\ 1 & 3 & 6 \end{bmatrix} R_2 \to R_2 - R_1$$

$$\begin{bmatrix} 1 & -\frac{3}{2} & 7 \\ 0 & \frac{9}{2} & -1 \end{bmatrix} R_2 \to \frac{2}{9} R_2$$

$$\begin{bmatrix} 1 & -\frac{3}{2} & 7 \\ 0 & 1 & -\frac{2}{9} \end{bmatrix} R_1 \to R_1 + \frac{3}{2} R_2$$

$$\begin{bmatrix} 1 & 0 & \frac{20}{3} \\ 0 & 1 & -\frac{2}{9} \end{bmatrix}$$

Thus, the corresponding basic solution is $(w, x, y, z) = \left(0,0,\frac{20}{3}, -\frac{2}{9}\right)$.

8. Consider the system:

$$w + 4x + 2y + 2z = 14$$
$$2w + 5x + 2y = 19$$
$$3w + 6x + 2y - 3z = 24$$

Verify that $(w, x, y, z) = (4,1,3,0)$ is a solution, but not a basic solution.

Solution: To verify it is a solution:

$$4 + 4(1) + 2(3) + 2(0) = 14$$
$$2(4) + 5(1) + 2(3) = 19$$
$$3(4) + 6(1) + 2(3) - 3(0) = 24$$

Writing the system in vector form:

$$w \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + y \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} + z \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 14 \\ 19 \\ 24 \end{bmatrix}.$$

Since $z = 0$, the solution is basic if and only if

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right\}$$

Is a basis of its span; that is, if S is linearly independent. However, observe that

$$1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \frac{3}{2} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}.$$

Since one element of S is a linear combination of two other vectors in S, S cannot be linearly independent (Remark 23b from the exercises at the end of the last section.) Thus, S is not a basis and the given solution is not a basic one.

9. Consider the system:

$$\begin{gathered} p+q+r+5s+t=100 \\ q+r+8s-3t=225 \\ r+10s+7t=169 \end{gathered}$$

Explain why any solution where $s=t=0$ must be a basic solution.

Solution: When $s=t=0$, we can write the system as:

$$\begin{gathered} p+q+r=100 \\ q+r=225 \\ r=169 \end{gathered}$$

In vector form, this is

$$p\begin{bmatrix}1\\0\\0\end{bmatrix}+q\begin{bmatrix}1\\1\\0\end{bmatrix}+r\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}100\\225\\169\end{bmatrix}$$

And the augmented coefficient matrix is

$$\begin{bmatrix}1 & 1 & 1 & 100\\ 0 & 1 & 1 & 225\\ 0 & 0 & 1 & 169\end{bmatrix}$$

It is clear that this system has a unique solution, since there are three leading 1's already (or, from using back substitution.) Thus, the columns of the coefficient matrix are a basis of their span, and so any such solution is a basic solution.

6. Some Non-linear Models

Solutions to exercises (and more music references):

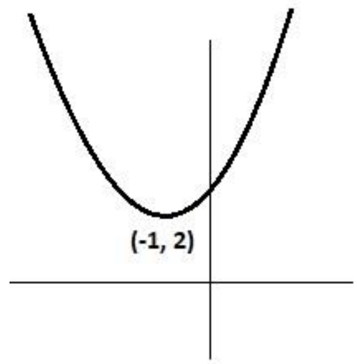
1. Put the curve in standard form, find the coordinates of the vertex, and sketch the curve:

$$y = x^2 + 2x + 3$$

Solution: Completing the square, we obtain

$$\begin{aligned} y &= x^2 + 2x + 1 + 2 \\ &= (x+1)^2 + 2 \end{aligned}$$

The vertex is located at $(-1,2)$. Sketch:



2. Same directions as exercise 1:

$$y = \frac{1}{4}x^2 - 2x$$

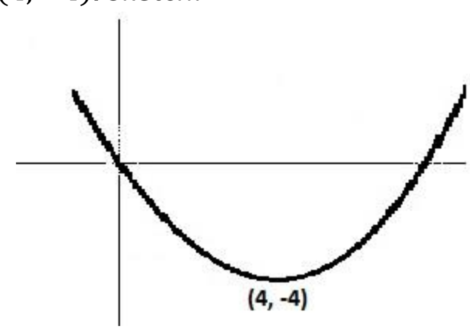
Solution: Completing the square:

$$y = \frac{1}{4}(x^2 - 8x)$$

$$y = \frac{1}{4}(x^2 - 8x + 16) - 4$$

$$y = \frac{1}{4}(x-4)^2 - 4$$

The vertex is located at $(4, -4)$. Sketch:



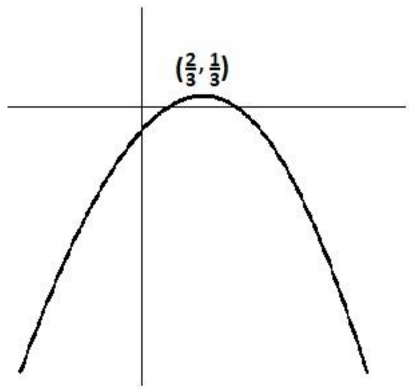
3. Same directions as exercise 1:

$$y = -3x^2 + 4x - 1$$

Solution: Completing the square:

$$y = -3x^2 + 4x - 1$$
$$y = -3\left(x^2 - \frac{4}{3}x\right) - 1$$
$$y = -3\left(x^2 - \frac{4}{3}x + \frac{4}{9}\right) - 1 + \frac{4}{3}$$
$$y = -3\left(x - \frac{2}{3}\right)^2 + \frac{1}{3}$$

The vertex is located at $\left(\frac{2}{3}, \frac{1}{3}\right)$. Sketch:



4. Find a pair of real numbers such that the first number plus twice the second sums to 12, and the product is as large as possible.

   Solution: Let the numbers be x and y. Their product is $p = xy$, which we want to be as large as possible. We must respect the constraint $x + 2y = 12$. Thus, $x = 12 - 2y$, and $p = (12 - 2y)y = 12y - 2y^2$. This is a parabola opening downwards, so the maximum value occurs at the vertex. To locate the vertex, complete the square:

$$p = -2y^2 + 12y$$
$$= -2(y^2 - 6y)$$
$$= -2(y^2 - 6y + 9) + 18$$
$$= -2(y - 3)^2 + 18$$

   The vertex is located at $(y, p) = (3,18)$. Thus, $x = 12 - 2(3) = 6$, so the numbers are $(x, y) = (6,3)$, with a maximum possible product of 18.

5. Find a pair of numbers such that the first number minus twice the second is 16, and the product is as small as possible.

   Solution: Here we must minimize $p = xy$, where $x - 2y = 16$. Thus, $x = 2y + 16$ and $p = (2y + 16)y = 2y^2 + 16y$. This is a parabola opening upwards, so the minimum value occurs at the vertex. Complete the square. You should obtain $p = 2(y + 4)^2 -$

32. The vertex is $(y, p) = (-4, -32)$. Thus $x = 2(-4) + 16 = 8, y = -4$, and the minimum value of the product is $p = -32$.

6. *Zimmerman's Millinery* is about to manufacture a run a leopard skin pillbox hats. It has been determined that if x hats are manufactured, then the cost of producing these hats is given by the formula $C = \frac{1}{10}x^2 - 80x + 31{,}000$ dollars, and the company believes this formula to be valid for any x between 200 and 500 hats. How many hats should they make in order to minimize the total cost? How much should they charge for a hat if they want to make \$40 profit per hat?

   Music Homage to folk/rock musician Bob Dylan, whose real name is Robert Zimmerman. His 1966 double album Blonde on Blonde (allegedly the first double album in rock music) included the song 'Leopard Skin Pillbox Hat', poking fun at the hats and the socialites who wore them in the 1960's. Dylan's many years of erudite lyrics earned him a Nobel prize in literature in 2019. (website: https://www.bobdylan.com/)

   Solution: The given formula is a parabola opening upwards, so the minimum is located at the vertex. Complete the square:

$$C = \frac{1}{10}x^2 - 80x + 31{,}000$$

$$C = \frac{1}{10}(x^2 - 800x) + 31{,}000$$

$$C = \frac{1}{10}(x^2 - 800x + 160{,}000) + 31{,}000 - 16{,}000$$

$$C = \frac{1}{10}(x - 400)^2 + 15{,}000$$

   Thus, the minimum cost of production is \$15,000, which occurs when 400 hats are manufactured. The cost amounts to \$37.50 per hat, so if they want to make a profit of \$40 per hat, they should sell the hats for \$77.50 each (for a total profit of \$16,000.)

7. *Yellow Brick* publishing house is selling astronaut John Nolte's new autobiography, 'Rocketman Across the Water.' They know they can sell 100,000 copies if they price the book at \$40. But they have done a market study which suggests that for each \$1 drop in price, they will generate 5,000 additional sales. What price should they set for the book which will maximize their total revenue? What is the total revenue if all the books are sold?

   Music homage to Elton John (anagram of John Nolte), who had successful albums in the early 1970s entitled Madman Across the Water and Goodbye Yellow Brick Road. His hit song 'Rocketman', from the album Honky Chateau, is also referenced. (website: https://www.eltonjohn.com/)

Solution: Let x be the number of $\$1$ price drops from the original $\$40$ proposed price. Then the price per book is $40 - x$, and the quantity of books sold at that price is $100{,}000 + 5{,}000x$. Thus, the total revenue is:

$$R = (40 - x)(100{,}000 + 5000x) = -5000x^2 + 100000x + 4000000$$

Completing the square:

$$R = -5000(x^2 - 20x) + 4{,}000{,}000$$
$$= -5000(x^2 - 20x + 100) + 4{,}500{,}000$$
$$= -5000(x - 10)^2 + 4{,}500{,}000$$

The vertex is located at $(10, 4500000)$, so the price should be set at $30 peer book to obtain a maximum revenue of $4,500,000.

8. *Elementary Penguin Farm* sells fresh eggs. They have a large hen house where they keep 300 hens. The hens are raised on organic feed and are well cared for. The average hen will produce 210 eggs in its lifetime. Recently *Walrus Animal Collective* published a study suggesting that overcrowding of hens can diminish egg production. Based on preliminary experiments that the head egg man on the farm has run, they believe that each decrease of 1 hen in their henhouse population will result in an increase of 1.5 eggs per hen (on average) over their lifetime. What is the ideal number of hens they should keep in the henhouse in order to maximize the total lifetime production of eggs?

   Music tip of the hat to the Beatles for their 1967 song 'I am the Walrus', the lyrics of which contained the phrases 'elementary penguin' and 'the egg man'. (Website: https://www.thebeatles.com/)

   Solution: Similar to the last problem (#7), it is convenient to let the variable be the 'number of drops' from, the proposed price of the book in that problem, or the proposed population of hens in this problem. That is, let x denote the difference between 300 and the ideal number of hens, so that the number of hens is $300 - x$. The reason for this convenience is that it is easy to express the total number of eggs produced over the hen's lifetime (on average) in terms of x: it is $210 + 1.5x$. Thus, the total number of eggs produced is

$$E = (300 - x)(210 + 1.5x) = -1.5x^2 + 240x + 63{,}000$$

This is a parabola opening downwards so the maximum is located at the vertex, which we find by completing the square:

$$E = -1.5(x^2 - 160x) + 63000$$
$$= -1.5(x^2 - 160x + 6400) + 63000 + 9600$$
$$= -1.5(x - 80)^2 + 72{,}600$$

Thus, the ideal number of hens is $300 - 80 = 220$ for a maximum total egg production of 72,600 eggs over their lifetime.

9. Use 'method 2' to verify the location of the saddle point in example 1.2 from subsection 1.6.3.

Solution: The equation is

$$z = -2xy + 3x - 8y + 17$$

Thus, $a = -2$, $b = -3$, $c = 8$, and $d = 17$. The conversion from general to standard form implies that $b = ak, c = ah$, and $d = ahk + \ell$. Thus

$$h = \frac{c}{a} = \frac{8}{-2} = -4$$
$$k = \frac{b}{a} = \frac{-3}{-2} = \frac{3}{2}$$
$$\ell = d - ahk = 17 - (-2)(-4)\left(\frac{3}{2}\right) = 5$$

Thus, the standard form is

$$z = -2(x + 4)\left(y - \frac{3}{2}\right) + 5$$

10. Put the following saddle surface of crossed type into standard form, and locate the coordinates of the saddle point.

$$z = 3xy + 12x - 3y - 2$$

Solution: (Details left for reader):

$$z = 3(x - 1)(y + 4) + 10$$

Saddle point at $(h, k, \ell) = (1, -4, 10)$.

11. Same directions as exercise 10 for the saddle surface:

$$z = -\frac{1}{5}xy - \frac{2}{5}y + 1$$

Solution: (Details left for reader):

$$z = -\frac{1}{5}(x+2)(y-0) + 1$$

Saddle point at $(h, k, \ell) = (-2,0,1)$.

12. Same directions as exercise 10 for the saddle surface:

$$z = 20xy + 100x + 40y + 120$$

Solution: (Details left for reader):

$$z = 20(x+2)(y+5) - 80$$

Saddle point at $(h, k, \ell) = (-2, -5, -80)$.

Chapter 2. Matrix Algebra

Section 2.1 Matrices

Solutions to Exercises.

1. Give a numerical example of each of the following:

a) A square matrix.

Solution: There are many possible correct answers. For example, any matrix of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where the letters can be replaced by any real numbers.

b) A row matrix.

Solution: There are many possible correct answers. Any matrix consisting of a single row, such as $(1,2,3)$ or $(0,0,0,0)$ works.

c) A diagonal matrix,

Solution: There are many possible correct answers. Any square matrix with each entry off the main diagonal equal to 0 works, such as $\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$ or $\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \pi \end{bmatrix}$.

2. Is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ considered to be a diagonal matrix? How about $\begin{bmatrix} 0 & 0 & 3 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$?

Solution: The first one is diagonal ($a_{ij} \neq 0$ implies $i = j$), while the second is not diagonal (for example, $a_{31} \neq 0.$)

3. Write out the general form of a 6×4 matrix with each a_{ij} in its correct location as we did in the text for a 3×4 matrix.

Solution:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \\ a_{51} & a_{52} & a_{53} & a_{54} \\ a_{61} & a_{62} & a_{63} & a_{64} \end{bmatrix}$$

4. Write out a 5×5 matrix where $a_{ij} = i + 1$ for all i and j.

Solution:

$$A = \begin{bmatrix} 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 & 4 \\ 5 & 5 & 5 & 5 & 5 \\ 6 & 6 & 6 & 6 & 6 \end{bmatrix}$$

5. Write out a 5×5 matrix where $a_{ij} = i + j$ for all i and j.

Solution:

$$A = \begin{bmatrix} 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \\ 6 & 7 & 8 & 9 & 10 \end{bmatrix}$$

6. Write out a 5×5 matrix where $a_{ij} = \max\{i, j\}$ for all i and j.

Solution:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 2 & 3 & 4 & 5 \\ 3 & 3 & 3 & 4 & 5 \\ 4 & 4 & 4 & 4 & 5 \\ 5 & 5 & 5 & 5 & 5 \end{bmatrix}$$

7. Write out a 5×5 matrix where $a_{ij} = |i - j|$ for all i and j.

Solution:

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 2 & 3 \\ 2 & 1 & 0 & 1 & 2 \\ 3 & 2 & 1 & 0 & 1 \\ 4 & 3 & 2 & 1 & 0 \end{bmatrix}$$

8. Write out a 5×5 matrix where $a_{ij} = i^2 + j$ for all i and j.

Solution:

$$A = \begin{bmatrix} 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 & 14 \\ 17 & 18 & 19 & 20 & 21 \\ 26 & 27 & 28 & 29 & 30 \end{bmatrix}$$

9. Write out a 5×5 matrix wihere $a_{ij} = (-1)^{i+j}$ for all i and j.

Solution:

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 \end{bmatrix}$$

10. Write out a 5×5 matrix where $a_{ij} = (-1)^{i+j}(i-1)(j-1)$ for all i and j.

Solution:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 3 & -4 \\ 0 & -2 & 4 & -6 & 8 \\ 0 & 3 & -6 & 9 & -12 \\ 0 & -4 & 8 & -12 & 16 \end{bmatrix}$$

11. A square matrix A is said to be upper triangular if $a_{ij} = 0$ whenever $i > j$. Likewise, A is said to be lower triangular if $a_{ij} = 0$ whenever $i < j$.

a). Give numerical examples (3×3 or larger) of upper triangular matrices. Also give examples of lower triangular matrices.

Solution: There are many correct answers, for example the first matrix below is upper triangular and the second is lower triangular:

$$\begin{bmatrix} 1 & 4 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

b) Give an example of a non-zero square matrix A (3×3 or larger) which is both upper triangular and lower triangular. What special type of matrix is A?

Solution: To be both upper and lower triangular, we must have $a_{ij} = 0$ for all $i \neq j$. But this is precisely the definition of a diagonal matrix. There are many examples, such as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

12. A square matrix P with the property that each row and column of P has exactly one nonzero entry, which has value 1, is called a *permutation matrix*. For example, an identity matrix I_n is a permutation matrix. Give at least two distinct exampl1es of a 3×3 permutation matrix which are not identity matrices. How many different 3×3 permutation matrices do you think there are? Can you prove your answer? Why do you think they are called permutation matrices?

Solution: Two examples are

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

There must be 6 different 3×3 permutation matrices, because there are 3 choices where to put the 1 in the first row, and for each of those, there are 2 choices where to put the 1 in the second row (because it cannot be in the same column as the 1 already selected), and this leave only one choice where to put the 1 in the third row. Thus, there are $3 \cdot 2 \cdot 1 = 6 = 3!$ such matrices. (The reader can easily write down all six of them now.) More generally, the same argument shows there are $n!$ distinct $n \times n$ permutation matrices. They are called permutation matrices because they can all be obtained by starting with the identity matrix I and permuting the rows (or columns) of I.

Section 2.2 operations on Matrices I

Music allusion in the text: The example of the *Grand Flick Railroad* movie theater alludes to the band Grand Funk Railroad. They released a successful album in 1970 entitled Closer to Home. (website: https://www.grandfunkrailroad.com/)

Solutions to exercises (and more music references):

1. Let

$$A = \begin{bmatrix} 1 & 0 \\ 4 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 3 \\ 2 & -3 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

a) Find $2A,\ 2A - 3B,\ 2A - 3B + 7C$.

Solution:

$$2A = 2\begin{bmatrix} 1 & 0 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 8 & -2 \end{bmatrix}$$

$$2A - 3B = 2\begin{bmatrix} 1 & 0 \\ 4 & -1 \end{bmatrix} - 3\begin{bmatrix} 2 & 3 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} -4 & -9 \\ 2 & 7 \end{bmatrix}$$

$$2A - 3B + 7C = \begin{bmatrix} -4 & -9 \\ 2 & 7 \end{bmatrix} + 7\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ 9 & 7 \end{bmatrix}$$

b) Find $A + B$ and $(A + B) + C$.

Solution:

$$A + B = \begin{bmatrix} 1 & 0 \\ 4 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 6 & -4 \end{bmatrix}$$

$$(A + B) + C = \begin{bmatrix} 3 & 3 \\ 6 & -4 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 7 & -4 \end{bmatrix}$$

c) Find $B + C$ and verify that $A + (B + C) = (A + B) + C$.

Solution:

$$B + C = \begin{bmatrix} 2 & 3 \\ 2 & -3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 3 & -3 \end{bmatrix}$$

$$A + (B + C) = \begin{bmatrix} 1 & 0 \\ 4 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 3 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 7 & -4 \end{bmatrix},$$

Which agrees with $(A + B) + C$ from part (b).

d) Find A^T, B^T and verify that $(A + B)^T = A^T + B^T$

Solution:

$$A^T = \begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix} \qquad and \qquad B^T = \begin{bmatrix} 2 & 2 \\ 3 & -3 \end{bmatrix}$$

Then

$$A^T + B^T = \begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 3 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 3 & -4 \end{bmatrix} = (A + B)^T$$

The last equality follows from part (b), where we computed $A + B$.

2. If T is a 4×7 matrix and ST is a 6×7 matrix, what size is S?

Solution: S must be 6×4 in order for the inner dimensions to match.

3. Let A, B and C be as in exercise 1. Compute the following products, if possible:

a) AB

Solution:

$$AB = \begin{bmatrix} 1 & 0 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 6 & 15 \end{bmatrix}$$

b) BA

Solution:

$$BA = \begin{bmatrix} 2 & 3 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 14 & -3 \\ -10 & 3 \end{bmatrix}$$

c) A^2

Solution:

$$A^2 = \begin{bmatrix} 1 & 0 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

d) C^3

Solution:

$$C^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = C$$

4. Let A be as in exercise 1, and let

$$D = \begin{bmatrix} 1 & 1 \\ -2 & 7 \\ 4 & 0 \end{bmatrix} \qquad E = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Compute the matrix products, if possible:

a) AD

Solution: AD is undefined

b) DA

Solution:

$$DA = \begin{bmatrix} 1 & 1 \\ -2 & 7 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 26 & -7 \\ 4 & 0 \end{bmatrix}$$

c) DE

Solution:

$$DE = \begin{bmatrix} 1 & 1 \\ -2 & 7 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 5 & 7 & 9 \\ 26 & 31 & 36 \\ 4 & 8 & 12 \end{bmatrix}$$

d) ED

Solution:

$$ED = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 7 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 15 \\ 18 & 39 \end{bmatrix}$$

5. Let A, B and C be as in exercise 1.

a) Verify that $AB + AC = A(B + C)$

Solution: We already know $AB = \begin{bmatrix} 2 & 3 \\ 6 & 15 \end{bmatrix}$. Also, $AC = \begin{bmatrix} 1 & 0 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 4 \end{bmatrix}$. Thus

$$AB + AC = \begin{bmatrix} 2 & 3 \\ 6 & 15 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 5 & 19 \end{bmatrix}$$

On the other hand, from problem (1c), we know $B + C = \begin{bmatrix} 2 & 4 \\ 3 & -3 \end{bmatrix}$. So

$$A(B+C) = \begin{bmatrix} 1 & 0 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 3 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 5 & 19 \end{bmatrix} = AB + AC$$

b) Verify that $BA + CA = (B + C)A$.

Solution: The reader can verify that both sides are equal to $\begin{bmatrix} 18 & -4 \\ -9 & 3 \end{bmatrix}$.

6. Let D and E be as in exercise 4.

a) Verify that $(DE)^T = E^T D^T$

Solution: Both are equal to $\begin{bmatrix} 5 & 26 & 4 \\ 7 & 31 & 8 \\ 9 & 36 & 12 \end{bmatrix}$.

b) Verify that $(ED)^T = D^T E^T$

Solution: Both are equal to $\begin{bmatrix} 9 & 18 \\ 15 & 39 \end{bmatrix}$

7. Let B, C and E be as above, and let $F = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 6 & 9 \\ 2 & -4 & 8 \end{bmatrix}$.

a) Verify that $(BE)F = B(EF)$.

Solution:

$$BE = \begin{bmatrix} 2 & 3 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 14 & 19 & 24 \\ -10 & -11 & -12 \end{bmatrix}$$

Also

$$EF = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 3 & 6 & 9 \\ 2 & -4 & 8 \end{bmatrix} = \begin{bmatrix} 13 & 0 & 44 \\ 31 & 6 & 101 \end{bmatrix}$$

Thus

$$(BE)F = \begin{bmatrix} 14 & 19 & 24 \\ -10 & -11 & -12 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 3 & 6 & 9 \\ 2 & -4 & 8 \end{bmatrix} = \begin{bmatrix} 119 & 18 & 391 \\ -67 & -18 & -215 \end{bmatrix}$$

And

$$B(EF) = \begin{bmatrix} 2 & 3 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 13 & 0 & 44 \\ 31 & 6 & 101 \end{bmatrix} = \begin{bmatrix} 119 & 18 & 391 \\ -67 & -18 & -215 \end{bmatrix},$$

Verifying the claim.

b) Verify that $(CE)F = C(EF)$.

Solution: Both are equal to $\begin{bmatrix} 31 & 6 & 101 \\ 13 & 0 & 44 \end{bmatrix}$.

8. Let F be as in exercise 7, and let

$$G = \begin{bmatrix} \frac{1}{2} & \frac{2}{3} & \frac{3}{4} \end{bmatrix} \qquad H = \begin{bmatrix} 30 \\ 6 \\ -12 \end{bmatrix}$$

Compute the following matrices, if possible:

a) GF

Solution:

$$GF = \begin{bmatrix} \frac{1}{2} & \frac{2}{3} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 3 & 6 & 9 \\ 2 & -4 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 13 \end{bmatrix}$$

b) GH

Solution:

$$GH = \begin{bmatrix} \frac{1}{2} & \frac{2}{3} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} 30 \\ 6 \\ -12 \end{bmatrix} = [10]$$

c) HG

Solution:

$$HG = \begin{bmatrix} 30 \\ 6 \\ -12 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{2}{3} & \frac{3}{4} \end{bmatrix} = \begin{bmatrix} 15 & 20 & \frac{45}{2} \\ 3 & 4 & \frac{9}{2} \\ -6 & -8 & -9 \end{bmatrix}$$

d) HF

Solution: Undefined

e) GFH

Solution: Using our calculation of GF from part (a):

$$GFH = \begin{bmatrix} 4 & 1 & 13 \end{bmatrix} \begin{bmatrix} 30 \\ 6 \\ -12 \end{bmatrix} = [-30]$$

9. a) Let $A = \begin{bmatrix} 1 & 2 & -6 \\ 0 & 5 & 1 \\ 4 & -3 & 8 \end{bmatrix}$. Show that $A + A^T$ is a symmetric matrix.

Solution:

$$A + A^T = \begin{bmatrix} 1 & 2 & -6 \\ 0 & 5 & 1 \\ 4 & -3 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 4 \\ 2 & 5 & -3 \\ -6 & 1 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 10 & -2 \\ -2 & -2 & 16 \end{bmatrix},$$

Which is visibly a symmetric matrix.

b) Prove that $A + A^T$ is always symmetric for all square matrices A.

Solution:

$$(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$$

Thus, $A + A^T$ is symmetric.

10. a) Let $A = \begin{bmatrix} 1 & 3 & 2 & 4 \\ 3 & 0 & -1 & 5 \end{bmatrix}$. Show that AA^T and A^TA are both symmetric matrices.

Solution:

$$AA^T = \begin{bmatrix} 1 & 3 & 2 & 4 \\ 3 & 0 & -1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 0 \\ 2 & -1 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 30 & 21 \\ 21 & 35 \end{bmatrix}$$

And

$$A^TA = \begin{bmatrix} 1 & 3 \\ 3 & 0 \\ 2 & -1 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 & 4 \\ 3 & 0 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 10 & 3 & -1 & 19 \\ 3 & 9 & 6 & 12 \\ -1 & 6 & 5 & 3 \\ 19 & 12 & 3 & 41 \end{bmatrix}$$

Both are symmetric matrices.

b) Let A be an arbitrary $m \times n$ matrix. Prove that AA^T and A^TA are always symmetric matrices. What are their dimensions?

Solution: Because A is $m \times n$, it follows that AA^T is $m \times m$ and A^TA is $n \times n$. Observe that:

$$(AA^T)^T = (A^T)^T A^T = AA^T$$

$$(A^TA)^T = A^T (A^T)^T = A^TA$$

Showing that both matrices are symmetric.

11. Consider the following system of equations:

$$\begin{aligned} 3x + 2y &= 21 \\ 2x - y &= 12 \\ 4x + y &= 33 \\ 7x + 2y &= 50 \end{aligned}$$

Write the system as a single matrix equation of the form $AX = B$, where A is the coefficient matrix of the system. What are the dimensions of the coefficient matrix.

Solution: The coefficient matrix is 4×2. The system is equivalent to the matrix equation:

$$\begin{bmatrix} 3 & 2 \\ 2 & -1 \\ 4 & 1 \\ 7 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 21 \\ 12 \\ 33 \\ 50 \end{bmatrix}$$

12. Let F and H be as in exercises 7 and 8, and let $X = \begin{bmatrix} r \\ s \\ t \end{bmatrix}$ be a variable matrix. Consider the matrix equation $FX = H$. Multiply out the left side and use it to write out the corresponding system of equations.

Solution:

$$FX = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 6 & 9 \\ 2 & -4 & 8 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} r + 2t \\ 3r + 6s + 9t \\ 2r - 4s + 8t \end{bmatrix}$$

Setting this equal to $H = \begin{bmatrix} 30 \\ 6 \\ -12 \end{bmatrix}$, we obtain the system of equations:

$$r + 2t = 30$$

$$3r + 6s + 9t = 6$$

$$2r - 4s + 8t = -12$$

13. Six students are taking part in a study of the effects of vitamins on the common cold. Each student is injected with a certain number of shots of one or two serums, Serum I and/or Serum II. The matrix A below indicates how many doses of each serum each student receives:

$$\begin{array}{l} \\ Mark \\ Ariel \\ Tina \\ Robert \\ Ivan \\ Xavier \end{array} \begin{array}{c} \begin{array}{cc} I & II \end{array} \\ \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 1 \\ 2 & 2 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} \end{array}$$

The matrix B below indicates how many mg of various types of vitamins are in each does of each serum:

$$\begin{array}{l} \\ Serum\ I \\ Serum\ II \end{array} \begin{array}{c} \begin{array}{ccccc} A & B_6 & B_{12} & C & D \end{array} \\ \begin{bmatrix} 300 & 100 & 0 & 1200 & 200 \\ 0 & 500 & 500 & 700 & 100 \end{bmatrix} \end{array}$$

Find the matrix AB and interpret the meaning of its entries.

Solution:

$$AB = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 1 \\ 2 & 2 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 300 & 100 & 0 & 1200 & 200 \\ 0 & 500 & 500 & 700 & 100 \end{bmatrix}$$

$$= \begin{bmatrix} 300 & 100 & 0 & 1200 & 200 \\ 300 & 600 & 500 & 1900 & 300 \\ 600 & 700 & 500 & 3100 & 500 \\ 600 & 1200 & 1000 & 3800 & 600 \\ 300 & 1100 & 1000 & 2600 & 400 \\ 0 & 500 & 500 & 700 & 100 \end{bmatrix}$$

The entries represent how many mg of each vitamin that each student receives, because the rows of AB correspond to the rows of A (students), while the columns of AB correspond to the columns of B (mg vitamins).

14. Rocky and Sadie are on a diet and on a budget. At a recent visit to the *Savoy Truffle Diner* for breakfast, Rocky had 2 eggs, 3 slices bacon, 2 slices buttered toast, 1 cup of coffee. Sadie had 1 egg, 2 slices ham, 1 slice buttered toast, 2 cups of coffee and 1 glass of orange juice.

Music homage to the Beatles. 'Rocky Racoon', 'Sexy Sadie', and 'Savoy Truffle' are songs from their 1968 eponymous double album (the "white album"). (Website: https://www.thebeatles.com/)

a) Put all the above information into a matrix A where there is a row for Rocky and a row for Sadie, and a column for each potential breakfast item. That is, a column for eggs, slices bacon, ham slices, buttered toast, coffee, and juice, in that order. Your matrix should be 2× 6.

Solution:

$$A = \begin{matrix} Rocky \\ Sadie \end{matrix} \begin{bmatrix} 2 & 3 & 0 & 2 & 1 & 0 \\ 1 & 0 & 2 & 1 & 2 & 1 \end{bmatrix}$$

b) The following matrix B represents the breakdown for each breakfast item in terms of calories, mg. cholesterol, and price (in cents):

$$\begin{matrix} \\ Egg \\ Slice\ Bacon \\ Slice\ Ham \\ Slice\ toast \\ Cup\ coffee \\ Glass\ juice \end{matrix} \begin{matrix} \begin{matrix} Cal. & Chol. & Cost \end{matrix} \\ \begin{bmatrix} 75 & 10 & 100 \\ 150 & 6 & 75 \\ 100 & 4 & 100 \\ 125 & 1 & 50 \\ 20 & 0 & 80 \\ 80 & 0 & 150 \end{bmatrix} \end{matrix}$$

Find the product matrix AB and use it to answer the questions:

Solution:

$$AB = \begin{bmatrix} 2 & 3 & 0 & 2 & 1 & 0 \\ 1 & 0 & 2 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 75 & 10 & 100 \\ 150 & 6 & 75 \\ 100 & 4 & 100 \\ 125 & 1 & 50 \\ 20 & 0 & 80 \\ 80 & 0 & 150 \end{bmatrix}$$

$$= \begin{bmatrix} 870 & 40 & 605 \\ 520 & 19 & 660 \end{bmatrix}$$

i) What did Rocky spend on his breakfast? \$6.05

ii) How many calories were in Sadie's breakfast? 520 calories

iii) Who consumed more cholesterol? Rocky consumed 40 mg and Sadie consumed 19 mg, so Rocky consumed more.

c) Consider the 1×2 matrix $S = [1 \quad 1]$. Find the matrix SAB and interpret the meaning of its entries.

Solution:

$$SAB = [1 \quad 1] \begin{bmatrix} 870 & 40 & 605 \\ 520 & 19 & 660 \end{bmatrix} = [1390 \quad 59 \quad 1265]$$

The entries are the total calories, mg cholesterol, and cost for both breakfasts together.

15. The *Locomotive Breath Rail Company* runs passenger trains between five cities. They do not run a train between every pair of cities, but it is possible to go from one city to any of the other five if you are willing to make the journey in several steps or legs, with an overnight layover between each leg. For simplicity, we assume that the trains all leave at the same time; say 9 AM every day. The diagram below shows how many trains g from each city to each other city. If there are no direct trains from one city to another, no arrow is shown in the diagram:

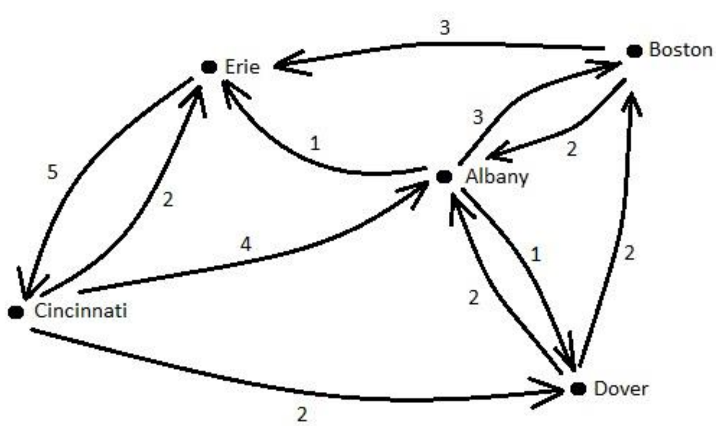


Figure 2.1 – a transit graph

Music allusion to the band Jethro Tull. 'Locomotive Breath' is a song from their 1971 album Aqualung. (Website: http://jethrotull.com)

a) Construct a 5×5 matrix T, where the rows and columns of T are labeled with the cities A, B, C, D and E (in that order), and where the entry t_{ij} is the number of trains from city j to city i. Thus, the cities of departure are the columns and the cities of arrival are the rows. For example, since there are 5 trains from Erie to Cincinnati, the 5 would be located in the column headed E and the row headed C; that is, $t_{35} = 5$.

Solution: Reading off the transit graph, we obtain:

$$T = \begin{bmatrix} 0 & 2 & 4 & 2 & 0 \\ 3 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 1 & 0 & 2 & 0 & 0 \\ 1 & 3 & 2 & 0 & 0 \end{bmatrix}$$

b) Find $T^2 = T \cdot T$, and interpret the meaning of its entries.

Solution:

$$T^2 = \begin{bmatrix} 0 & 2 & 4 & 2 & 0 \\ 3 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 1 & 0 & 2 & 0 & 0 \\ 1 & 3 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 & 4 & 2 & 0 \\ 3 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 1 & 0 & 2 & 0 & 0 \\ 1 & 3 & 2 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 & 4 & 4 & 20 \\ 2 & 6 & 16 & 6 & 0 \\ 5 & 15 & 10 & 0 & 0 \\ 0 & 2 & 4 & 2 & 10 \\ 9 & 2 & 4 & 8 & 10 \end{bmatrix}$$

The ij -entry of T^2 is the number of possible ways of going from city j to city i in two steps (with one layover).

c) Find $T^3 = T \cdot T \cdot T$, and interpret the meaning of its entries.

Solution:

$$T^3 = T \cdot T^2 = \begin{bmatrix} 0 & 2 & 4 & 2 & 0 \\ 3 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 1 & 0 & 2 & 0 & 0 \\ 1 & 3 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 8 & 0 & 4 & 4 & 20 \\ 2 & 6 & 16 & 6 & 0 \\ 5 & 15 & 10 & 0 & 0 \\ 0 & 2 & 4 & 2 & 10 \\ 9 & 2 & 4 & 8 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 24 & 76 & 80 & 16 & 20 \\ 24 & 4 & 20 & 16 & 80 \\ 45 & 10 & 20 & 40 & 50 \\ 18 & 30 & 24 & 4 & 20 \\ 24 & 48 & 72 & 22 & 20 \end{bmatrix}$$

The ij -entry of T^3 is the number of possible ways of going from city j to city i in three steps (with two layovers). More generally, the ij -entry of T^k is the number of possible ways of going from city j to city i in k steps.

d) What is the meaning of the entries in $T + T^2 + T^3$?

Solution: The ij -entry is the number of possible ways of going from city j to city i in at most three steps.

16 Let A and B be as in exercise 1.

a) Show that $(A+B)^2 \neq A^2 + 2AB + B^2$.

Solution:

$$(A+B)^2 = \begin{bmatrix} 3 & 3 \\ 6 & -4 \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 6 & -4 \end{bmatrix} = \begin{bmatrix} 27 & -3 \\ -6 & 34 \end{bmatrix}$$

On the other hand,

$$A^2 = \begin{bmatrix} 1 & 0 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 3 \\ 6 & 15 \end{bmatrix} \ (see\ exercise\ 3a)$$

$$B^2 = \begin{bmatrix} 2 & 3 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 10 & -3 \\ -2 & -3 \end{bmatrix}$$

Therefore

$$A^2 + 2AB + B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2\begin{bmatrix} 2 & 3 \\ 6 & 15 \end{bmatrix} + \begin{bmatrix} 10 & -3 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} 15 & 3 \\ 10 & 28 \end{bmatrix}$$

Which differs from $(A+B)^2$.

b) Show that $(A+B)(A-B) \neq A^2 - B^2$.

Solution: $A + B = \begin{bmatrix} 3 & 3 \\ 6 & -4 \end{bmatrix}$, and $A - B = \begin{bmatrix} -1 & -3 \\ 2 & 2 \end{bmatrix}$. Thus,

$$(A+B)(A-B) = \begin{bmatrix} 3 & 3 \\ 6 & -4 \end{bmatrix} \begin{bmatrix} -1 & -3 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -14 & -26 \end{bmatrix}$$

On the other hand, (see part a):

$$A^2 - B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 10 & -3 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} -9 & 3 \\ 2 & 4 \end{bmatrix}$$

Which differs from $(A+B)(A-B)$.

c) What conditions on square matrices M, N would be necessary and sufficient to make $(M+N)(M-N) = M^2 - N^2$?

Solution: Each line holds if and only if the previous holds:

$$(M+N)(M-N) = M^2 - N^2$$

$$M^2 + NM - MN - N^2 = M^2 - N^2$$

$$NM - MN = 0$$

$$NM = MN$$

Thus, the identity holds if and only if M, N commute with each other.

17. Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ -3 & 4 & 5 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & -6 & 3 \\ 1 & 4 & 0 \\ -3 & 4 & 5 \end{bmatrix}$$

Show that $AB = AC$, yet $B \neq C$. This demonstrates that the cancellation law does not always hold for matrices.

Solution:

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ -3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} -5 & 14 & 18 \\ 3 & -4 & -5 \\ -15 & 20 & 25 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 2 & -6 & 3 \\ 1 & 4 & 0 \\ -3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} -5 & 14 & 18 \\ 3 & -4 & -5 \\ -15 & 20 & 25 \end{bmatrix}$$

Thus, $AB = AC$.

18. A square matrix N is said to be nilpotent if some power of it is the zero matrix; that is, if $N^k = 0$ for some positive integer k. Verify that the following two matrices are nilpotent:

$$N = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad M = \begin{bmatrix} 0 & 0 & \sqrt{3} & \pi & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution:

$$N^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$N^3 = NN^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$N^4 = 0$$

Also

$$M^2 = \begin{bmatrix} 0 & 0 & \sqrt{3} & \pi & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & \sqrt{3} & \pi & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & -\pi \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M^3 = MM^2 = \begin{bmatrix} 0 & 0 & \sqrt{3} & \pi & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 0 & -\pi \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = 0$$

19. A square matrix E is said to be idempotent if $E^2 = E$.

a) Verify that $E = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $F = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ are idempotent.

Solution:

$$E^2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = E$$

And

$$F^2 = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} = F$$

b) Prove that if E is idempotent, so is $I - E$, where I is the identity matrix of the same size as E.

Solution: Since I and E are both idempotent, we have

$$(I - E)^2 = I^2 - IE - EI + E^2$$

$$= I - E - E + E$$

$$= I - E$$

In some of the exercises above, parts of theorem 2.3 were illustrated. In the following exercise, we ask you to prove some parts of this theorem. As an example, here is a proof of part (i) of that theorem: $r(sA) = (rs)A$, where A is an arbitrary matrix and r and s are real numbers. Recall that the entry of A in the ith row and jth column is denoted a_{ij}. The matrix sA has every entry scaled up by a factor of s, so the ij entry of sA is sa_{ij}. Similarly, the entries in $r(sA)$ are obtained from those in sA by scalaing by a factor of r, so the ij entry of $r(sA)$ is $r(sa_{ij}) = (rs)(a_{ij})$. On the other hand, the entries of $(rs)A$ are obtained from those in A by scaling by a factor of rs, so the ij entry of $(rs)A$ is also $(rs)a_{ij}$. Because this is true for all i, j, it follows that as matrices $r(sA) = (rs)A$.

20. Using similar arguments and ideas as the proof above, prove the following parts of theorem 2.3:

a) Part (a): $A + B = B + A$

Solution: The ij entry of $A + B$ is $a_{ij} + b_{ij} = b_{ij} + a_{ij}$, which is the ij entry in $B + A$. Since this holds for all i and j, it follows that $A + B = B + A$.

b) Part (b): $(A + B) + C = A + (B + C)$.

Solution: The ij entry of $(A + B) + C$ is $(a_{ij} + b_{ij}) + c_{ij} = a_{ij} + (b_{ij} + c_{ij})$, which is the ij entry in $A + (B + C)$. Since this holds for all i and j, it follows that $(A + B) + C = A + (B + C)$.

c) Part (c): $s(A + B) = sA + sB$.

Solution: The ij entry of $s(A + B)$ is $s(a_{ij} + b_{ij}) = sa_{ij} + sb_{ij}$, which is the ij entry in $sA + sB$. Since this holds for all i and j, it follows that $s(A + B) = sA + sB$.

d) Part (g): $(A + B)^T = A^T + B^T$.

Solution: The ij entry of $(A + B)^T$ is the ji entry of $A + B$; namely $a_{ji} + b_{ji}$. But a_{ji} the ji entry in A (and the ij entry of A^T) and similarly b_{ji} is the ij entry of B^T. Since this holds for all i and j, it follows that $(A + B)^T = A^T + B^T$.

21. In some of the exercises above, you saw parts of theorem 2.6 illustrated. In this exercise, give proofs of the following parts of theorem 2.6:

a) Part (b): $A(B + C) = AB + AC$.

Solution: By definition of matrix multiplication, the ij entry of $A(B + C)$ is the dot product of the ith row of A and the jth column of $B + C$. Assuming the inner dimensions match (A has n columns and $B + C$ has n rows), we can write this as

$$\sum_{k=1}^{n} a_{ik}(b_{kj} + c_{kj}) = \sum_{k=1}^{n} a_{ik}b_{kj} + \sum_{k=1}^{n} a_{ik}c_{kj}$$

But this is precisely the sum of the ij entry in AB with the ij entry in AC. Thus, $A(B + C)\ and\ AB + AC$ both have the same ij entries for all i and j, so they are the same matrix.

b) Part (c): $(B + C)A = BA + CA$.

Solution: The ij entry of $(B + C)A$ is

$$\sum_{k=1}^{n} (b_{ik} + c_{ik})a_{kj} = \sum_{k=1}^{n} b_{ik}a_{kj} + \sum_{k=1}^{n} c_{ik}a_{kj}$$

Which is precisely the sum of the ij entry of BA plus the ij entry of CA. Thus $(B + C)A$ and $BA + CA$ have the same ij entry for all i and j. Thus $(B + C)A = BA + CA$.

c) Part (d): $s(AB) = (sA)B = A(sB)$.

Solution: We see all the ij entries agree:

$$s\sum_{k=1}^{n} a_{ik}b_{kj} = \sum_{k=1}^{n}(sa_{ik})b_{kj} = \sum_{k=1}^{n} a_{ik}(sb_{kj})$$

Since this holds for all i and j, it follows that the matrices are the same.

d) Part (e): $(AB)^T = B^T A^T$.

Solution: The ij entry of $(AB)^T$ is the ji entry of AB, which is

$$\sum_{k=1}^{n} a_{jk}b_{ki}$$

Let us write $b_{ki} = b'_{ik}$ and $a_{jk} = a'_{kj}$, so that these are the ik and kj entries in B^T and A^T, respectively. Then the above sum is

$$\sum_{k=1}^{n} a'_{kj}b'_{ik} = \sum_{k=1}^{n} b'_{ik}a'_{kj}$$

Which is precisely the ij entry of $B^T A^T$. Since they have the same ij entries for all i and j, they are the same matrix.

Section 2.3 operations on Matrices II: Matrix Inversion

Solutions to Exercises:

1. Compute the determinant of the following 2×2 matrices:

Solutions:

a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ determinant $= -2$

b) $\begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}$ determinant $= 0$

c) $\begin{bmatrix} 4 & 3 \\ 5 & 5 \end{bmatrix}$ determinant $= 5$

d) $\begin{bmatrix} 0 & 7 \\ 7 & 100 \end{bmatrix}$ determinant $= -49$

2. For each of the matrices in exercise 1, determine whether or not the matrix is invertible. If it is, find the inverse matrix, and verify your answer by checking that $AA^{-1} = I_2$.

Solution: a) A is invertible since $detA$ is not 0. The inverse is

$$\frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

b) Not invertible since $detA = 0$.

c) Invertible. The inverse is

$$\frac{1}{5}\begin{bmatrix} 5 & -3 \\ -5 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{3}{5} \\ -1 & \frac{4}{5} \end{bmatrix}$$

d) Invertible. The inverse is

$$\frac{1}{-49}\begin{bmatrix} 100 & -7 \\ -7 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{100}{49} & \frac{1}{7} \\ \frac{1}{7} & 0 \end{bmatrix}$$

3. For what value(s) of x is the following matrix not invertible?

$$\begin{bmatrix} x-1 & 3 \\ 8 & x-3 \end{bmatrix}$$

Solution: The matrix is not invertible if and only if its determinant is 0, so

$$(x-1)(x-3) - 24 = 0$$

$$x^2 - 4x - 21 = 0$$

$$(x-7)(x+3) = 0$$

Thus, the matrix is not invertible if (and only if) $x = 7$ or $x = -3$.

4. a) Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 \\ 5 & 5 \end{bmatrix}$. Find AB and show that, in this example, $det(AB) = det(A) \cdot det(B)$.

Solution:

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 5 & 5 \end{bmatrix} = \begin{bmatrix} 14 & 13 \\ 32 & 29 \end{bmatrix}$$

$$\det(AB) = 14(29) - 32(13) = -10$$

On the other hand, $detA = -2$ and $detB = 5$, so $\det(A) \cdot \det(B) = (-2)5 = -10 = \det(AB)$.

b) Show that $det(AB) = det(A) \cdot det(B)$ for arbitrary 2×2 matrices. (It can be shown that this is true for all $n \times n$ matrices as well.)

Solution: Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$$

Be arbitrary 2×2 matrices. Then $\det(A) = ad - bc$ and $\det(B) = ps - rq$. Also

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{bmatrix}$$

$$\det(AB) = (ap+br)(cq+ds) - (aq+bs)(cp+dr)$$

$$= (apcq + apds + brcq + brds) - (cpaq + cpbs + draq + drbs)$$

$$= apds + brcq - cpbs - draq$$

$$= ps(ad - bc) - rq(ad - bc)$$

$$= (ad - bc)(ps - rq) = detA \cdot detB$$

5. a) It can be shown that $detI_n = 1$. Verify it for the case $n = 2$.

Solution: Left for the reader to verify.

b) In theorem 2.15 in this section, we showed (in the 2×2 case) that if $detA \neq 0$, then A is invertible. This can be strengthened to say A is invertible if and only if $det\ A \neq 0$. (This is theorem 2.14.) To prove the converse direction, show that if $det\ A = 0$, then A could not possibly have an inverse. [**Hint**: Use exercise (4b) and part (a) of this exercise.]

Solution: Observe that if $B = A^{-1}$, then $AB = I$ by definition of inverses. However, by exercise (4b), this implies:

$$1 = \det(I) = \det(AB) = \det(A) \cdot \det(B)$$

If $detA = 0$, this becomes

$$1 = 0 \cdot detB$$

Which is impossible no matter what B is, since the right side is 0. Thus $B = A^{-1}$ cannot exist if $detA = 0$.

6. Suppose A is invertible, and suppose B and C are two inverses of A. Prove that $B = C$ so that part (a) of theorem 2.16 will be proved. [**Hint**: One possible approach is to consider the product BAC and evaluate it in two different ways.]

Solution: Following the hint, and using the associative law:

$$B = BI = B(AC) = (BA)C = IC = C$$

Another approach:

$$AB = I = AC$$

$$AB - AC = I - I = 0$$

$$A(B - C) = 0$$

Now multiply both sides of this by an inverse of A on the left (for example, B or C or any inverse…) to obtain:

$$I(B - C) = 0$$

$$B - C = 0$$

$$B = C$$

7) Let $A = \begin{bmatrix} -3 & 7 \\ -2 & 1 \end{bmatrix}$. Verify parts (b) and (c) of theorem 2.16 for A.

Solution: The determinant of A is 11, so

$$A^{-1} = \frac{1}{11}\begin{bmatrix} 1 & -7 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} \frac{1}{11} & -\frac{7}{11} \\ \frac{2}{11} & -\frac{3}{11} \end{bmatrix}$$

Now the determinant of A^{-1} is $\frac{1}{11}$, so

$$(A^{-1})^{-1} = 11\begin{bmatrix} -\frac{3}{11} & \frac{7}{11} \\ \frac{-2}{11} & \frac{1}{11} \end{bmatrix} = \begin{bmatrix} -3 & 7 \\ -2 & 1 \end{bmatrix} = A$$

Verifying part (b).

For part (c), note that:

$$A^T = \begin{bmatrix} -3 & -2 \\ 7 & 1 \end{bmatrix}$$

Which also has determinant 11, so

$$(A^T)^{-1} = \frac{1}{11}\begin{bmatrix} 1 & 2 \\ -7 & -3 \end{bmatrix} = \begin{bmatrix} \frac{1}{11} & \frac{2}{11} \\ -\frac{7}{11} & -\frac{3}{11} \end{bmatrix} = (A^{-1})^T$$

Verifying part (c).

8. a) Prove part c of Theorem 2.16 for 2×2 matrices using equation (2.11) of Theorem 2.15.

Solution: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ so that $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$. By equation (2.11) we have

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$(A^{-1})^T = \frac{1}{ad - bc}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

On the other hand, equation (2.11) applied directly to A^T gives:

$$(A^T)^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

Which agrees with $(A^{-1})^T$, proving the claim.

b) Prove part (d) of theorem 2.16.

Solution: Notice you are asked to prove it without restriction on the size of the matrix. Thus, theorem 2.15 is of no help in the general case. Instead, just observe that if A^{-1} and B^{-1} both exist, then

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I,$$

A similar calculation shows that $(B^{-1}A^{-1})(AB) = I$. Thus, $B^{-1}A^{-1}$ satisfies the definition of the inverse of AB. Since we know inverses are unique (exercise 6), it follows that $(AB)^{-1} = B^{-1}A^{-1}$.

9. Show by example that $A + B$ need not be invertible, if both A and B are invertible.

Solution: There are many counterexamples. Here's one:

$$A = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B = -A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Then A and B are both invertible (in fact, they are equal to their own inverses.) Yet, $A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$, which is not invertible.

In the text, it was mentioned that in lieu of theorem 2.15, an alternate method exists for finding the inverse of a matrix based on solving systems of equations. The next few exercises explore this approach.

10. Let $A = \begin{bmatrix} 3 & 6 \\ 2 & 5 \end{bmatrix}$. Since $\det(A) \neq 0$, A is invertible. Let $B = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$ be the unknown inverse, so that $AB = I$.

a) Multiply out AB and set it equal to I.

Solution:

$$AB = \begin{bmatrix} 3 & 6 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 3w + 6y & 3x + 6z \\ 2w + 5y & 2x + 5z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

b) Notice that in your answer to part (a), the first column of AB only involves the variables w and y, and the second column only involves the variables x and z. Thus, by setting the first column of AB equal to the first column of I, you should obtain a system of two linear equations involving the unknowns w and y. Write out this system of equations. Similarly, write out the system of equations involving x and z obtained when setting the second column of AB to the second column of I.

Solution: The two systems of equations are:

$$3w + 6y = 1$$

$$2w + 5y = 0$$

And

$$3x + 6z = 0$$

$$2w + 5y = 1$$

c) Solve the first of the two systems by pivoting the augmented coefficient matrix of the system $\begin{bmatrix} 3 & 6 & 1 \\ 2 & 5 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & w \\ 0 & 1 & y \end{bmatrix}$ as we learned in Chapter One.

Solution:

$$\begin{bmatrix} 3 & 6 & 1 \\ 2 & 5 & 0 \end{bmatrix} R_1 \to \frac{1}{3} R_1$$

$$\begin{bmatrix} 1 & 2 & \frac{1}{3} \\ 2 & 5 & 0 \end{bmatrix} R_2 \to R_2 - 2R_1$$

$$\begin{bmatrix} 1 & 2 & \frac{1}{3} \\ 0 & 1 & -\frac{2}{3} \end{bmatrix} R_1 \to R_1 - 2R_2$$

$$\begin{bmatrix} 1 & 0 & \frac{5}{3} \\ 0 & 1 & -\frac{2}{3} \end{bmatrix}$$

This is reduced echelon for, so the solution is $w = \frac{5}{3}$ and $y = -\frac{2}{3}$.

d) Notice that the coefficient matrix of both systems is the same matrix – namely A. This means that solving the second system by pivoting, you would use exactly the same elementary row operations in the same order that you used in part (c) to solve the first system. Therefore, in yet another instance of using matrices to efficiently handle itemized calculations, you can solve both systems at once by annexing a fourth column $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to the augmented matrix you used in part (c). That is, we pivot $\begin{bmatrix} 3 & 6 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & w & x \\ 0 & 1 & y & z \end{bmatrix}$ using the exact same steps you used in part (c). Perform the pivoting and thereby find all four variables w, x, y, z. Thus, you have found B, the inverse of A. Check you answer by using equation (2.11).

Solution:

$$\begin{bmatrix} 3 & 6 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{bmatrix} R_1 \to \frac{1}{3} R_1$$

$$\begin{bmatrix} 1 & 2 & \frac{1}{3} & 0 \\ 2 & 5 & 0 & 1 \end{bmatrix} R_2 \to R_2 - 2R_1$$

$$\begin{bmatrix} 1 & 2 & \frac{1}{3} & 0 \\ 0 & 1 & -\frac{2}{3} & 1 \end{bmatrix} R_1 \to R_1 - 2R_2$$

$$\begin{bmatrix} 1 & 0 & \frac{1}{3} & -2 \\ 0 & 1 & -\frac{2}{3} & 1 \end{bmatrix}$$

Thus, in addition to the solution to the first system we also obtain $x = -2$ and $z = 1$. Thus, we have found the inverse:

$$B = \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -2 \\ -\frac{2}{3} & 1 \end{bmatrix}$$

The reader can verify that this is the inverse using equation (2.11), or by direct multiplication to see that $AB = I$.

11. In problem 10, you showed that in order to find the inverse of $A = \begin{bmatrix} 3 & 6 \\ 2 & 5 \end{bmatrix}$, you annex an identity matrix I_2 to form an augmented matrix, and pivot until the two left columns become the identity matrix I_2, at which point the inverse matrix A^{-1} automatically appears in the two right columns.

a) In this exercise, show that you can do the same thing for ay invertible 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. That is, to find A^{-1}, row reduce (pivot) $[A|I_2] \to [I_2|A^{-1}]$. [**Hint**: Treat a,b,c,d as known and follow the steps you used in exercise 10. You will need to know that $detA = ad - bc \neq 0$.]

Solution: Assuming $ad - bc \neq 0$, it follows that either a or c must be nonzero. We assume $a \neq 0$ in the proof below. The argument when $c \neq 0$ is similar after interchanging the two rows.

$$\begin{bmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{bmatrix} R_1 \to \frac{1}{a} R_1$$

$$\begin{bmatrix} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{bmatrix} R_2 \to R_2 - cR_1$$

$$\begin{bmatrix} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & \frac{ad - bc}{a} & -\frac{c}{a} & 1 \end{bmatrix} R_2 \to \frac{a}{ad - bc} R_2$$

$$\begin{bmatrix} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & -\frac{c}{ad - bc} & \frac{a}{ad - bc} \end{bmatrix} R_1 \to R_1 - \frac{b}{a} R_2$$

$$\begin{bmatrix} 1 & 0 & \frac{d}{ad - bc} & \frac{-b}{ad - bc} \\ 0 & 1 & \frac{-c}{ad - bc} & \frac{a}{ad - bc} \end{bmatrix}$$

Thus

$$A^{-1} = \begin{bmatrix} \dfrac{d}{ad-bc} & \dfrac{-b}{ad-bc} \\ \dfrac{-c}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Which agrees with equation (2.11).

b) Use this method to find the inverse of $\begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}$ and check your answer either by using equation (2.11) or multiplying AA^{-1} to obtain the identity I.

Solution:

$$\begin{bmatrix} 3 & 5 & 1 & 0 \\ 4 & 7 & 0 & 1 \end{bmatrix} R_1 \to \frac{1}{3}R_1$$

$$\begin{bmatrix} 1 & \frac{5}{3} & \frac{1}{3} & 0 \\ 4 & 7 & 0 & 1 \end{bmatrix} R_2 \to R_2 - 4R_1$$

$$\begin{bmatrix} 1 & \frac{5}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & -\frac{4}{3} & 1 \end{bmatrix} R_2 \to 3R_2$$

$$\begin{bmatrix} 1 & \frac{5}{3} & \frac{1}{3} & 0 \\ 0 & 1 & -4 & 3 \end{bmatrix} R_1 \to R_1 - \frac{5}{3}R_2$$

$$\begin{bmatrix} 1 & 0 & 7 & -5 \\ 0 & 1 & -4 & 3 \end{bmatrix}$$

Thus,

$$A^{-1} = \begin{bmatrix} 7 & -5 \\ -4 & 3 \end{bmatrix}$$

We leave the reader to check this.

c) What happens if you try to apply this method to a matrix which is not invertible, such as $\begin{bmatrix} 2 & -4 \\ 9 & -18 \end{bmatrix}$?

Solution: The reduced echelon form of $[A|I]$ will have a row of all zero's where the A is (to the left of dividing line), but something nonzero where the I is (to the right of the dividing line.) Thus, the system has no solution. Try it and see with the given example!

12. The method of exercise 11 actually works for any $n \times n$ matrix, and the reason is essentially the same as the one given in exercises 10 and 11 for the case $n = 2$. Use this method to find the inverses of the following 3×3 matrices, and check your answers by multiplying AA^{-1} to obtain I_3.

a) $\begin{bmatrix} 3 & 1 & -4 \\ 1 & 3 & 2 \\ 5 & -1 & -1 \end{bmatrix}$

Solution:

$$\begin{bmatrix} 3 & 1 & -4 & 1 & 0 & 0 \\ 1 & 3 & 2 & 0 & 1 & 0 \\ 5 & -1 & -1 & 0 & 0 & 1 \end{bmatrix} R_2 \leftrightarrow R_2$$

$$\begin{bmatrix} 1 & 3 & 2 & 0 & 1 & 0 \\ 3 & 1 & -4 & 1 & 0 & 0 \\ 5 & -1 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 5R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 3 & 2 & 0 & 1 & 0 \\ 0 & -8 & -10 & 1 & -3 & 0 \\ 0 & -16 & -11 & 0 & -5 & 1 \end{bmatrix} R_2 \to -\frac{1}{8} R_2$$

$$\begin{bmatrix} 1 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & \frac{10}{8} & -\frac{1}{8} & \frac{3}{8} & 0 \\ 0 & -16 & -11 & 0 & -5 & 1 \end{bmatrix} \begin{matrix} R_1 \to R_1 - 3R_2 \\ \\ R_3 \to R_3 + 16R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & -\frac{14}{8} & \frac{3}{8} & -\frac{1}{8} & 0 \\ 0 & 1 & \frac{10}{8} & -\frac{1}{8} & \frac{3}{8} & 0 \\ 0 & 0 & 9 & -2 & 1 & 1 \end{bmatrix} R_3 \to \frac{1}{9} R_3$$

$$\begin{bmatrix} 1 & 0 & -\frac{14}{8} & \frac{3}{8} & -\frac{1}{8} & 0 \\ 0 & 1 & \frac{10}{8} & -\frac{1}{8} & \frac{3}{8} & 0 \\ 0 & 0 & 1 & -\frac{2}{9} & \frac{1}{9} & \frac{1}{9} \end{bmatrix} \begin{matrix} R_1 \to R_1 + \frac{14}{8} R_3 \\ R_2 \to R_2 - \frac{10}{8} R_2 \\ \\ \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{72} & \frac{5}{72} & \frac{14}{72} \\ 0 & 1 & 0 & \frac{11}{72} & \frac{17}{72} & -\frac{10}{72} \\ 0 & 0 & 1 & -\frac{2}{9} & \frac{1}{9} & \frac{1}{9} \end{bmatrix}$$

Thus,

$$A^{-1} = \begin{bmatrix} -\frac{1}{72} & \frac{5}{72} & \frac{14}{72} \\ \frac{11}{72} & \frac{17}{72} & -\frac{10}{72} \\ -\frac{2}{9} & \frac{1}{9} & \frac{1}{9} \end{bmatrix}$$

This checks:

$$AA^{-1} = \begin{bmatrix} 3 & 1 & -4 \\ 1 & 3 & 2 \\ 5 & -1 & -1 \end{bmatrix} \begin{bmatrix} -\frac{1}{72} & \frac{5}{72} & \frac{14}{72} \\ \frac{11}{72} & \frac{17}{72} & -\frac{10}{72} \\ -\frac{2}{9} & \frac{1}{9} & \frac{1}{9} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b) $\begin{bmatrix} 4 & 6 & 2 \\ 2 & 3 & 2 \\ 1 & 2 & \frac{1}{2} \end{bmatrix}$

Solution: We leave the pivoting for the reader. The final result is

$$\begin{bmatrix} 4 & 6 & 2 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ 1 & 2 & \frac{1}{2} & 0 & 0 & 1 \end{bmatrix} \to\to$$

Reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & \frac{5}{4} & -\frac{1}{2} & -3 \\ 0 & 1 & 0 & -\frac{1}{2} & 0 & 2 \\ 0 & 0 & 1 & -\frac{1}{2} & 1 & 0 \end{bmatrix}$$

Thus,

$$A^{-1} = \begin{bmatrix} \frac{5}{4} & -\frac{1}{2} & -3 \\ -\frac{1}{2} & 0 & 2 \\ -\frac{1}{2} & 1 & 0 \end{bmatrix}$$

13. Find, if possible, the inverse of the following 4×4 matrices, using the method of exercises 11 and 12.

a) $\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Solution:

$$\left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 \to R_1 + R_2 \\ \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 \to R_1 + R_3 \\ R_2 \to R_2 + R_3 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 \to R_1 + R_4 \\ R_2 \to R_2 + R_4 \\ R_3 \to R_3 + R_4 \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

Thus,

$$A^{-1} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Check:

$$AA^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

b) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 \\ 2 & 4 & 6 & 8 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

Solution: We leave the pivoting for the reader. The final result is

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 2 & 4 & 6 & 8 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow$$

Reduced echelon form:

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 0 & -2 & \frac{1}{2} & -\frac{2}{3} \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} & 0 \end{bmatrix}$$

Notice that the bottom row represents an equation for which there is no solution (using the fifth column), so this matrix is not invertible.

14. Recall from exercise 15 of the last section that a square matrix N is *nilpotent* if $N^k = 0$ for some integer $k > 0$.

a) Suppose N is nilpotent with $k = 4$. Show that $I - N$ is invertible by direct calculation as follows:

$$(I - N)(I + N + N^2 + N^3) = I$$

And so $(I - N)^{-1} = I + N + N^2 + N^3$.

Solution: Indeed,

$$(I - N)(I + N + N^2 + N^3) = (I + N + N^2 + N^3) - N - N^2 - N^3 - N^4$$
$$= I - N^4 = I$$

Because $N^4 = 0$.

b) What matrix N should be used to show that part (a) gives an alternate way to find the inverse of the matrix in part (a) of exercise 13?

Solution: If we wish $A = I - N$, then

$$N = I - A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This works. Indeed,

$$N^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$N^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

And $N^4 = 0$. Then indeed $A^{-1} = I + N + N^2 + N^3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, which agrees with our answer to exercise (13a).

c) Generalize part (a) to arbitrary $k > 0$, What is the inverse of $I - N$ if $N^k = 0$?

Solution: The same argument as above shows that

$$(I - N)^{-1} = I + N + N^2 + \cdots + N^{k-1}.$$

15. Recall from exercise 16 of the last section that a square matrix E is called *idempotent* if $E^2 = E$. Explain why the only invertible idempotent $n \times n$ matrix is I_n.

Solution: If E is invertible, then multiply both sides of $E^2 = E$ by E^{-1} to obtain:

$$E^{-1}(EE) = E^{-1}E$$

$$E = I.$$

Section 2.4 Solving Linear Systems by Matrix Inversion

Music Reference in the Text: The bongo drum example is an homage to rock musician, composer, and outspoken social critic Frank Zappa. Bongo Fury was the title of an album released in 1975 that was a collaboration between Zappa, the Mothers of Invention, and Captain Beefheart. Zappa also released albums entitled Chunga's Revenge (1970), Waka Jawaka (1972) and an album of orchestral music entitled the Yellow Shark (1993), his final release about a month before he passed away from cancer. Futhermore, 'tar shot' is an anagram of 'Hot Rats', the title of his 1969 album (on which Captain Beefheart also appears as a vocalist for one song.) (Website: https://www.zappa.com)

Solutions to Exercises (and more music references):

1. Solve the following system using matrix inversion:

$$5x + y = 8$$

$$2x - y = 13$$

Solution: Write the system in terms of the coefficient matrix A, the multiply by A^{-1}.

$$\begin{bmatrix} 5 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 13 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ 13 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-7} \begin{bmatrix} -1 & -1 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 8 \\ 13 \end{bmatrix} = -\frac{1}{7} \begin{bmatrix} -21 \\ 49 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$$

2. Solve the following system using matrix inversion:

$$x + 6y = 4$$

$$5x - 2y = 4$$

Solution: The coefficient matrix is $A = \begin{bmatrix} 1 & 6 \\ 5 & -2 \end{bmatrix}$, so

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-32} \begin{bmatrix} -2 & -6 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$$

3. Solve the following system using matrix inversion:

$$-x + y = 3$$

$$9x + 3y = 13$$

Solution: The coefficient matrix is $A = \begin{bmatrix} -1 & 1 \\ 9 & 3 \end{bmatrix}$, so

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-12}\begin{bmatrix} 3 & -1 \\ -9 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 13 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{10}{3} \end{bmatrix}$$

4. Solve the following system using matrix inversion:

$$x + 2y = 1$$

$$5x + 9y = 16$$

Solution: The coefficient matrix is $A = \begin{bmatrix} 1 & 2 \\ 5 & 9 \end{bmatrix}$, so

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-1}\begin{bmatrix} 9 & -2 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 16 \end{bmatrix} = \begin{bmatrix} 23 \\ -11 \end{bmatrix}$$

5. a) Given that $A = \begin{bmatrix} 10 & -1 & 2 \\ 5 & 0 & 3 \\ 7 & -2 & -2 \end{bmatrix}$, verify that

$$A^{-1} = \begin{bmatrix} \frac{6}{9} & -\frac{6}{9} & -\frac{3}{9} \\ \frac{31}{9} & -\frac{34}{9} & -\frac{20}{9} \\ -\frac{10}{9} & \frac{13}{9} & \frac{5}{9} \end{bmatrix}$$

By checking that $AA^{-1} = I_3$.

Solution: Left for the reader to check!

b) Solve the following system using matrix inversion:

$$10x - y + 2z = 34$$

$$5x + 3z = 31$$

$$7x - 2y - 2z = 0$$

Solution: Since the coefficient matrix is A, and we know A^{-1} from part (a), we have

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{6}{9} & -\frac{6}{9} & -\frac{3}{9} \\ \frac{31}{9} & -\frac{34}{9} & -\frac{20}{9} \\ -\frac{10}{9} & \frac{13}{9} & \frac{5}{9} \end{bmatrix}\begin{bmatrix} 34 \\ 31 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 7 \end{bmatrix}$$

6. In the *Bongo Fury* Company example from the text, how many sets of each type of drum should they produce if the supply of tar shot increases to 1000 square feet, while the mahogany and steel supplies remain fixed at 1400 lbs. and 400 lbs., respectively.

Solution: Using the inverse of the coefficient matrix which was derived in the text, we have

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{5}{4} & -\frac{1}{2} & -3 \\ -\frac{1}{2} & 0 & 2 \\ -\frac{1}{2} & 1 & 0 \end{bmatrix} \begin{bmatrix} 1400 \\ 1000 \\ 400 \end{bmatrix} = \begin{bmatrix} 50 \\ 100 \\ 300 \end{bmatrix}$$

So, they should produce 50 Yellow Shark Bongos, 100 Jawaka Congas, and 300 Chunga Timbales.

7. The *Green Onions Natural Food Grocery* store orders all their dairy products from *Maggie's Farm* and from *Dixie's Chicken Farm*. On any given day, Maggie's farm can deliver 50 dozen eggs and 100 gallons of milk. On any given day, Dixie's Chicken Farm can deliver 40 dozen eggs and 70 gallons milk.

Music References: 'Green Onions' alludes to the song of the same title from the 1962 debut album by Booker T & the MGs. 'Maggie's Farm' is a song from Bob Dylan's 1965 album Bringing it All Back Home, and Dixie's Chicken Farm alludes to the title track 'Dixie Chicken' of a 1973 album by the band Little Feat. (Websites: https://en.wikipedia.org/wiki/Booker_T._andtheMGs, http://www.bobdylan.com/songs/maggies-farm/, and https://www.littlefeat.net/)

a) Each week, the store can sell 350 dozen eggs and 650 gallons milk. How many days per week should each farm make a delivery to the store?

Solution: Let x be the number of delivery days per week for Maggie's Farm and let y be the number of delivery days per week for Dixie's Chicken Farm. Assuming the store wants to carry the exact number of dozens of eggs and gallons of milk it can sell each week, we see we must solve the following system:

$$50x + 40y = 350$$

$$100x + 70y = 650$$

In matrix form:

$$\begin{bmatrix} 50 & 40 \\ 100 & 70 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 350 \\ 650 \end{bmatrix}$$

The inverse of the coefficient matrix is

$$\frac{1}{-500} \begin{bmatrix} 70 & -40 \\ -100 & 50 \end{bmatrix} \begin{bmatrix} 350 \\ 650 \end{bmatrix} = \begin{bmatrix} -\frac{7}{50} & \frac{2}{25} \\ \frac{1}{5} & -\frac{1}{10} \end{bmatrix}$$

Thus,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{7}{50} & \frac{2}{25} \\ \frac{1}{5} & -\frac{1}{10} \end{bmatrix} \begin{bmatrix} 350 \\ 650 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

So, Maggie's Farm should deliver 3 says per week and Dixie's Chicken Farm should deliver 5 days per week.

b) Suppose the demand increases to 400 dozen eggs and 750 gallons of milk. Now how many days should each farm make a delivery?

Solution: Using the inverse matrix from part (a),

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{7}{50} & \frac{2}{25} \\ \frac{1}{5} & -\frac{1}{10} \end{bmatrix} \begin{bmatrix} 400 \\ 750 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

So, 4 deliveries from Maggie's Farm and 5 from Dixie's Chicken Farm.

c) Same question if the demand changed to 380 dozen eggs per week and 690 gallons of milk per week.

Solution:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{7}{50} & \frac{2}{25} \\ \frac{1}{5} & -\frac{1}{10} \end{bmatrix} \begin{bmatrix} 380 \\ 690 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

So, 2 deliveries from Maggie's Farm and 7 from Dixie's Chicken Farm.

8. The *Aoxomoxoa Chemical Company* mixes nitric acid for industrial use. In their warehouse, they keep a large supply of nitric acid in two concentrations – one vat has a 50% concentration and one vat has a 20% concentration.

Music homage to the Grateful Dead. Aoxomoxoa is the title of an album they released in 1969. (Website: https://www.dead.net/)

a) They receive an order for 240 gallons of nitric acid at 30% concentration. How many gallons from each vat do they need to mix to fill the order?

Solution: Let x denote the number of gallons of the 50% solution and y the number of gallons of the 20% solution. Then

$$x + y = 240$$

$$.5x + .2y = .3(240) = 72$$

In matrix form:

$$\begin{bmatrix} 1 & 1 \\ \frac{1}{2} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 240 \\ 72 \end{bmatrix}$$

The inverse of the coefficient matrix is

$$\frac{1}{-\frac{3}{10}} \begin{bmatrix} \frac{1}{5} & -1 \\ -\frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} & \frac{10}{3} \\ \frac{5}{3} & -\frac{10}{3} \end{bmatrix}$$

Thus

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} & \frac{10}{3} \\ \frac{5}{3} & -\frac{10}{3} \end{bmatrix} \begin{bmatrix} 240 \\ 72 \end{bmatrix} = \begin{bmatrix} 80 \\ 160 \end{bmatrix}$$

Thus, mix 80 gallons of the 50% solution and 160 gallons of the 20% solution.

b) They also receive a second order for 240 gallons at 40% concentration. How many gallons from each vat do they need to mix to fill this order?

Solution: The only difference with part (a) is the right side of the equation, which is now $\begin{bmatrix} 240 \\ .4(240) \end{bmatrix} = \begin{bmatrix} 240 \\ 96 \end{bmatrix}$. Thus

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} & \frac{10}{3} \\ \frac{5}{3} & -\frac{10}{3} \end{bmatrix} \begin{bmatrix} 240 \\ 96 \end{bmatrix} = \begin{bmatrix} 160 \\ 80 \end{bmatrix}$$

So, mix 160 gallons of the 50% solution with 80 gallons of the 20% solution.

c) They also receive an order for 180 gallons at 25% solution. How many gallons from each vat do they need to mix to fill this order?

Solution:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} & \frac{10}{3} \\ \frac{5}{3} & -\frac{10}{3} \end{bmatrix} \begin{bmatrix} 180 \\ 45 \end{bmatrix} = \begin{bmatrix} 30 \\ 150 \end{bmatrix}$$

So, mix 30 gallons of 50% solution with 150 gallons of 20% solution.

d) Finally, they receive a fourth order for 400 gallons at 40% strength. Set up a single matrix multiplication that will solve for all four orders (from parts (a)-(d)) at once.

Solution: The following calculation solves all four orders (itemized by column – one column for each of the four orders):

$$\begin{bmatrix} -\frac{2}{3} & \frac{10}{3} \\ \frac{5}{3} & -\frac{10}{3} \end{bmatrix} \begin{bmatrix} 240 & 240 & 180 & 400 \\ 72 & 96 & 45 & 160 \end{bmatrix} = \begin{bmatrix} 80 & 160 & 30 & \frac{800}{3} \\ 160 & 80 & 150 & \frac{400}{3} \end{bmatrix}$$

9. The *Low Spark Body Shop* has a bay that specializes in repairs to sedans and to pick-up trucks due to rust. Each sedan repair takes 4 hours on the lift and 7 hours in the paint room. Each pick-up truck repair requires 5 hours on the lift and 6 hours in the paint room.

Music Reference to the rock band Traffic for their 1971 album entitled the Low Spark of High Heeled Boys. (Website: https://www.wikiwand.com/en/Traffic. See also https://www.stevewinwood.com/news/5766)

a) Suppose they have 33 hours available on the lift and 55 hours available in the paint room. How many repairs of each type should they schedule?

Solution: Let x be the number of sedan repairs and y the number of pick-up ruck repairs. Assuming we wish to use up all available hours, we must solve this system:

$$4x + 5y = 33$$

$$7x + 6y = 55$$

In matrix form:

$$\begin{bmatrix} 4 & 5 \\ 7 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 33 \\ 55 \end{bmatrix}$$

The inverse of the coefficient matrix is

$$\frac{1}{-11} \begin{bmatrix} 6 & -5 \\ -7 & 4 \end{bmatrix} = \begin{bmatrix} -\frac{6}{11} & \frac{5}{11} \\ \frac{7}{11} & -\frac{4}{11} \end{bmatrix}$$

Thus,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{6}{11} & \frac{5}{11} \\ \frac{7}{11} & -\frac{4}{11} \end{bmatrix} \begin{bmatrix} 33 \\ 55 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

Thus, they should book 7 sedan repairs and 1 pick-up truck repair.

b) Suppose they have 35 hours available on the lift and 53 hours available in the paint room. How many repairs of each type should they schedule?

Solution:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{6}{11} & \frac{5}{11} \\ \frac{7}{11} & -\frac{4}{11} \end{bmatrix} \begin{bmatrix} 35 \\ 53 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Thus, they should book 5 sedan repairs and 3 truck repairs.

c) Suppose they have 44 hours available on the lift and 66 hours available in the paint room. How many of each type of repair should they schedule?

Solution:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{6}{11} & \frac{5}{11} \\ \frac{7}{11} & -\frac{4}{11} \end{bmatrix} \begin{bmatrix} 44 \\ 66 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

Thus, they should book 6 sedan repairs and 4 truck repairs.

d) One week, they repaired 6 sedans and 3 pick-up trucks. How many hours did they use on the lift in total and how many hours in the paint room? (Use matrix multiplication!)

Solution: In this problem we are given x and y, and we want to know B, so we should multiply by A instead of by A^{-1}, since $AX = B$:

$$\begin{bmatrix} 4 & 5 \\ 7 & 6 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 39 \\ 60 \end{bmatrix}$$

So, they used 39 hours on the lift and 60 hours in the paint room.

10. Solve the following system by matrix inversion:

$$w - x = 10$$

$$x - y = 20$$

$$y - z = 40$$

$$z = 100$$

[**Hint**: See exercise 13a or exercise 14 from the last section to invert the 4×4 coefficient matrix.]

Solution: In matrix form the equation reads

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \\ 40 \\ 100 \end{bmatrix}$$

Following the hint, we look up the inverse of the coefficient matrix and multiply both sides by it to find the solution:

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 20 \\ 40 \\ 100 \end{bmatrix} = \begin{bmatrix} 170 \\ 160 \\ 140 \\ 100 \end{bmatrix}$$

Section 2.5 Some Applications to Cryptography and Economics

2.5.1 A Matrix Cipher

We reproduce here Table 2 from the text:

Blank space	A	B	C	D	E	F	G	H
0	1	2	3	4	5	6	7	8
I	J	K	L	M	N	O	P	Q
9	10	11	12	13	14	15	16	17
R	S	T	U	V	W	X	Y	Z
18	19	20	21	22	23	24	25	26

Table 2 – converting between letters and numbers

Music homage in the text. The cipher example in the text about "Captain Daltrey" pays homage to the British Rock band The Who. The original band members were Roger Daltrey, Pete Townshend, John Entwistle, and Keith Moon. A 1967 hit for the band was 'I Can See For Miles' (Website: https://www.thewho.com/)

2.5.2 Leontief Input-Output Models

Music homage in the text. Our fictional country of Sunhillow was named after the 1976 solo album Olias of Sunhillow, from Jon Anderson, the vocalist for the progressive rock band Yes. (Website: https://www.jonanderson.com/)

Solutions to Exercises and more music references:

In Exercises 1-3, messages (which are all famous quotes) were encoded with a given matrix A. In each case, A^{-1} is given. Verify that $AA^{-1} = I$, and then use A^{-1} to decode the message.

1. Attributed to Jack Kerouac:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

A^{-1} is given in the text in this section.

$$AM = \begin{bmatrix} 22 & 13 & -13 & 44 & 12 & -10 & 30 & 6 \\ -21 & 31 & 51 & -7 & -16 & 39 & 18 & -9 \\ 18 & 43 & 25 & 60 & 8 & 19 & 59 & 3 \end{bmatrix}$$

Solution: $A^{-1} = \begin{bmatrix} -7 & -4 & 5 \\ -4 & -2 & 3 \\ 2 & 1 & -1 \end{bmatrix}$. We recover the message matrix as follows:

$$M = A^{-1}(AM)$$

$$= \begin{bmatrix} -7 & -4 & 5 \\ -4 & -2 & 3 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 22 & 13 & -13 & 44 & 12 & -10 & 30 & 6 \\ -21 & 31 & 51 & -7 & -16 & 39 & 18 & -9 \\ 18 & 43 & 25 & 60 & 8 & 19 & 59 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & 0 & 12 & 20 & 20 & 9 & 13 & 9 \\ 8 & 15 & 25 & 18 & 8 & 19 & 21 & 3 \\ 5 & 14 & 0 & 21 & 0 & 0 & 19 & 0 \end{bmatrix}$$

Next, we convert to letters, using the conversion table above:

$$\begin{bmatrix} T & & L & T & T & I & M & I \\ H & O & Y & R & H & S & U & C \\ E & N & & U & & & S & \end{bmatrix}$$

Thus, the secret message is “The only truth is music”.

2. Attributed to Oscar Wilde:

$$A = \begin{bmatrix} 5 & -5 & 3 & 0 & 2 & 0 \\ 12 & -13 & 7 & 3 & 6 & 0 \\ -10 & 11 & -5 & -2 & -5 & 0 \\ -2 & 2 & -1 & 0 & -1 & 0 \\ 21 & -22 & 13 & 2 & 10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 2 & 3 & -1 & 0 & 0 \\ -1 & 2 & 4 & 0 & 1 & 0 \\ -1 & 0 & 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ -3 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

And

$$AM = \begin{bmatrix} 59 & -43 & 1 & 16 & 167 & 75 & 136 & 11 & 147 & 75 & 75 \\ 139 & -113 & 27 & 98 & 428 & 246 & 393 & 86 & 380 & 246 & 180 \\ -113 & 97 & -18 & -67 & -357 & -192 & -320 & -57 & -317 & -192 & -150 \\ -21 & 21 & 0 & -8 & -66 & -29 & -53 & -6 & -58 & -29 & -30 \\ 248 & -184 & 35 & 107 & 744 & 359 & 630 & 86 & 660 & 359 & 315 \\ 3 & 8 & 5 & 5 & 0 & 7 & 5 & 5 & 0 & 7 & 0 \end{bmatrix}$$

Solution: We leave the matrix multiplication to the reader. The solution message is “Some cause happiness wherever they go; others, whenever they go”.

3. Attributed to Bertrand Russell:

$$A = \begin{bmatrix} 1 & -1 & 2 & 0 & 0 & 0 \\ -2 & 3 & -1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & -2 & 3 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -7 & -4 & 5 & 0 & 0 & 0 \\ -4 & -2 & 3 & 0 & 0 & 0 \\ 2 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -7 & -4 & 5 \\ 0 & 0 & 0 & -4 & -2 & 3 \\ 0 & 0 & 0 & 2 & 1 & -1 \end{bmatrix}$$

$$AM = \begin{bmatrix} 58 & -14 & 6 & 32 & -7 & 23 & 25 & -7 & 19 \\ -61 & 47 & 7 & -13 & 29 & -16 & 0 & 29 & -15 \\ 37 & 19 & 14 & 37 & 15 & 23 & 38 & 15 & 17 \\ 26 & 39 & 51 & 51 & -10 & -12 & 71 & -10 & 20 \\ -3 & -3 & -43 & -33 & 39 & 44 & -73 & 39 & -40 \\ 34 & 55 & 41 & 46 & 19 & 20 & 46 & 19 & 0 \end{bmatrix}$$

Solution: The message is "War does not determine who is right; only who is left."

4. Let $B = \begin{bmatrix} 2 & 1 & 0 & 0 & 3 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & -1 & -2 \end{bmatrix}$

Verify that $B^{-1} = \begin{bmatrix} -2 & 5 & 3 & -8 & -11 \\ 2 & -4 & -3 & 7 & 10 \\ 0 & 0 & 0 & 1 & 1 \\ -2 & 4 & 2 & -6 & -9 \\ 1 & -2 & -1 & 3 & 4 \end{bmatrix}$ by checking that $BB^{-1} = I_5$.

Solution: Left for the reader to verify.

The following messages in exercises 5-11 were all encoded using the matrix B from exercise 4. Use B^{-1} to decode them. (They are all quotes from rock song lyrics.)

5.

$$BM = \begin{bmatrix} 89 & 88 & 71 & 27 \\ 25 & 36 & 9 & 27 \\ 0 & -35 & -13 & -19 \\ 62 & 24 & 47 & 47 \\ -50 & -24 & -47 & -32 \end{bmatrix}$$

Solution: The message is "All you need is love" from the 1967 song "All You Need is Love" by the Beatles. (Website: https://en.wikipedia.org/wiki/All_You_Need_Is_Love)

6.

$$BM = \begin{bmatrix} 73 & 78 & 29 & 59 & 13 & 111 & 44 & 53 \\ 19 & 13 & 23 & 21 & 24 & 50 & 35 & 28 \\ -10 & -14 & -6 & -27 & -1 & 4 & -5 & 16 \\ 42 & 47 & 64 & 36 & 41 & 74 & 31 & 55 \\ -38 & -47 & -43 & -36 & -22 & -51 & -16 & -33 \end{bmatrix}$$

Solution: The message is "And she is buying the stairway to heaven" from the 1971 song 'Stairway to Heaven' by Led Zeppelin (Website: https://lz50.ledzeppelin.com/)

7.

$$BM = \begin{bmatrix} 87 & 60 & 42 & 60 & 77 & 67 & 27 & 41 \\ 33 & 10 & 36 & 33 & 30 & 18 & 26 & 43 \\ 0 & -30 & 1 & -3 & -7 & 1 & 12 & -14 \\ 50 & 55 & 29 & 13 & 49 & 35 & 38 & 29 \\ -38 & -55 & -14 & -8 & -38 & -30 & -18 & -14 \end{bmatrix}$$

Solution: The message is "I'll see you on the dark side of the moon" from the song 'Brain Damage' on the 1973 Pink Floyd album Dark Side of the Moon. (Website: https://www.pinkfloyd.com/home.php)

8.

$$BM = \begin{bmatrix} 51 & 101 & 15 & 81 & 14 & 117 & 32 \\ 33 & 56 & 28 & 50 & 16 & 33 & 27 \\ -3 & -1 & 10 & -1 & -5 & -3 & -10 \\ 7 & 64 & 23 & 60 & 23 & 51 & 4 \\ -2 & -41 & -4 & -37 & -14 & -46 & 0 \end{bmatrix}$$

Solution: The message is "The answer is blowing in the wind" from the 1963 song 'Blowing in the Wind' by Bob Dylan. (Website: https://en.wikipedia.org/wiki/Blowin_in_the_Wind)

9.

$$BM = \begin{bmatrix} 53 & 49 & 10 & 89 & 75 & 36 & 22 & 51 & 17 & 38 & 33 & 100 & 14 \\ 27 & 25 & 28 & 42 & 38 & 43 & 34 & 46 & 21 & 14 & 36 & 57 & 7 \\ -45 & 11 & 18 & 5 & 8 & -14 & 18 & 8 & 5 & 4 & -8 & -17 & 0 \\ 36 & 55 & 28 & 60 & 68 & 41 & 28 & 23 & 18 & 46 & 29 & 61 & 0 \\ -36 & -35 & -5 & -40 & -45 & -21 & -5 & -5 & -6 & -33 & -14 & -43 & 0 \end{bmatrix}$$

Solution: The message is "By the time we got to Woodstock, we were half a million strong" from the 1970 song 'Woodstock' by Joni Mitchell (Website: https://jonimitchell.com/)

10.

$$BM = \begin{bmatrix} 108 & 41 & 32 & 79 & 14 & 33 & 61 & 95 & 21 & 120 & 41 \\ 33 & 28 & 24 & 42 & 20 & 20 & 36 & 35 & 29 & 56 & 33 \\ -3 & -19 & -13 & 13 & 8 & -22 & -17 & -40 & 1 & -1 & -25 \\ 45 & 24 & 13 & 60 & 18 & 19 & 19 & 50 & 18 & 66 & 0 \\ -40 & -19 & -8 & -37 & -5 & -18 & -14 & -50 & -4 & -46 & 0 \end{bmatrix}$$

Solution: The message is "The trees are drawing me near, I've got to find out why" from the 1967 song 'Tuesday Afternoon' by the Moody Blues. (Website: https://www.moodybluestoday.com/)

11.

$$BM = \begin{bmatrix} 59 & 94 & 105 & 17 & 1 & 63 & 98 & 25 & 60 & 4 \\ 29 & 25 & 28 & 14 & 15 & 33 & 33 & 31 & 12 & 19 \\ -25 & 4 & -17 & -11 & 9 & -21 & -1 & -2 & 2 & 11 \\ 25 & 42 & 47 & 43 & 18 & 33 & 69 & 21 & 32 & 15 \\ -24 & -37 & -47 & -31 & -4 & -28 & -55 & -7 & -29 & 0 \end{bmatrix}$$

Solution: The message is "Planet Earth is blue and there's nothing I can do" from the 1969 song 'A Space Oddity' by David Bowie. (Website: https://livemap.davidbowie.com/)

12. The following message (also a lyric from a rock song) was coded using the matrix A given. Check the given inverse and decode the message.

$$A = \begin{bmatrix} 3 & 4 & 0 & 0 & 0 & 0 & 0 \\ 4 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 2 & 0 & 0 \\ 0 & 0 & -2 & 3 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 3 & 7 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -5 & 4 & 0 & 0 & 0 & 0 & 0 \\ 4 & -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -7 & -4 & 5 & 0 & 0 \\ 0 & 0 & -4 & -2 & 3 & 0 & 0 \\ 0 & 0 & 2 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7 & -2 \\ 0 & 0 & 0 & 0 & 0 & -3 & 1 \end{bmatrix}$$

$$AM = \begin{bmatrix} 102 & 32 & 60 & 51 & 31 & 122 & 137 & 135 & 131 & 59 \\ 131 & 40 & 80 & 64 & 40 & 156 & 176 & 174 & 166 & 74 \\ 30 & -14 & 3 & -13 & 59 & 37 & 7 & 63 & 14 & 0 \\ -25 & 47 & -1 & 51 & -58 & -56 & 4 & -69 & -7 & 0 \\ 26 & 19 & 5 & 25 & 40 & 12 & 13 & 38 & 17 & 0 \\ 28 & 37 & 28 & 24 & 33 & 45 & 54 & 19 & 26 & 0 \\ 91 & 126 & 89 & 77 & 104 & 153 & 182 & 65 & 91 & 0 \end{bmatrix}$$

Solution: The message is "Nothing he's got he really needs; twenty first century schizoid man" from the 1969 song 'Twenty First Century Schizoid Man' by King Crimson (Website: https://www.dgmlive.com/king-crimson)

13. In the economy for Sunhillow in the example in the text, suppose the demand matrix is as follows:

$$D = \begin{bmatrix} 5 \\ 30 \\ 15 \\ 10 \end{bmatrix}$$

Find the production levels for each sector that will meet the demand.

Solution: Using $(I - A)^{-1}$, which was computed in the text:

$$X = (I - A)^{-1}D$$

$$= \begin{bmatrix} 1.7089 & 1.1421 & 1.0391 & 2.0567 \\ 0.77286 & 1.9236 & 1.2237 & 1.8849 \\ 0.76428 & 1.0133 & 2.8768 & 1.9751 \\ 0.66982 & 1.0004 & 1.0605 & 2.9669 \end{bmatrix} \begin{bmatrix} 5 \\ 30 \\ 15 \\ 10 \end{bmatrix}$$

$$= \begin{matrix} Chemicals \\ Electricity \\ Services \\ Steel \end{matrix} \begin{bmatrix} 78.961 \\ 98.777 \\ 97{,}123 \\ 78.938 \end{bmatrix}$$

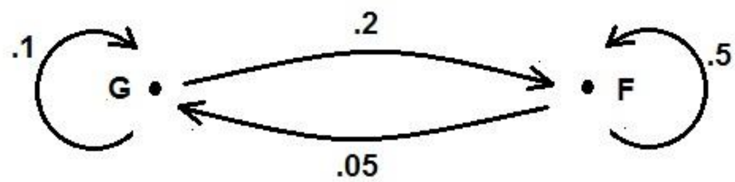
14. The fictional country of *Mekano* hired the *Mirror and Door* consulting firm to study its use of energy. For the purposes of the study, they only consider two sectors of the economy – Fossil fuels, denoted F (which includes energy produced from gas, oil, and coal), and Green Energy, denoted G (which includes energy produced by solar, wind, and geothermal sources.) All other sectors of the economy are considered to be part of the consumer demand. The result is the following 2×2 input-output matrix for the annual dollar output of the economy.

$$A = \begin{matrix} & F & G \\ F & .5 & .05 \\ G & .2 & .1 \end{matrix}$$

Music Reference to the Canadian "Rock-in-opposition" group Miriodor. Their 2001 album was entitled Mekano. (Website: http://miorodor.com/)

a) Draw the directed graph showing the flow of economic output between these two sectors of the economy. Explain why this is an open Leontief model.

Solution: It is an open Leontief model because the sum of all the numbers emanating from a vertex is strictly smaller than 1 (indicating that a portion of the output is available for consumer demand.) Here is the diagram:



b) Find the matrix $(I-A)^{-1}$

Solution:

$$I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} .5 & .05 \\ .2 & .1 \end{bmatrix} = \begin{bmatrix} .5 & -.05 \\ -.2 & .9 \end{bmatrix}$$

$$(I-A)^{-1} = \frac{1}{.44}\begin{bmatrix} .9 & .05 \\ .2 & .5 \end{bmatrix} = \begin{bmatrix} 2.0455 & 0.11364 \\ 0.45455 & 1.1364 \end{bmatrix}$$

c) Suppose the annual demand (in millions of dollars) from consumers and all other economic sectors is given by

$$D = \begin{bmatrix} 500 \\ 200 \end{bmatrix}$$

Find the production levels that will meet this demand.

Solution:

$$X = (I-A)^{-1}D = \begin{bmatrix} 2.0455 & 0.11364 \\ 0.45455 & 1.1364 \end{bmatrix}\begin{bmatrix} 500 \\ 200 \end{bmatrix} = \begin{bmatrix} 1045.5 \\ 454.56 \end{bmatrix}$$

Thus, the Fossil Fuel production F should be $1,045,500,000 and the Green Energy production G should be $454,560,000.

d) In the near future, Mekano estimates that its total annual demand for energy will increase by 200 units (that is, by $200,000,000). Determine the production levels that will meet the demand under two hypothetical scenarios -first assuming that the additional demand is placed entirely on the Fossil Fuel sector F, so that the demand becomes $D = \begin{bmatrix} 700 \\ 200 \end{bmatrix}$. Secondly, assuming the additional demand is placed entirely with the Green Energy sector, so that the demand becomes $D = \begin{bmatrix} 500 \\ 400 \end{bmatrix}$. If Mekano could control where the additional demand could be placed, which scenario would be better for Mekano, and why?

Solution: First suppose the additional demand was placed entirely on Fossil Fuels F. Then the demand matrix changes to $D_1 = \begin{bmatrix} 700 \\ 200 \end{bmatrix}$. Secondly, if the additional demand is placed entirely in the Green Energy sector, the new demand would be $D_2 = \begin{bmatrix} 500 \\ 400 \end{bmatrix}$. In either case, we find the required production by left multiplying by $(I - A)^{-1}$. We can solve both scenarios at once by combining the two demand vectors into a single matrix with two columns:

$$(I - A)^{-1}[D_1\ D_2] = \begin{bmatrix} 2.0455 & 0.11364 \\ 0.45455 & 1.1364 \end{bmatrix} \begin{bmatrix} 700 & 500 \\ 200 & 400 \end{bmatrix} = \begin{bmatrix} 1454.6 & 1068.2 \\ 545.47 & 681.84 \end{bmatrix}$$

In the first column (additional demand on F), the production vector is $X_1 = \begin{matrix} F \\ G \end{matrix} \begin{bmatrix} 1454.6 \\ 545.47 \end{bmatrix}$, for a total output of \$2,000.07 million. In the second column, the production vector is $X_2 = \begin{matrix} F \\ G \end{matrix} \begin{bmatrix} 1068.2 \\ 681.84 \end{bmatrix}$, for a total output of \$1750.04 million. Assuming the infrastructure of the economy can handle either scenario, it would be better if Mekano could place the additional demand all in the Green energy sector G, since that results in a savings in production of over \$250 million per year while still meeting the demand.

15. Suppose you are given the input-output matrix A and the demand matrix D shown below. Assuming an open Leontief model, determine the production levels that will meet the demand.

$$A = \begin{bmatrix} .1 & .25 \\ .5 & .3 \end{bmatrix} \qquad D = \begin{bmatrix} 400{,}000 \\ 500.000 \end{bmatrix}$$

Solution:

$$I - A = \begin{bmatrix} .9 & -.25 \\ -.5 & .7 \end{bmatrix}$$

$$(I - A)^{-1} = \frac{1}{.505} \begin{bmatrix} .7 & .25 \\ .5 & .9 \end{bmatrix} = \begin{bmatrix} 1.3861 & 0.49505 \\ 0.99010 & 1.7822 \end{bmatrix}$$

Thus

$$X = (I - A)^{-1}D = \begin{bmatrix} 1.3861 & 0.49505 \\ 0.99010 & 1.7822 \end{bmatrix} \begin{bmatrix} 400{,}000 \\ 500.000 \end{bmatrix} = \begin{bmatrix} 801{,}970 \\ 1{,}287{,}100 \end{bmatrix}$$

16. Suppose you are given the input-output matrix A and the demand matrix D shown below. Assuming an open Leontief model, determine the production levels that will meet the demand.

$$A = \begin{bmatrix} .1 & .5 & .2 \\ .3 & .4 & .2 \\ 0 & .2 & .3 \end{bmatrix} \qquad D = \begin{bmatrix} 500 \\ 200 \\ 400 \end{bmatrix}$$

(You will need to find the inverse of a 3×3 matrix. You may use the technique of exercise 11 from section 2.4, or you may use a computer algebra system such as *Mathematica*.)

Solution:

$$I - A = \begin{bmatrix} .9 & -.5 & -.2 \\ -.3 & .6 & -.2 \\ 0 & -.2 & .7 \end{bmatrix}$$

Using software, we obtain:

$$(I - A)^{-1} = \begin{bmatrix} 1.6889 & 1.7333 & 0.97778 \\ 0.93333 & 2.8 & 1.0667 \\ 0.26667 & 0.80 & 1.7333 \end{bmatrix}$$

Thus.

$$X = \begin{bmatrix} 1.6889 & 1.7333 & 0.97778 \\ 0.93333 & 2.8 & 1.0667 \\ 0.26667 & 0.80 & 1.7333 \end{bmatrix} \begin{bmatrix} 500 \\ 200 \\ 400 \end{bmatrix} = \begin{bmatrix} 1582.2 \\ 1453.3 \\ 986.66 \end{bmatrix}$$

17. a) Any mathematical model has assumptions built into it. Consider Leontief's original analysis of the US economy in the late 1940's broken down into 500 sectors. Would you expect the predictions of Leontief's model to remain valid for more than a year or two into the future? Why or why not? What assumptions were made in the model?

Solution: The assumptions that were built into the model include that the entries of the input-output matrix do not change from year to year. Consequently, we would not expect the analysis of the US economy to be valid for more than a year or two into the future (because over time, we expect the entries of the input-output matrix to vary somewhat.)

b) Suppose that during the 1940's, a similar analysis was made of the economy of the USSR (Soviet Union.) Assume similar levels of accuracy and the same number of sectors of the economy -500. Speculate on whether the predictions of Leontief's model would be more accurate for the USSR for farther into the future than the predictions of the study of the US economy, or whether they would be less accurate. Give reasons for your answers.

Solution: In the USSR economy, decisions were made by a central committee of the government, including (presumably) how much of each output is sent to various other sectors of the economy. Therefore, it seems more likely that the numbers in the input-output matrix would remain the same over a several year period, unlike a free-market economy. So, one might expect the analysis of the future of the Soviet economy to be more accurate for farther into the future than the analysis of the US economy.

Chapter 3. Graphical Linear Programming

Section 3.1 Introduction and Graphical Solutions

Music reference in the text: Example 3.2 alludes to the new wave band the Cars. Rick Ocasek was the vocalist in the band. 'Moving in Stereo' was a song from their debut album in 1978 (Website: https://thecars.org/)

Solutions to Exercises (and more music references):

The first six exercises deal with the following system of equations and accompanying graph:

$$5x + 2y = 60$$

$$3x + 6y = 60$$

$$5x + 4y = 90$$

$$x = 14$$

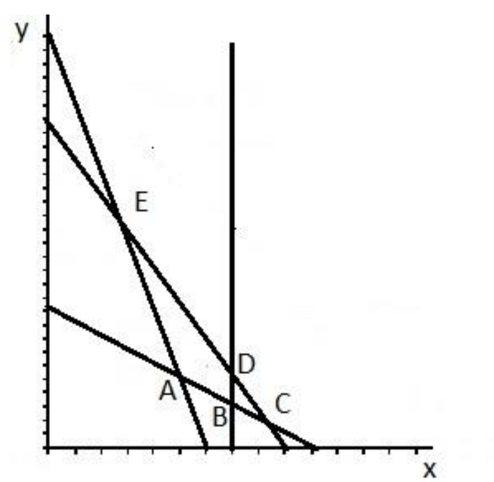


Figure 3.4

1. Find the coordinates of the seven x and y intercepts for the four lines in the above system, and thereby identify which equation goes with each line in the graph.

Solution: The line $x = 14$ is vertical and has no y intercept. The x intercept is $(14,0)$. From the picture, there is only one line with a smaller x intercept, and it is the line fist line $5x + 2y = 60$, with x intercept $(12,0)$ and y intercept $(0,30)$. The second line $3x + 6y = 60$ has x intercept $(20,0)$ and y intercept $(0,10)$. In the picture it is the line with the least steepness (slope), The third line $5x + 4y = 90$ has x intercept $(8,0)$ and y intercept $\left(0, \frac{45}{2}\right)$.

2. Find the coordinates of the five labeled points of intersection A, B, C, D, E in the graph above.

Solution: The point A is the intersection of the two lines

$$5x + 2y = 60$$

$$3x + 6y = 60$$

Thus

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{24}\begin{bmatrix} 6 & -2 \\ -3 & 5 \end{bmatrix}\begin{bmatrix} 60 \\ 60 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

The point B is the intersection of the lines

$$3x + 6y = 60$$

$$x = 14$$

The solution is $(x, y) = (14,3)$.

The point C is the intersection of the lines

$$3x + 6y = 60$$

$$5x + 4y = 90$$

The solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-18}\begin{bmatrix} 4 & -6 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 60 \\ 90 \end{bmatrix} = \begin{bmatrix} \frac{50}{3} \\ \frac{5}{3} \end{bmatrix}.$$

The point D is the intersection of the lines:

$$5x + 4y = 90$$

$$x = 14$$

The solution is $(x, y) = (14,5)$.

The point E is the intersection of the lines

$$5x + 2y = 60$$

$$5x + 4y = 90$$

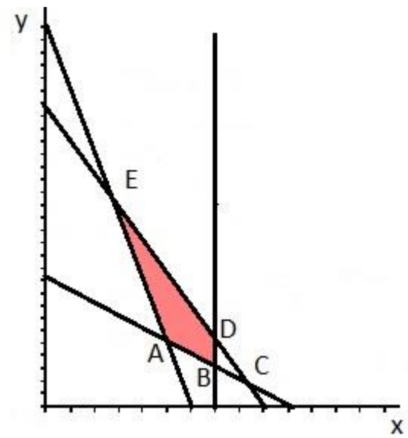
The solution is $(x, y) = (6,15)$.

3. Determine which region of the graph should be shaded in to indicate the solution set to the system of inequalities:

$$5x + 2y \geq 60$$

$$3x + 6y \geq 60$$
$$5x + 4y \leq 90$$
$$x \leq 14$$
$$x \geq 0, \quad y \geq 0$$

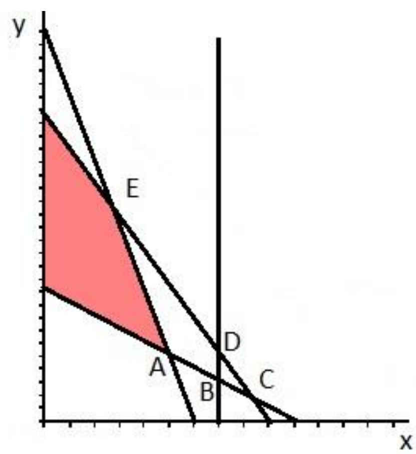
Solution:



4. Determine which region of the graph should be shaded in to indicate the solution set to the system of inequalities:

$$5x + 2y \leq 60$$
$$3x + 6y \geq 60$$
$$5x + 4y \leq 90$$
$$x \leq 14$$
$$x \geq 0, \quad y \geq 0$$

Solution:



5. Determine which region of the graph should be shaded in to indicate the solution set to the system of inequalities:

$$5x + 2y \leq 60$$

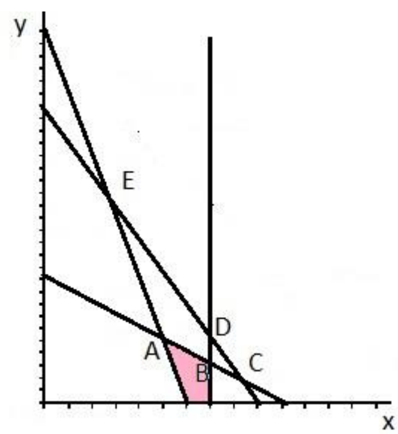
$$3x + 6y \geq 60$$
$$5x + 4y \leq 90$$
$$x \geq 14$$
$$x \geq 0,\;\; y \geq 0$$

Solution: The solution set is empty.

6. Determine which region of the graph should be shaded in to indicate the solution set to the system of inequalities:

$$5x + 2y \geq 60$$
$$3x + 6y \leq 60$$
$$5x + 4y \leq 90$$
$$x \leq 14$$
$$x \geq 0,\;\; y \geq 0$$

Solution:



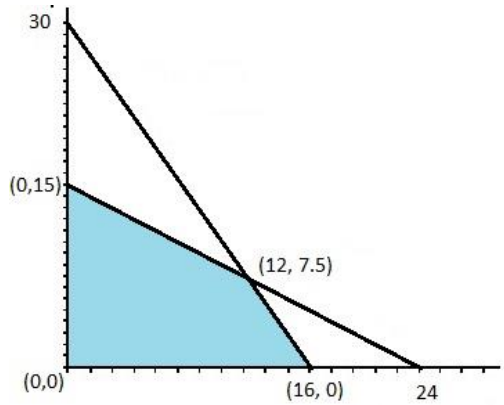
7. Graph the system of inequalities, shade in the solution set, and find the coordinates of the points of intersection:

$$10x + 16y \leq 240$$

$$15x + 8y \leq 240$$

$$x \geq 0,\;\; y \geq 0$$

Solution:



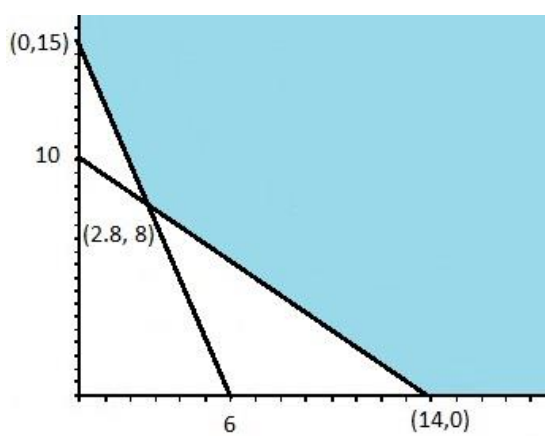
8. Graph the system of inequalities, shade in the solution set, and find the coordinates of the points of intersection:

$$5x + 2y \geq 30$$

$$5x + 7y \geq 70$$

$$x \geq 0, \quad y \geq 0$$

Solution:



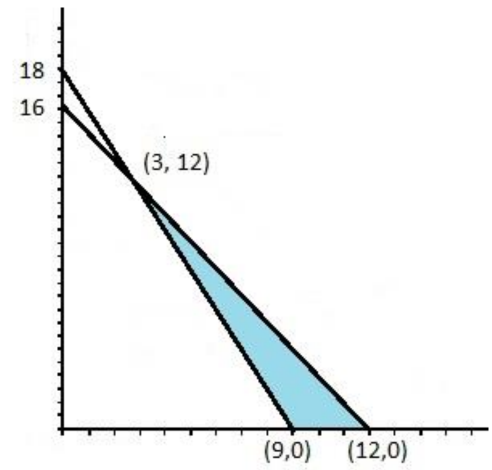
9. Graph the system of inequalities, shade in the solution set, and find the coordinates of the points of intersection:

$$4x + 2y \geq 36$$

$$4x + 3y \leq 48$$

$$x \geq 0, \ y \geq 0$$

Solution:



10. Graph the system of inequalities, shade in the solution set, and find the coordinates of the points of intersection:

$$4x + 7y \leq 120$$

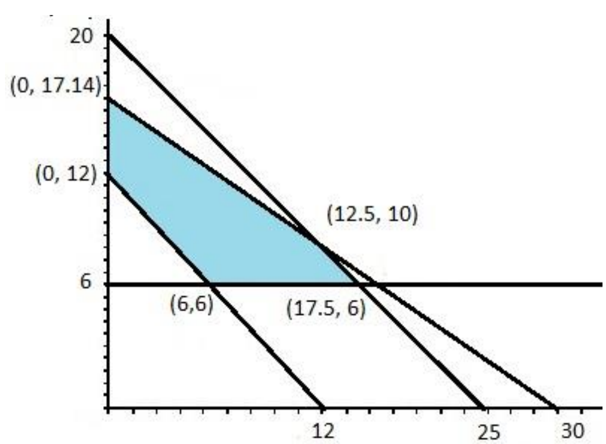
$$20x + 25y \leq 500$$

$$x + y \geq 12$$

$$y \geq 6$$

$$x \geq 0, \quad y \geq 0$$

Solution: Note that the inequality $y \geq 0$ is superfluous, since we already know $y \geq 6$.



Set up the following problems as linear programming problems. Be sure to label the decision variables, clearly state the objective function, and record all the constraints. (Do not graph the feasible region or try to solve the problems.)

11. Bruce Jax is responsible for constructing end tables and kitchen tables for the *White Room Woodshop*. Each end table uses 2 square yards of $\frac{3}{4}$ inch oak boards and takes two hours to complete. Each kitchen table uses 4 square yards of oak boards and takes 3 hours to complete. This week he has available 36 square yards of oak boards and 32 hours of time. Other resources are unlimited. How many of each item should he make if he is paid $70 for each end table and $100 for each kitchen table?

Music homage to the band Cream. Their vocalist and bass player was Jack Bruce, and 'White Room' was a 1968 hit from the double album Wheels of Fire. (Website: https://en.wikipedia.ord/wiki/Cream)

Solution: Let x be the number of end tables and y the number of kitchen tables. We assume the objective is to maximize the total amount P that Bruce is paid.

$$Maximize\ P = 70x + 100y$$

Subject to:

$$\begin{gathered} 2x + 4y \leq 36\ (square\ yards\ of\ wood) \\ 2x + 3y \leq 32\ (hours\ of\ time) \\ x \geq 0, \quad y \geq 0 \end{gathered}$$

(**Remark**: for a realistic solution, we'd also require x, y to be integers.)

12. Joni is putting her designing skills to work at the *Court and Sparkle Jewelry Emporium*. She makes two signature design bracelet models. Each Dawntreader bracelet uses 1 ruby, 6 pearls, and 10 opals. Each Hejira bracelet uses 3 rubies, 3 pearls, and 15 opals. She has 54 rubies, 120 pearls, and 300 opals to work with. If either model results in a profit of $1,800 for Joni, how many of each type should she make?

Music homage to Joni Mitchell. Her 1974 album is entitled Court and Spark, and her 1976 album is entitled Hejira. Also, "The Dawntreader" is a song from her 1968 debut album. (Website: https://www.jonimitchell.com/)

Solution: Let x be the number of Dawntreader bracelets, and let y the number of Hejira bracelets.

$$Maximize\ Profit\ P = 1800x + 1800y$$

Subject to:

$$\begin{gathered} x + 3y \leq 54\ (rubies) \\ 6x + 3y \leq 120\ (pearls) \\ 10x + 15y \leq 500\ (opals) \\ x \geq 0, \quad y \geq 0 \end{gathered}$$

(**Remark**: for a realistic solution, we'd also require x, y to be integers.)

13. *Toys in the Attic, Inc*. operates two workshops to build toys for needy children. Mr. Tyler's shop can produce 36 *Angel* dolls, 16 *Kings and Queens* board games, and 16 *Back in the Saddle* rocking horses each day it operates. Mr. Perry's shop can produce 10 *Angel* dolls, 10 *Kings and Queens* board games, and 20 *Back in the Saddle* rocking horses each day it operates. It costs \$144 to operate Mr. Tyler's shop for one day and \$166 to operate Mr. Perry's shop for one day. Suppose the company receives an order from *Kids Dream On* Charity Foundation for at least 720 *Angel* dolls, at least 520 *Kings and Queens* board games, and at least 640 *Back in the Saddle* rocking horses. How many days should they operate each shop in order to fill the order at least possible cost?

Music homage to the rock band Aerosmith. Two of the band members are vocalist Steven Tyler and guitarist Joe Perry. They released songs entitled 'Toys in the Attic' (1975), 'Back in the Saddle' (1976), 'Angel' (1987), and 'Kings and Queens' (1977), as well as their smash breakout hit 'Dream On' (1973). (Website: https://www.aerosmith.com/)

Solution: Let x be the number of days to operate Mr. Tyler's shop and y the number of days to operate Mr. Perry's shop.

$$Minimize\ Cost\ C = 144x + 166y$$

Subject to:

$$\begin{aligned} 36x + 10y &\geq 720\ (Angel\ dolls) \\ 16x + 10y &\geq 520\ (Kings\ and\ Queens\ board\ games) \\ 16x + 20y &\geq 640\ (Back\ in\ the\ Saddle\ rocking\ horses) \\ x \geq 0 &\quad y \geq 0 \end{aligned}$$

14. The ocean liner *Gigantic* is being refitted with new life boats and safety equipment. There are three types of lifeboats available. Each wooden lifeboat holds 40 people, weighs 1,000 lbs., and comes equipped with 6 first aid kits and 40 life jackets. The coast for a wooden boat is \$6,000. Each large rubber raft holds 24 people, weighs 400 lbs., and comes equipped with 4 first aid kits and 30 life jackets. Each small rubber raft holds 10 people, weighs 250 lbs., and comes equipped with 1 first aid kit and 8 life jackets. The cost for a large rubber raft is \$4,000 and the cost for a small rubber raft is \$1,500. The Gigantic has 40 storage berths for these lifeboats. Each berth can hold either 1 wooden lifeboat or 2 large rubber rafts or 4 small rubber rafts. According to commercial marine regulations, all of the following conditions must be met:

- The total weight of the lifeboats must not exceed 38,000 lbs.
- The 40 storage berths must all be filled completely
- They must have the ability to float at least 1,600 people in the lifeboats
- They must carry a total of at least 1,500 life jackets
- They must carry a total of at least 200 first aid kits.

In addition, due to contract agreements with CSK Wooden Ships, the manufacturer, they are obligated to purchase:

- At least 12 wooden lifeboats

- At least a total of 80 rubber rafts (regardless of size)

How many of each type of lifeboat should be purchased in order to minimize the total cost?

Music Reference to the 1969 anti-war song 'Wooden Ships'. The song was co-written by Paul Kantner of Jefferson Airplane and David Crosby and Steven Stills of Crosby, Stills & Nash. Hence the "CSK" in the exercise. Jefferson Airplane released a recording of the song on their 1969 album Volunteers, and Crosby, Stills, and Nash recorded their own version which appeared on their 1969 self-titled debut album. (Websites: https://jeffersonairplane.com and https://www.crosbystillsnash.com/)

Solution: Let x be the number of wooden lifeboats, y the number of large rubber rafts, and z the number of small rubber rafts. Then:

$$Minimize\ cost \quad C = 6000x + 4000y + 1500z$$

Subject to:

$$\begin{gathered} 1000x + 400y + 250z \leq 38{,}000\ (lbs.) \\ x + \frac{1}{2}y + \frac{1}{4}z = 40\ (storage\ berths) \\ 40x + 24y + 10z \geq 1600\ (people) \\ 40x + 30y + 8z \geq 1500\ (life\ jackets) \\ 6x + 4y + z \geq 200\ (first\ aid\ kits) \\ x \geq 12\ (wooden\ boats) \\ y + z \geq 80\ (rubber\ rafts) \\ x \geq 0, \quad y \geq 0, \quad z \geq 0 \end{gathered}$$

(**Remark**: In this problem we would also require that x, y, z be integers.)

15. Arnold and Penny Layne have up to $60,000 to invest for a one-year period. There are three investment options they are considering. Certificates of Deposit, which have an expected annual return of 3%, Municipal bonds (the proceeds of which go to the cleaning and upkeep of local busses, police cars, and fire engines), which have an expected annual return of 5%, and Stocks on women's apparel, with an expected annual return of 11%. They would like to maximize their expected return for the year, but are adhering to the following guidelines suggested by their financial advisor:

- The amount invested in Municipal Bonds should be at least $10,000.
- Because of stock volatility, the amount invested in Stocks should be at most $10,000 more than the amount invested in Bonds.
- Because of the reliability of Certificates of Deposit, at least $\frac{1}{3}$ of the total investment should be in Certificates of Deposit.

Be sure you have all of the constraints.

Music references to the Beatles for the song 'Penny Lane' (which contains the line "he likes to keep his fire engine clean, it's a clean machine", and to Pink Floyd for their debut single song 'Arnold Layne' about an odd character who steals women's clothing. Both songs are from 1967. (Websites: https://en.wikipedia.org/wiki/Penny_Lane and https://en.wikipedia.org/wiki/Arnold_Layne)

Solution: Let x be the amount invest in Certificates of Deposit, y the amount invested in Municipal Bonds, and z the amount invested in Stocks of women's apparel.

$$Maximize\ interest \quad R = .03x + .05y + .11z$$

Subject to:

$$\begin{aligned} x + y + z &\leq 60{,}000\ (amount\ invested) \\ y &\geq 10000\ (Municipal\ Bonds) \\ -y + z &\leq 10000\ \ (stock\ volatility) \\ 2x - y - z &\geq 0\ (reliability\ of\ CD's) \\ x \geq 0, \quad y &\geq 0, \quad z \geq 0 \end{aligned}$$

Remarks: It is not hard to see that the maximum return can only occur if they invest the entire $60,000. Thus, it would be equivalent to replace the first constraint by an exact equality $x + y + z = 60{,}000$. Also, there are alternate ways of writing the fourth constraint coming from the reliability of CDs. Start out writing $x \geq \frac{1}{3}(x + y + z)$ and rearrange it as you prefer. If you replace the first constraint with the exact equation $x + y + z = 60{,}000$, then the total amount invested is $60,000 instead of $x + y + z$, so the fourth constraint in this case could be written as $x \geq 20000$.

The next three exercises all relate to the following scenario. The buffet bar at the *Acquiring the Taste Bistro* offers five different items, which they sell individually by weight. Customers may add whatever they want to their plate, but pay for each item separately. Because the bistro customers are health-conscious, the following nutritional information is posted at the buffet bar:

Per ounce	Garden salad w/dressing	Grilled vegetables	Pasta w/tomato sauce	Meatballs	Curried chicken salad
Fat (g)	**1**	**1**	**3**	**6**	**4**
Protein (g)	**0**	**1**	**2**	**10**	**12**
Carbohydrates (g)	**3**	**4**	**12**	**6**	**3**
Sodium (mg)	**10**	**12**	**15**	**25**	**20**
Sugar (mg)	**12**	**5**	**6**	**4**	**1**
Calories	**15**	**20**	**40**	**60**	**50**
Cost (cents)	**25**	**40**	**30**	**50**	**60**

Table 3 – Acquiring the Taste Nutritional Information

Music homage to the British progressive rock band Gentle Giant. Acquiring the Taste is the title of their second album, released in 1971. Three of the band members were brothers: Derek, Ray, and Phil Shulman. The protagonists in the next three exercises are named in their honor. Other band members and songs will be referenced in later exercises. (Website: https://gentlegiantmusic.com/GG/Gentle_Giant)

16. Derek just moved to town and has a new job, so is most concerned with saving money, so when he eats at the Bistro, his objective is to minimize costs. He does have some other requirements, however. Namely:

- He would like to limit his fat intake to 40 g or less.
- He wants a minimum of 80g protein.
- He wants a minimum of 60g carbohydrates.
- With high blood pressure, Derek must limit his sodium intake to at most 200 mg.
- He would like to limit his calorie intake to at most 700 calories.

Solution: Let s be the number of ounces of garden salad, v the number of ounces of grilled vegetables, p the number of ounces of pasta, m the number of ounces of meatballs, and x the number of ounces of curried chicken salad. Then:

$$Minimize\ Cost\ \ C = 25s + 40v + 30p + 50m + 60x$$

Subject to:

$$\begin{gathered} s + v + 3p + 6m + 4x \leq 40\ (grams\ fat) \\ v + 2p + 10m + 12x \geq 80\ (grams\ protein) \\ 3s + 4v + 12p + 6m + 3x \geq 60\ (grams\ carbs) \\ 10s + 12v + 15p + 25m + 20x \leq 200\ (mg\ sodium) \\ 15s + 20v + 40p + 60m + 50x \leq 700\ (calories) \\ s \geq 0, \quad v \geq 0, \quad p \geq 0, \quad m \geq 0, \quad x \geq 0 \end{gathered}$$

17. Ray is a body builder, so his objective is to maximize protein intake. Ray's other requirements are:

- He would like to limit his fat intake to at most 60 g.
- He wants a minimum of 120 g carbohydrates.
- He would like to have between 90 mg and 200 mg sugar.
- He is on a budget and cannot spend more than $7.50 for lunch.

Solution: Keep the same variable names as in problem 16. Then Ray must

$$Maximize\ Protein\ \ P = v + 2p + 10m + 12x$$

Subject to:

$$\begin{gathered} s + v + 3p + 6m + 4x \leq 60\ (grams\ fat) \\ 3s + 4v + 12p + 6m + 3x \geq 120\ (grams\ carbs) \\ 12s + 5v + 6p + 4m + x \geq 90\ \ (mg\ sugar) \\ 12s + 5v + 6p + 4m + x \leq 200\ \ (mg\ sugar) \\ 25s + 40v + 30p + 50m + 60x \leq 750\ (cost) \\ s \geq 0, \quad v \geq 0, \quad p \geq 0, \quad m \geq 0, \quad x \geq 0 \end{gathered}$$

18. Phil is on a weight-loss diet, and his objective is to minimize calorie intake. Phil's other requirements are:

- He would like to limit his fat intake to at most 20 g.

- He would like to limit his carbohydrate intake to at most 50 g.
- He would like to limit his sugar intake to at most 88 mg.
- He would like to limit his cost to at most $6.00
- He needs to eat at least 12 oz. food altogether, in order to avoid being hungry before his next meal.

Solution: Phil must

$$Minimize\ calories\ \ Z = 15s + 20v + 40p + 60m + 50x$$

Subject to:

$$s + v + 3p + 6m + 4x \leq 20\ (grams\ fat)$$
$$3s + 4v + 12p + 6m + 3x \leq 50\ \ (grams\ carbs)$$
$$12s + 5v + 6p + 4m + x \leq 88\ \ (mg\ sugar)$$
$$25s + 40v + 30p + 50m + 60x \leq 600\ (cost)$$
$$s + v + p + m + x \geq 12\ (oz\ food)$$
$$s \geq 0, \quad v \geq 0, \quad p \geq 0, \quad m \geq 0, \quad x \geq 0$$

19. *Green's Heavy Metal Foundry* mixes three different alloys composed of copper, zinc, and iron. Each 100 lb. unit of Alloy I consists of 50 lbs. copper, 50 lbs. zinc, and no iron. Each 100 lb. unit of Alloy II consists of 30 lbs. copper, 30 lbs. zinc, and 40 lbs. iron. Each 100 lb. unit of Alloy III consists of 50 lbs. copper, 20 lbs. zinc, and 30 lbs. iron. Each unit of Alloy I generates $100 profit, each unit of Alloy II generates $80 profit, and each unit of Alloy III generates $40 profit. There are 12,000 lbs. copper, 10,000 lbs. zinc, and 12,000 lbs. iron available. The foundry has hired Gary Giante, and outside consultant from *Mortimore, Weathers, and Smith, Ltd*., to help them maximize their profit. What problem does Mr. Giante need to solve in order to advise the foundry?

The Music allusion to Gentle Giant continues. Gary Green was the guitarist for the band and they had three drummers pass through their ranks: Martin Smith for the first two albums, Malcolm Mortimore for the third, and John Weathers for the rest of their career.

Solution: Let x be the number of (100 lb.) units of Alloy I, y the number of units of Alloy II, and z the number of units of Alloy III. Then Mr. Giante must solve the following linear programming problem:

$$Mazimize\ Profit\ \ P = 100x + 80y + 40z$$

Subject to:

$$50x + 30y + 50z \leq 12000\ (lbs.copper)$$
$$50x + 30y + 20z \leq 10000\ (lbs.zinc)$$
$$40y + 30z \leq 12000\ (lbs.iron)$$
$$x \geq 0, \quad y \geq 0, \quad z \geq 0$$

20. The *Creative Thought Matters Biological Research Company* has developed a promising new antibiotic that has been shown to be effective in fighting strains of hepatitis infection that are resistant to other antibiotic agents. They have stockpiled a supply of this new drug at two research hospitals.

The *Huey News Hospital* in Atlanta, GA has 330 kg. of the agent available to ship (beyond what they will keep there), and the *Lewis Memorial Hospital* in Boston, MA has 450 kg. of it available to ship (beyond what they will keep there.) In order to fight a growing national problem of resistant infection, the company wants to ship the available units to three other hospital clinics across the country – one in Chicago, IL, one in Denver, CO, and one in San Francisco, CA. Chicago has requested at least 300 kg., Denver has requested at least 120 kg., and San Francisco has requested at least 360 kg. The drug must be kept cold during shipment, so shipping is expensive. The table below indicates the cost (in dollars) of shipping a one kg. unit of the drug from each supply hospital to each of the three clinics at the recipient hospitals.

	Chicago	**Denver**	**San Francisco**	**Shipping Capacities**
Huey News Hospital (Atlanta)	**$30**	**$45**	**$80**	**330 kg**
Lewis Memorial Hospital (Boston)	**$40**	**$40**	**$70**	**450 kg**
Clinic Requests	**300 kg**	**120 kg**	**360 kg**	

Table 4

If the Creative Thought Matters Company wants to ship the drugs at minimum cost, how many kg. should they ship from each hospital to each clinic? Set this up as a linear programming problem.

Music Reference: to Huey Lewis and the News, an American band from the west coast during the 1980's who released a song entitled 'I Want a New Drug' in 1983. (Website: https://hueylewisandthenews.com/) The name of the biological research company which is developing the new drug in this exercise, "Creative Thought Matters" is not a musical reference – it is a motto used by Skidmore College where I teach.

Solution: Let u be the number of kilograms shipped from Atlanta to Chicago, v the number of kilograms shipped from Atlanta to Denver, w the number of kilograms shipped from Atlanta to San Francisco, x the number of kilograms shipped from Boston to Chicago, y the number of kilograms shipped from Boston to Denver, and z the number of kilograms shipped from Boston to San Francisco. Then:

$$Minimize\ Cost\ \ C = 30u + 45v + 80w + 40x + 40y + 70z$$

Subject to:

$$u + v + w \leq 330\ (kg. stored\ in\ Atlanta)$$
$$x + y + z \leq 450\ (kg. stored\ in\ Boston)$$
$$u + x \geq 300\ (kg. requested\ by\ Chicago)$$
$$v + y \geq 120\ (kg. requested\ by\ Denver)$$
$$w + z \geq 360\ (kg. requested\ by\ San\ Francisco)$$
$$u \geq 0, \quad v \geq 0, \quad w \geq 0, \quad x \geq 0, \quad y \geq 0, \quad z \geq 0$$

21. Set up the following brain-teaser as a linear programming problem. Find the largest three-digit number N such that the sum of the digits of N is at least 12, the hundreds digit minus the units' digit is no larger than the tens digit plus 4, and the number N minus the number M obtained from N by reversing the digits is no larger than 693. Be sure to include all the constraints.

Solution: The way to view this problem is to think of the unknowns as the digits of N. Let x be the hundreds digit, y the tens digit, and z the units' digit. Then $N = 100x + 10y + z$ is what is to be maximized. The number $M = 100z + 10y + x$. Notice that $N - M = 99x - 99z$, and $99x - 99z \leq 693$ is equivalent to $x - z \leq 7$. Finally note that digits must be at most 9, while x cannot be 0 since we stated that N is a three-digit number. Thus:

$$Maximize\ N = 100x + 10y + z$$

Subject to:

$$x + y + z \leq 12$$
$$x - y - z \leq 4$$
$$x - z \leq 7$$
$$x \leq 9$$
$$y \leq 9$$
$$z \leq 9$$
$$x \geq 1$$
$$(x \geq 0), \quad y \geq 0, \quad z \geq 0$$

(**Remark**: The inequality in parenthesis is superfluous since we already know $x \geq 1$. Also, in this problem, x, y, z must be integers.)

Section 3.2 The Decision Space: $m \times 2$ problems

No new music references or exercises in this section.

Section 3.3 Convex Sets and the Corner Point Theorem

No new music references or exercises in this section.

Section 3.4 Problems with No Solutions or Infinitely Many Solutions.

Solutions to Exercises and more music references:

1. Consider Angie and Kiki's lemonade stand problem (3.1; in final form on page 107). Suppose the girls decide to increase the prices of their lemonade to $1.50 for a glass for sweet lemonade and $2.50 for a glass of tart lemonade. With no other changes to the problem, determine the optimal lemonade production plan now. Be sure to indicate the leftover resources in your solution.

Solution: The feasible set is unchanged, so we need only evaluate the new objective function at the known corner points:

corner	(x, y)	$R = 1.5x + 2.5y$
A	$(0,0)$	$\$0$
B	$(15,0)$	$\$22.50$
C	$(12,6)$	$\$33.00$
D	$(4,12)$	$\$36.00$
E	$(0,14)$	$\$35.00$

Thus, the point D is optimal. At this point, all the lemons and limes are used up, but there are $10\ Tbsp$ sugar leftover. Thus, the solution is:

$$x = 4\ glasses\ sweet\ lemonade$$
$$y = 12\ glasses\ tat\ lemonade$$
$$R = \$36.00\ \ (maximum\ revenue)$$
$$S_1 = 0\ (leftover\ lemons)$$
$$S_2 = 0\ (leftover\ limes)$$
$$S_3 = 10\ (leftover\ Tbsp. sugar)$$

2. Solve the following linear programming problem. Be sure to give the complete solution.

a)

$$Maximize\ R = 10x + 3y$$

Subject to:

$$5x + 2y \geq 60$$
$$3x + 6y \geq 60$$
$$5x + 4y \leq 90$$
$$x \leq 14$$
$$x \geq 0, \qquad y \geq 0$$

(see exercise 3 from section 3.1)

Solution: The system of inequalities which defines the feasible set is the same system as exercise 3 from section 3.1, and we have already graphed the solution. Furthermore, in exercise 2 from section 3.1, we already found the coordinates of the corner points A, B, D & E of this region. We evaluate the objective function at these corners:

Corner	(x, y)	$R = 10x + 3y$
A	$(10,5)$	115
B	$(14,3)$	149
D	$(14,5)$	155
E	$(6,15)$	105

Thus, D is the optimal point. The complete solution is:

$$x = 14$$
$$y = 5$$
$$R = 155\ (maximized)$$
$$S_1 = 0$$
$$S_2 = 48$$
$$S_3 = 0$$
$$S_4 = 0$$

b) Keep the same objective function and constraints as in part a, but now Minimize R.

Solution: The feasible set and the table from part (a) are unchanged. The only difference is we take the corner with the minimum value of R, which is corner E. Thus,

$$x = 6$$
$$y = 15$$
$$R = 105\ (minimized)$$
$$S_1 = 20$$
$$S_2 = 12$$
$$S_3 = 0$$
$$S_4 = 8$$

3. Solve the maximization problem (see exercise 7 from section 3.1):

Maximize $P = x + 3y$
Subject to:
$$10x + 16y \leq 240$$
$$15x + 8y \leq 240$$
$$x \geq 0, \qquad y \geq 0$$

Solution: The system of inequalities is exactly the same as exercise 7 from section 3.1. We computed the coordinates of the corners of the feasible set in that exercise. Thus, we need only evaluate the objective function at these corners:

(x, y)	$P = x + 3y$
$(0,0)$	0
$(0,15)$	45
$(16,0)$	16
$(12,7.5)$	34.5

The optimal point is $(0,15)$, so the complete solution is:

$$x = 0$$
$$y = 15$$
$$P = 45\ (maximized)$$
$$S_1 = 0$$
$$S_2 = 120$$

4. Solve the minimization problem (see exercise 8 from section 3.1):

Minimize $C = 20x + 12y$
Subject to
$$5x + 2y \geq 30$$
$$5x + 7y \geq 70$$
$$x \geq 0, \qquad y \geq 0$$

Solution: The system of inequalities is exactly the same as exercise 8 from section 3.1. We computed the coordinates of the corners of the feasible set in that exercise. Thus, we need only evaluate the objective function at these corners:

(x, y)	$C = 20x + 12y$
$(0,15)$	180
$(2.8, 8)$	152
$(14,0)$	280

Thus, the optimal point is (2.8, 8). The complete solution:

$$x = 2.8 = \frac{14}{5}$$
$$y = 8$$
$$C = 152\ (minimized)$$
$$S_1 = 0$$
$$S_2 = 0$$

5. a) Solve the maximization problem (see exercise 10 from section 3.1):

Maximize $z = 500x + 440y$
Subject to
$$4x + 7y \leq 120$$

$$20x + 25y \leq 500$$
$$x + y \geq 12$$
$$y \geq 6$$
$$x \geq 0, \qquad y \geq 0$$

Solution: The system of inequalities is exactly the same as exercise 10 from section 3.1. We computed the coordinates of the corners of the feasible set in that exercise. Thus, we need only evaluate the objective function at these corners:

(x, y)	$z = 500x + 440y$
$(0,12)$	$5{,}280$
$\left(0, \frac{120}{7}\right) \approx (0, 17.14)$	$7{,}542.86$
$(12.5, 10)$	$10{,}650$
$(17.5, 6)$	$11{,}390$
$(6,6)$	$5{,}640$

The optimal point is $(17.5,6)$, so the complete solution is:

$$x = 17.6$$
$$y = 8$$
$$z = 11{,}390 \ (maximized)$$
$$S_1 = 8$$
$$S_2 = 0$$
$$S_3 = 11.5$$
$$S_4 = 0$$

b) Minimize z, subject to the same constraints.

Solution: The feasible set and corner points are unchanged, so we merely pick out the minimum value of z instead of the maximum in the table from part (a). The solution is:

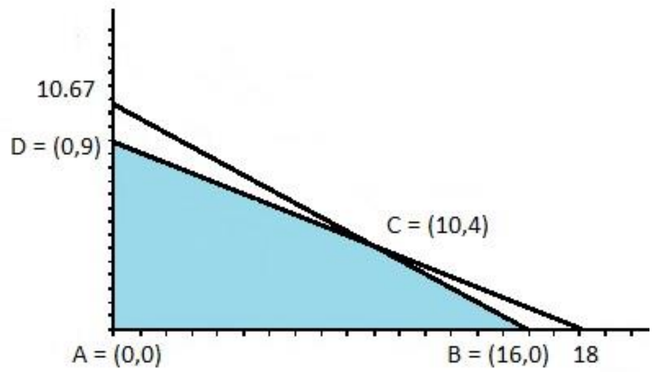
$$x = 0$$
$$y = 12$$
$$z = 5{,}280 \ (minimized)$$
$$S_1 = 36$$
$$S_2 = 200$$
$$S_3 = 0$$
$$S_4 = 0$$

6. Solve the problem which was set up in exercise 11 from section 3.1, which we reproduce here for convenience: Bruce Jax is responsible for constructing end tables and kitchen tables for the *White Room Woodshop*. Each end table uses 2 square yards of $\frac{3}{4}$ inch oak boards and takes two hours to complete. Each kitchen table uses 4 square yards of oak boards and takes 3 hours to complete. This week he has available 36 square yards of oak boards and 32 hours of time. Other resources are unlimited. How many of each item should he make if he is paid $70 for each end table and $100 for each kitchen table? Be sure to include the complete data set for the solution.

Solution: Recall the set-up from exercise 11, section 3.1. Let x be the number of end tables, and y the number of kitchen tables.

$$\text{Maximize Pay } P = 70x + 100y$$
$$\text{Subject to}$$
$$2x + 4y \leq 36 \ \ (square\ yards\ wood)$$
$$2x + 3y \leq 32\ (hours)$$
$$x \geq 0, \qquad y \geq 0$$

The feasible set:



We evaluate the objective function at the corner points:

(x, y)	$P = 70x + 100y$
$A \quad (0,0)$	0
$B \quad (16,0)$	1120
$C \quad (10,4)$	1100
$D \quad (0,9)$	900

Thus, the solution is:

$$x = 16\ end\ tables$$
$$y = 0\ kitchen\ tables$$
$$P = \$1120\ (maximized)$$
$$S_1 = 4\ leftover\ square\ yards\ wood$$
$$S_2 = 0\ leftover\ hours\ time$$

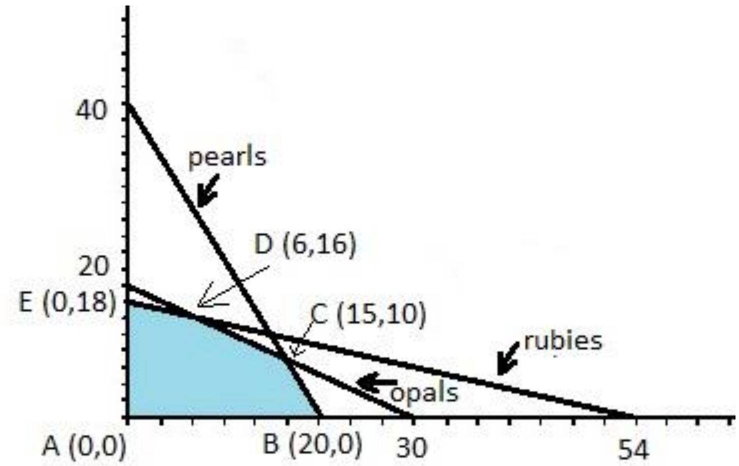
7. Solve the following problem (see exercise 12, section 3.1). Joni is putting her designing skills to work at the *Court and Sparkle Jewelry Emporium*. She makes two signature design bracelet models. Each Dawntreader bracelet uses 1 ruby, 6 pearls, and 10 opals. Each Hejira bracelet uses 3 rubies, 3 pearls, and 15 opals. She has 54 rubies, 120 pearls, and 300 opals to work with. If either model results in a profit of $1,800 for Joni, how many of each type should she make in order to maximize her profit? Be sure to include the complete data set for the solution.

Solution: From exercise 12, section 3.1, the set-up is this: Let x be the number of Dawntreader bracelets, and y the number of Hejira bracelets.

$$\text{Maximize Profit } P = 1800x + 1800y$$
$$\text{Subject to:}$$

$$x + 3y \leq 54 \ (rubies)$$
$$6x + 3y \leq 120 \ (pearls)$$
$$10x + 15y \leq 300 \ (opals)$$
$$x \geq 0, \qquad y \geq 0$$

The feasible set:



We evaluate the objective function at the corner points:

(x, y)	$P = 1800x + 1800y$
(0,0)	0
(0,18)	\$32,400
(6,16)	\$39,600
(15,10)	\$45,000
(20,0)	\$36,000

The optimal solution is:

$$x = 15 \ Dawntreader\ bracelets$$
$$y = 10 \ Hejira\ bracelets$$
$$P = \$45{,}000 \ (maximized)$$
$$S_1 = 9 \ leftover\ rubies$$
$$S_2 = 0 \ leftover\ pearls$$
$$S_3 = 0 \ leftover\ opals$$

8. Solve the following problem (see exercise 12, section 3.1). *Toys in the Attic, Inc*. operates two workshops to build toys for needy children. Mr. Tyler's shop can produce 36 *Angel* dolls, 16 *Kings and Queens* board games, and 16 *Back in the Saddle* rocking horses each day it operates. Mr. Perry's shop can produce 10 *Angel* dolls, 10 *Kings and Queens* board games, and 20 *Back in the Saddle* rocking horses each day it operates. It costs \$144 to operate Mr. Tyler's shop for one day and \$166 to operate Mr. Perry's shop for one day. Suppose the company receives an order from *Kids Dream On* Charity Foundation for at least 720 *Angel* dolls, at least 520 *Kings and Queens* board games, and at least 640 *Back in the Saddle* rocking horses. How many days should they operate each shop in order to fill the order at least possible cost?

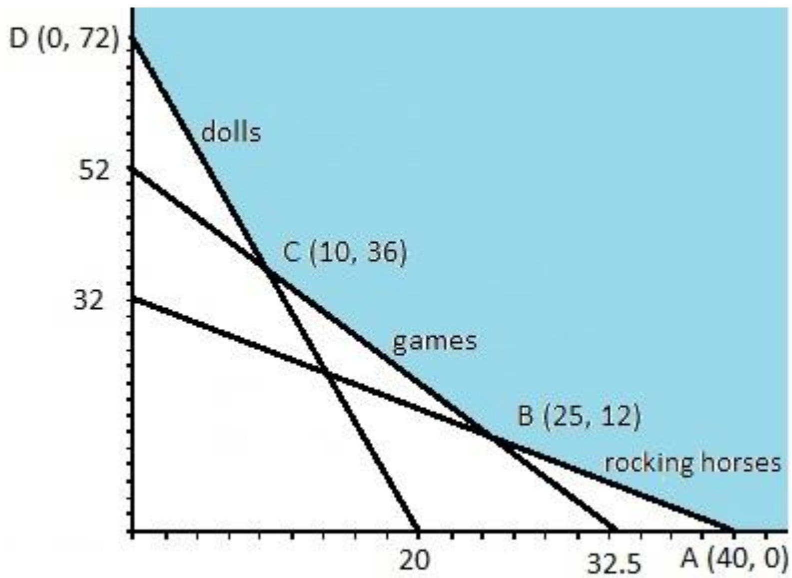
Solution: From exercise 13, section 3.1, we have the set-up: Let x be the number of days to operate Mr. Tyler's shop and y the number of days to operate Mr. Perry's shop.

$$Minimize\ Cost\ C = 144x + 166y$$

Subject to:

$$36x + 10y \geq 720\ (Angel\ dolls)$$
$$16x + 10y \geq 520\ (Kings\ and\ Queens\ board\ games)$$
$$16x + 20y \geq 640\ (Back\ in\ the\ Saddle\ rocking\ horses)$$
$$x \geq 0 \quad y \geq 0$$

The feasible set:



Evaluating the objective function at the corner points:

(x, y)	$C = 144x + 166y$
(0,72)	\$11,952
(10,36)	\$7,416
(25,12)	\$5,592
(40,0)	\$5,760

Thus, the optimal solution is

$$x = 25\ days\ for\ Mr. Tyler's\ shop$$
$$y = 12\ days\ for\ Mr. Perry's\ shop$$
$$C = \$5,592\ (minimized)$$
$$S_1 = 300\ surplus\ Angel\ Dolls$$
$$S_2 = 0\ surplus\ Kings\ and\ Queens\ Board\ Games$$
$$S_3 = 0\ surplus\ Back\ in\ the\ Saddle\ rocking\ horses$$

9. Solve the following diet mix problem. *Kerry's Kennel* is mixing two commercial brands of dog food for its canine guests. A bag of *Dog's Life Canine Cuisine* contains 3 lbs. fat, 2 lbs. carbohydrates, 5 lbs. protein, and 3 oz. vitamin C. A giant size bag of *Way of Life Healthy Mix* contains 1 lb. fat, 5 lbs. carbohydrates, 10 lbs. protein, and 7 oz. vitamin C. The requirements for a weeks' supply of food for the kennel are that there should be at most 21 lbs. fat, at mist 40 lbs. carbohydrates, and at least 21 oz. vitamin C. How many bags of each type of food should be mixed in order to design a diet that maximizes protein?

Music homage to British progressive rock group Gentle Giant. Kerry Minnear was the keyboard player for the band (he also played cello and vibraphone.) 'Dogs' Life' is a song rom their 1972 album Octopus. 'Way of Life' is a song from their 1973 album In a Glass House. (Website: https://gentlegiantmusic.com/GG/Gentle_Giant)

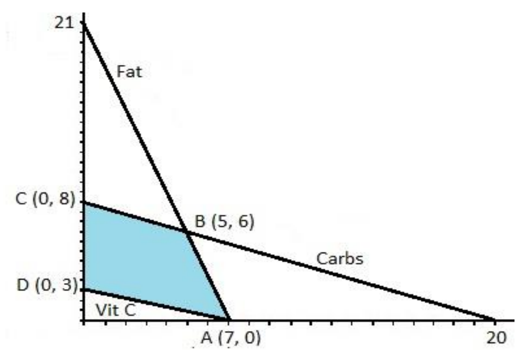
Solution: Let x be the number of bags of Dog's Life Canine Cuisine, and y the number of bags of Way of Life Healthy Mix.

$$\text{Maximize Protein} \quad P = 5x + 10y$$

Subject to:

$$3x + y \leq 21\ lbs.fat$$
$$2x + 5y \leq 40\ lbs.carbohydrates$$
$$3x + 7y \geq 21\ oz.Vitamin\ C$$
$$x \geq 0, \qquad y \geq 0$$

The feasible set:



Evaluating the objective function at the corners:

(x, y)	Protein $P = 5x + 10y$
$A\ (7,0)$	35 lbs.
$B\ (5,6)$	85 lbs.
$C\ (0,8)$	80 lbs.
$D\ (0,3)$	30 lbs.

The optimal solution is:

$$x = 5\ bags\ A\ Dog's\ Life\ Canine\ Cuisine$$
$$y = 6\ bags\ Way\ of\ Life\ Healthy\ Mix$$
$$P = 85\ lbs.protein\ (mximized)$$
$$S_1 = 0\ lbs.fat$$
$$S_2 = 0\ lbs.carbohydrates$$
$$S_3 = 36\ oz.surplus\ Vitamin\ C$$

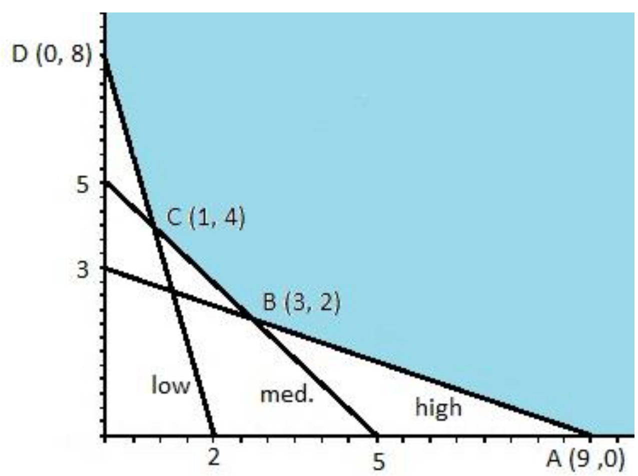
10. Solve the following scheduling problem. The *Poseidon's Wake* petroleum company operates two refineries. The *Cadence Refinery* can produce 40 units of low grade oil, 10 units medium grade oil, and 10 units high grade oil in a single day. (Each unit is 1000 barrels.) The *Cascade Refinery* can produce 10 units low grade oil, 10 units medium grade oil, and 30 units high grade oil in a single day. They receive an order from the Mars Triangle Oil Retailers for at least 80 units low grade oil, at least 50 units medium grade oil, and at least 90 units high grade oil. If it costs Poseidon's Wake $1,800 to operate the Cadence refinery for a day, and $2,000 to operate the Cascade Refinery for a day, how many days should they operate each refinery to fill the order at least cost?

Music homage to influential British progressive rock band King Crimson. In the Wake of Poseidon was the title of their second album, released in 1970. It contained songs entitled 'Cadence and Cascade' and 'Devil's Triangle' (the latter based on the classical composition 'Mars' from Holst's The Planets.) (Website: https://en.wikipedia.org/wiki/King_Crimson)

Solution: Let x be the number of days to operate the Cadence refinery, and y the number of days to operate the Cascade refinery. Then

$$\text{Minimize Cost } C = 1800x + 2000y$$
$$\text{Subject to:}$$
$$40x + 10y \geq 80 \ (units\ low\ grade\ oil)$$
$$10x + 10y \geq 50 \ (units\ medium\ grade\ oil)$$
$$10x + 30y \geq 90 \ (units\ high\ grade\ oil)$$
$$x \geq 0, \qquad y \geq 0$$

The feasible set:



Evaluating the objective function at the corners:

(x, y)	$C = 1800x + 2000y$
$A\ (9,0)$	\$16,200
$B\ (3,2)$	\$9,400
$C\ (1,4)$	\$9,800
$D\ (0,8)$	\$16,000

The optimal solution is at point B:

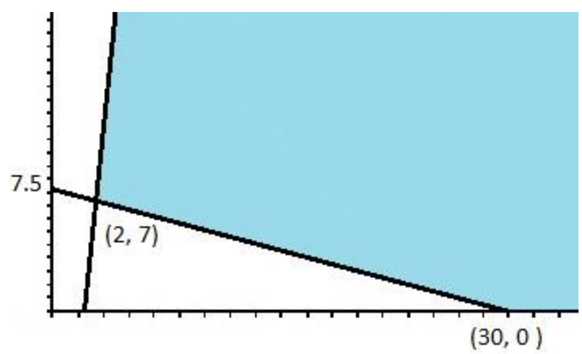
$$x = 3 \text{ days to run Cadence Refinery}$$
$$y = 2 \text{ days to run Cascade Refinery}$$
$$C = \$9{,}400 \text{ (minimized)}$$
$$S_1 = 60 \text{ units surplus low grade oil}$$
$$S_2 = 0 \text{ units surplus medium grade oil}$$
$$S_3 = 0 \text{ units surplus high grade oil}$$

11. Consider the following linear programming problem:

$$\text{Minimize } R = 10x - 3y$$
$$\text{Subject to:}$$
$$x + 4y \geq 30$$
$$27x - 3y \geq 33$$
$$x \geq 0, \qquad y \geq 0$$

Show that the feasible region is unbounded, and that R has neither a maximum nor a minimum ove this region.

Solution: The feasible region:

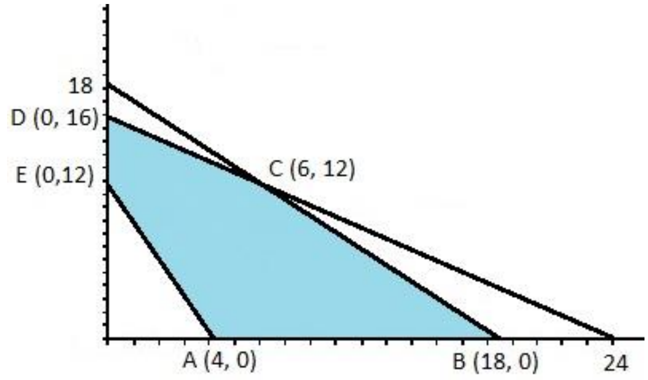


The region is unbounded. Indeed, all points of the form $(x, 0)$ where $x \geq 30$ are feasible, and the objective function has value $R(x, 0) = 10x$ at such a point. Clearly, x can be chosen arbitrarily large, so R has no maximum. Also, consider all points on the line $27x - 3y = 33$ with $x \geq 2$, which are also feasible points. Solving for y in terms of x, we see all such points have the form $(x, 9x - 11)$. The objective function has value $R(x, 9x - 11) = 10x - 3(9x - 11) = 33 - 17x$ at this point. Since $x \geq 2$ can be chosen arbitrarily large, we see R has no minimum (R can be made negative with arbitrarily large absolute value – it approaches minus infinity.)

12 a) Graph the feasible region X given by the constraints:

$$2x + 3y \leq 48$$
$$x + y \leq 18$$
$$3x + y \geq 12$$
$$x \geq 0, \qquad y \geq 0$$

Solution:



b) Find the minimum value of $C = 14x + 21y$ over the region X. List the complete solution for the optimal point.

Solution: Evaluating C at the corners:

(x, y)	$C = 14x + 21y$
$A\ (4,0)$	56
$B(18,0)$	252
$C\ (6,12)$	336
$D\ (0,16)$	336
$E\ (0,12)$	252

The optimal solution is at point A:

$$x = 4$$
$$y = 0$$
$$C = 56\ (minimized)$$
$$S_1 = 40$$
$$S_2 = 14$$
$$S_3 = 0$$

c) Find the maximum value of C over the region X. List the complete optimal data.

The maximum value of C is 336 from the table above. However, it occurs are two distinct corners, C and D. Thus, any point on the segment CD is optimal with $C = 336$. We give the complete optimal data for the two corners only.

At point C:

$$x = 6$$
$$y = 12$$
$$C = 336\ (maximized)$$
$$S_1 = 0$$
$$S_2 = 0$$
$$S_3 = 18$$

At point D:

$$x = 0$$
$$y = 16$$
$$C = 336\ (maximized)$$
$$S_1 = 0$$
$$S_2 = 2$$
$$S_3 = 4$$

d) Suppose you wanted to maximize C over the region X, and you also have a secondary objective (less important than maximizing C) of maximizing $z = x + y$. What point would you consider to be optimal now?

Solution: Since maximizing C is the primary objective, we already know from part (c) that the solution must lie on the line segment CD. In effect, this segment becomes the feasible set for the secondary objective of maximizing z. So we evaluate z at each of its corners, which are the two endpoints C and D.

(x, y)	$z = x + y$
C (6,12)	18
D (0,16)	16

Thus, the optimal point is point C as it maximizes both C and z. The complete date for this point is given in part (c), plus the additional information that $z = 18$ (maximized).

13. Solve the following linear programming problem. The *Jefferson Plastic Fantastic Assembly Corporation* manufactures gadgets and widgets for airplanes and starships. Each case of gadgets uses 2 kg. steel and 5 kg. plastic. Each case of widgets uses 2 kg. steel and 3 kg. plastic. The profit for a case of gadgets is $360 and the profit for a case of widgets is $200. Suppose they have 80 kg. steel available and 150 kg. plastic available on a daily basis, and can sell everything they manufacture. How many cases of each should they manufacture if they are obligated to produce at least 10 cases of widgets per day? The objective is to maximize daily profit.

Music homage to American West coast band Jefferson Airplane (and their later incarnation/spinoff Jefferson Starship). Their 1967 album entitled A Surrealistic Pillow contained the song 'Plastic Fantastic Lover'. (Websites: https://jeffersonairplane.com/ and https://www.jeffersonstarship.com/)

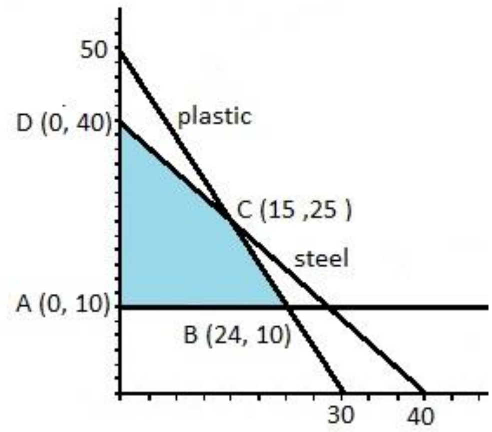
Solution: Let x be the number of cases of gadgets produced per day, and let y be the number of cases of widgets produced per day. Then

Maximize Profit $P = 360x + 200y$

Subject to:

$$2x + 2y \leq 80\ (kg.steel)$$
$$5x + 3y \leq 150\ (kg.plastic)$$
$$y \geq 10\ (contractural\ obligation)$$
$$x \geq 0, \quad y \geq 0$$

The feasible set:



Evaluating the objective function at the corners:

(x, y)	$P = 360x + 200y$
$A\ (0,10)$	\$2,000
$B\ (24,10)$	\$10,640
$C(15,25)$	\$10,400
$D\ (0,40)$	\$8,000

The optimal solution is at point B:

$$x = 24\ cases\ gadgets\ daily$$
$$y = 10\ cases\ widgets\ daily$$
$$P = \$10{,}640\ daily\ profit\ (maximized)$$
$$S_2 = 12\ kg. leftover\ steel$$
$$S_2 = 0\ kg. leftover\ plastic$$
$$S_3 = 0\ surplus\ cases\ widgets$$

14. Mr. Cooder, a farmer in the Purple Valley, has at most 400 acres to devote to two crops: rye and barley. Each acre of rye yields \$100 profit per week, while each acre of barley yields \$80 profit per week. Due to local demand, Mr. Cooder must plant at least 100 acres of barley. The federal government provides a subsidy to grow these crops in the form of tax credits. They credit Mr. Cooder 4 units for each acre of rye and 2 units for each acre of barley. The exact value of a 'unit' of tax credit is not important. Mr. Cooder has decided that he needs at least 600 units of tax credits in order to be able to afford his loan payments on a new harvester. How many acres of each crop should he plant in order to maximize his profit?

Music homage to American guitarist and songwriter Ry Cooder. His 1972 album entitled Into the Purple Valley contains a song '(The Taxes on) the Farmer Feeds Us All.' (Website https://rycooder.com/)

Solution: Let x be the number of acres of rye and y the number of acres of barley. Then

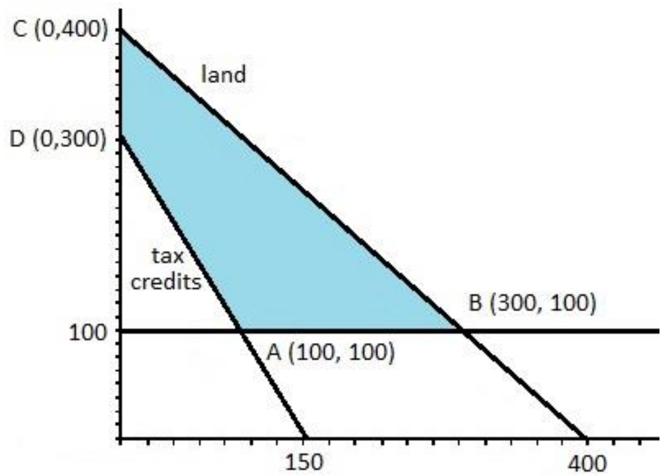
Maximize weekly profit $P = 100x + 80y$

Subject to:

$$x + y \leq 400\ (acres\ land)$$

$$y \geq 100 \ (acres\ barley\ demand)$$
$$4x + 2y \geq 600 \ (unit\ tax\ credit)$$
$$x \geq 0, \qquad y \geq 0$$

The feasible set:



Evaluating the objective function at the corner points:

(x, y)	$P = 100x + 80y$
A (100,100)	\$18,000
B (300,100)	\$38,000
C(0,400)	\$32,000
D (0,300)	\$24,000

The optimal solution is at point B:

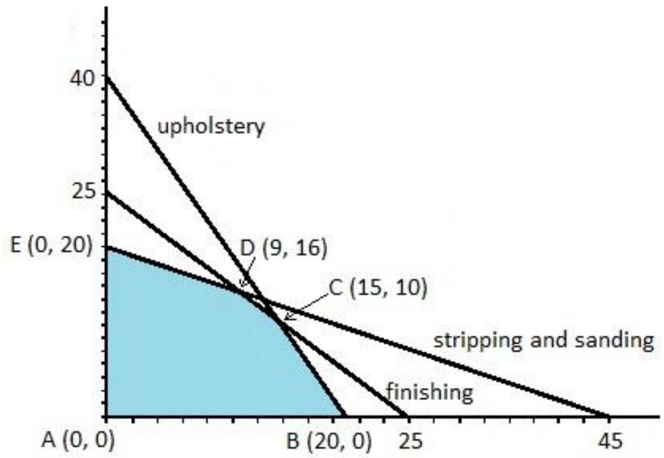
$$x = 300 \ acres\ rye$$
$$y = 100 \ acres\ barley$$
$$P = \$38{,}000 \ (maximized)$$
$$S_1 = 0 \ leftover\ acres\ land$$
$$S_2 = 0 \ surplus\ acres\ barley$$
$$S_3 = 800 \ surplus\ units\ tax\ credit$$

15. *Joanne's Antique Restoration Emporium* refurbishes Victorian furniture. Each dining room set requires 16 hours stripping and sanding time, 4 hours refinishing time, and 4 hours in the upholstery shop. Each bedroom set requires 36 hours stripping and sanding time, 4 hours refinishing time, and 2 hours in the upholstery shop. Each month, they have 720 person-hours in the stripping and sanding shop, 100 person-hours in the refinishing shop, and 80 person-hours in the upholstery shop. Each dining room set generates \$600 profit and each bedroom set generates \$900 profit. How many of each type of furniture set should Joanne accept to work on each month in order to maximize her profit?

Solution: Let x be the number of dining room sets per month, and y the number of bedroom sets per month. Then

$$\text{Maximize Profit } P = 600x + 900y$$
Subject to:
$$16x + 36y \leq 720 \; (hours\ stripping\ and\ sanding)$$
$$4x + 4y \leq 100 \; (hours\ refinishing)$$
$$4x + 2y \leq 80 \; (hours\ for\ upholstery\ shop)$$
$$x \geq 0, \qquad y \geq 0$$

The feasible set:



Evaluating the objective function at the corner points:

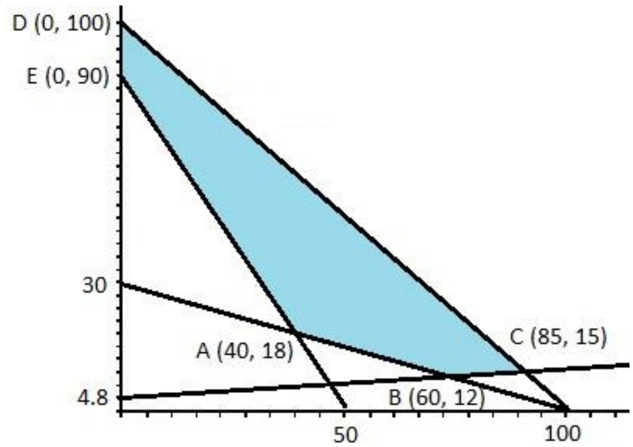
(x, y)	$P = 600x + 900y$
$A\ (0,0)$	\$0
$B\ (20,0)$	\$12,000
$C\ (15,10)$	\$18,000
$D\ (9,16)$	\$19,800
$E\ (0,20)$	\$18,000

The optimal solution is at point D:
$$x = 9\ dining\ room\ sets$$
$$y = 16\ bedroom\ sets$$
$$P = \$19{,}800\ (maximized)$$
$$S_1 = 0\ leftover\ hours\ stripping\ and\ sanding\ time$$
$$S_2 = 0\ leftover\ hours\ refinishing\ time$$
$$S_3 = 12\ leftover\ hours\ in\ the\ upholstery\ shop$$

16. Solve the linear programming problem:

$$\text{Minimize } z = 15x + 51y$$
Subject to:
$$x + y \leq 100$$
$$3x + 10y \geq 300$$
$$18x + 10y \geq 900$$
$$-6x + 50y \geq 240$$
$$x \geq 0, \qquad y \geq 0$$

Solution: The feasible set:



Evaluating the objective function at the corners:

(x, y)	$z = 15x + 51y$
$A\ (40{,}18)$	1,518
$B\ (60{,}12)$	1,512
$C\ (85{,}15)$	2,040
$D\ (0{,}100)$	5,100
$E\ (0{,}90)$	4,590

The optimal solution is at point B:

$$x = 60$$
$$y = 12$$
$$z = 1{,}512\ (minimized)$$
$$S_1 = 28$$
$$S_2 = 0$$
$$S_3 = 300$$
$$S_4 = 0$$

17. As a result of a federal discrimination lawsuit, the town of Yankee, New York is required to build a low-income housing project. The outcome of the lawsuit specifies that Yankee should build enough units to be able to house at least 44 adults and at least 72 children. They must also meet a separate requirement to be able to house at least 120 people altogether. They have available up to 54,000 square feet on which to build. Each townhouse requires 1,800 square feet and can house 6 people (2 adults and 4 children.) Each apartment requires 1,500 square feet and can house 4 people (2 adults and 2 children.) Each townhouse costs $100,000 to build and each apartment costs $80,000 to build. How many of each type of housing unit should they build in order to minimize the total cost?

Solution: Let x be the number of townhouses and y the number of apartments. Then the town must

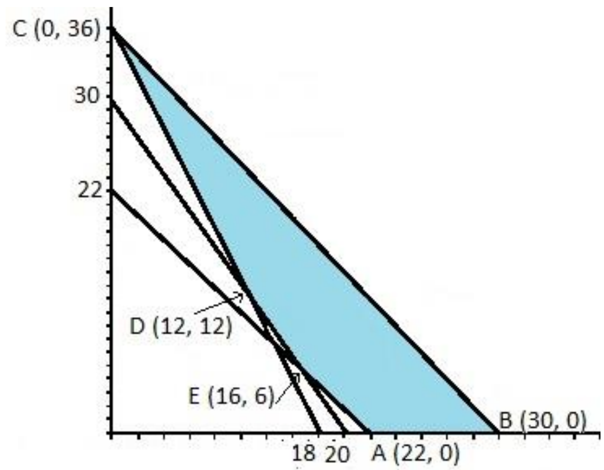
Minimize Cost $C = 100{,}000x + 80{,}000y$

Subject to:

$$2x + 2y \geq 44\ (adults\ housed)$$
$$4x + 2y \geq 72\ (children\ housed)$$

$$6x + 4y \geq 120 \ (people\ housed)$$
$$1800x + 1500y \leq 54{,}000 \ (square\ feet\ of\ land)$$
$$x \geq 0, \qquad y \geq 0$$

Also, for a realistic model, x and y must be integers. The feasible set is:



Evaluating the objective function at the corner points:

(x, y)	$C = 100{,}000x + 80{,}000y$
A (22,0)	\$2,200,000
B (30,0)	\$3,000,000
C (0,36)	2,880,000
D (12,12)	\$2,160,000
E (16,6)	\$2,080,000

The optimal solution is at point E:

$$x = 16 \ townhouses$$
$$y = 6 \ apartments$$
$$C = \$2{,}080{,}000 \ (minimized)$$
$$S_1 = 0 \ surplus\ adults\ housed$$
$$S_2 = 4 \ surplus\ children\ housed$$
$$S_3 = 0 \ surplus\ people\ housed$$
$$S_4 = 16{,}200 \ leftover\ square\ feet\ of\ land$$

18. Solve the linear programming problem:

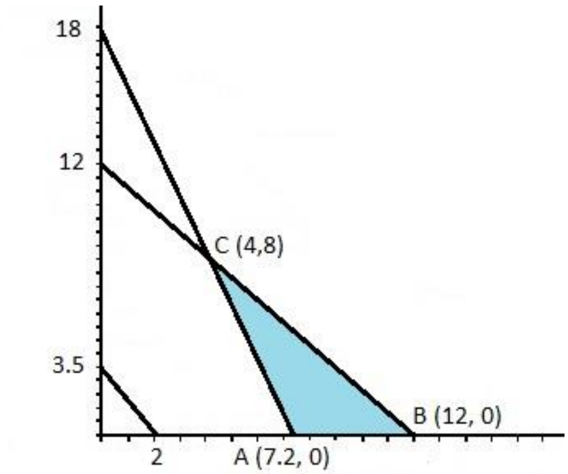
Maximize $z = 3x + 4y$

Subject to:

$$x + y \leq 12$$
$$5x + 2y \geq 36$$
$$7x + 4y \geq 14$$
$$x \geq 0, \qquad y \geq 0$$

Do you notice something unusual about one of the constraints? What is it?

Solution: The feasible set:



Evaluating the objective function at the corner points:

(x, y)	$z = 3x + 4y$
$A\ (7.2, 0)$	21.6
$B\ (12{,}0)$	36
$C\ (4{,}8)$	44

The optimal solution is at point C:

$$x = 4$$
$$y = 8$$
$$z = 44\ (maximized)$$
$$S_1 = 0$$
$$S_2 = 0$$
$$S_3 = 46$$

What is unusual is that the third constraint can be deleted without changing the feasible set. Such a constraint is called *superfluous*.

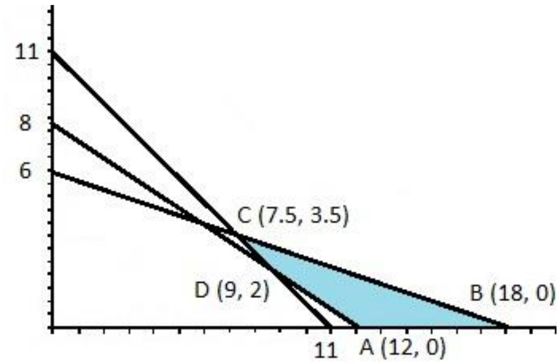
19. Solve the following linear programming problem:

Maximize $z = 5x + 2y$

Subject to:

$$x + y \geq 11$$
$$2x + 3y \geq 24$$
$$x + 3y \leq 18$$
$$x \geq 0, \qquad y \geq 0$$

Solution: The feasible set is:



Evaluating the objective function at the corner points:

(x, y)	$z = 5x + 2y$
$A\ (12,0)$	60
$B\ (18,0)$	90
$C\ (7.5, 3.5)$	44.5
$D\ (9,2)$	49

The optimal solution is at point B:

$$x = 18$$
$$y = 0$$
$$z = 90\ (maximized)$$
$$S_1 = 7$$
$$S_2 = 12$$
$$S_3 = 0$$

20. The *Topographic Starship Tour Company* has been offering extended tours of the dark side of the moon's surface ever since the *Roundabout Interplanetary Mining Company* set up a lune base in the year 2025. They have two types of vehicles – a land rover (called a *Relayer*), and a flying jet shuttle (called a *Khatru*.) Each Relayer can hold 6 first class passengers, 10 economy class passengers, and 600 kg. luggage and supplies. Each Khatru can hold 10 first class passengers, 4 economy class passengers, and 400 kg. luggage and supplies. Each Relayer costs $8,000 to run a two-week tour, and each Khatru costs $11,500 to run a two-week tour. A group of research geologists and potential mining investors want to take a tour.

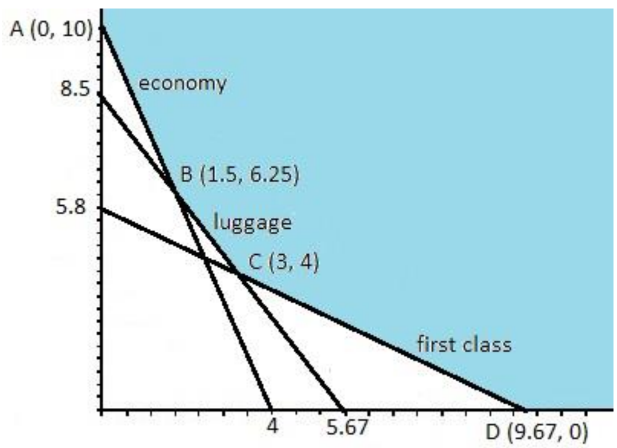
Music homage to the British progressive rock band Yes. They released albums entitled Tales from Topographic Oceans (1973) and Relayer (1974). They also released songs entitled 'Roundabout' (from the album Fragile, 1971), and 'Siberian Khatru' (from the album Close to the Edge, 1972). (Website: http://yesworld.com/)

a) There are 58 first class tourists and 40 economy class tourists in the group, and they have a total of 3400 kg. of luggage and supplies to bring along. How many of each vehicle should they charter in order to minimize the total cost?

Solution: Let x be the number of Relayers and y the number of Khatrus. Then:

$$\text{Minimize Cost } C = 8000x + 11500y$$
$$\text{Subject to:}$$
$$6x + 10y \geq 58\ (first\ class\ tourists)$$
$$10x + 4y \geq 40\ (economy\ class\ tourists)$$
$$600x + 400y \geq 3400\ (kg. luggage\ and\ supplies)$$
$$x \geq 0, \qquad y \geq 0$$

(and $x\ and\ y$ should be integers.) The feasible set:



Evaluating the objective function at the corner points:

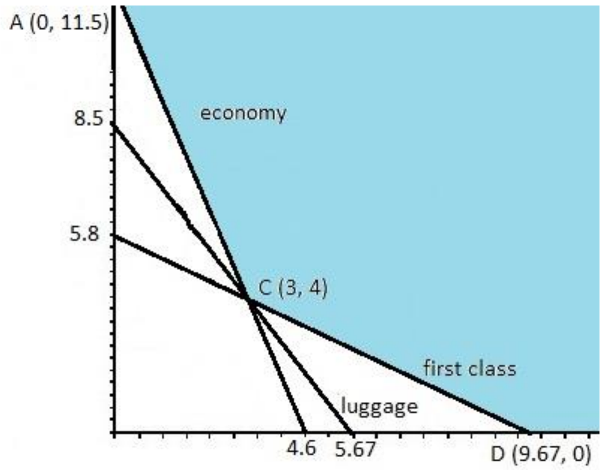
(x, y)	$C = 8000x + 11500y$
$A\ (0,10)$	\$115,000
$B(\ (1.5, 6.25)$	\$83,875
$C\ (3,4)$	\$70,000
$D\ (9.67, 0)$	\$77,333.33

The optimal solution is at point C:

$$x = 3\ Relayers$$
$$y = 4\ Khatrus$$
$$C = \$70{,}000\ (minimized)$$
$$S_1 = 0\ surplus\ first\ class\ tourist\ seats$$
$$S_2 = 6\ (surplus\ economy\ class\ tourist\ seats)$$
$$S_3 = 0; space\ for\ spare\ kg.\ luggage\ and\ supplies$$

b) Suppose there were 46 economy class tourists instead of 40, with no other changes from part (a). Find the optimal solution now. What is different about the feasible set in part (b)? (**Remark**: When this happens, the problem is said to be **degenerate**. A more formal definition is given later in the chapter.)

Solution: The second constraint changes from $10x + 4y \geq 40$ to $10x + 4y \geq 46$. This changes the feasible set to:



Notice that all three constraint lines are concurrent – they all pass through the point $C = (3,4)$. Normally, corner points are determined by just two constraint lines. When a corner is determined by three or more constraint lines, the $(m \times 2)$ problem is said to be degenerate.

Clearly, the cost at (0,11.5) - the new point A – is greater than the cost at the old point A – $(0,10)$, which was already too high to be optimal. The other two corners haven't changed, so their cost is also unchanged. Thus, the optimal point is still $C = (3,4)$. The only other difference is that now $S_2 = 0$ instead of 6, since the second constraint passes through the point C now. The rest of the optimal data is unchanged.

Section 3.5 The Constraint Space and $2 \times n$ Problems.

Solutions to exercises (and more music references):

1. Consider the following vector equation:

$$x\begin{bmatrix}2\\7\end{bmatrix}+y\begin{bmatrix}4\\1\end{bmatrix}+S_1\begin{bmatrix}1\\0\end{bmatrix}+S_2\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}44\\50\end{bmatrix}$$

a) Find the basic solution corresponding to the basic variables x, y.

Solution: Since S_1 and S_2 are nonbasic variables, they will have value 0. Thus the equation reduces to:

$$x\begin{bmatrix}2\\7\end{bmatrix}+y\begin{bmatrix}4\\1\end{bmatrix}=\begin{bmatrix}44\\50\end{bmatrix}$$

In matrix form, this reads

$$\begin{bmatrix}2 & 4\\7 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}44\\50\end{bmatrix},$$

Which has solution

$$\begin{bmatrix}x\\y\end{bmatrix}=\frac{1}{-26}\begin{bmatrix}1 & -4\\-7 & 2\end{bmatrix}\begin{bmatrix}44\\50\end{bmatrix}=\begin{bmatrix}6\\8\end{bmatrix}$$

Thus, the basic solution we seek is $(x, y, S_1, S_2) = (6,8,0,0)$.

We remark that to be a basic solution, the vectors $\begin{bmatrix}2\\7\end{bmatrix}, \begin{bmatrix}4\\1\end{bmatrix}$ must be a basis of $\mathbb{R}^2$. It can be shown that this is equivalent to the matrix $\begin{bmatrix}2 & 4\\7 & 1\end{bmatrix}$ (formed using these vectors as columns) being invertible. Since we did find the inverse of this matrix to solve the above system, this means the columns are indeed a basis.

b) Find the basic solution corresponding to the basic variables x, S_1.

Solution: As in part (a), we set the nonbasic variables equal to zero and solve the resulting system. The final result is $(x, y, S_1, S_2) = \left(\frac{50}{7}, 0, \frac{208}{7}, 0\right)$. As in part (a), the vectors corresponding to x and S_1 are a basis because the matrix with these vectors as columns is invertible. So this is truly a basic solution.

c) Find the basic solution corresponding to the basic variables y, S_1.

Solution: $(x, y, S_1, S_2) = (0,50,-156,0)$.

d) Find the basic solution corresponding to the basic variables S_1, S_2.

Solution: $(x, y, S_1, S_2) = (0,0,44,50)$.

e) If this vector equation described the solution to a standard form maximization linear programming problem, one of the above basic solutions would not be feasible. Which one, and why not?

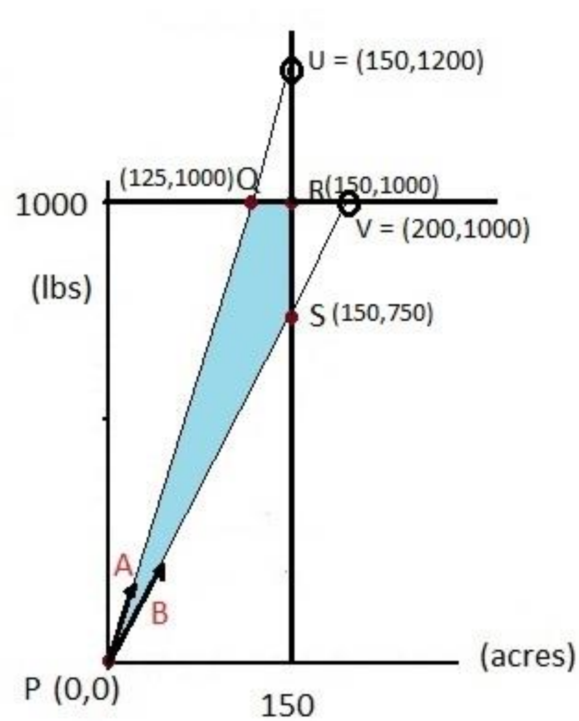
Solution: The basic solution in part (c) is not feasible, because $S_1 < 0$.

2. In the table of basic solutions to the 2×2 farming problem discussed in this section, two of the basic solutions were not feasible. In the graph of this problem in the constraint space, determine exactly where these infeasible points are located. Do the same thing for the graph of this problem in the decision space.

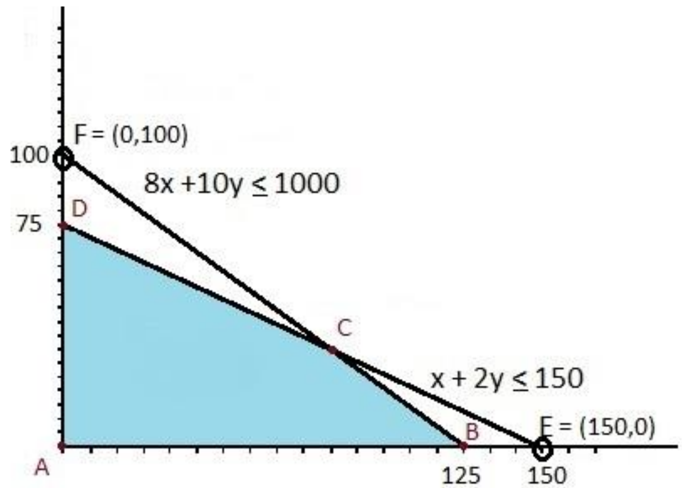
Solution: The two basic solutions in question are:

$$150A + 0B + 0U_1 + (-200)U_2 = v, and$$
$$0A + 100B + (-50)U_1 + 0U_2 = v,$$

where $v = \begin{bmatrix} 150 \\ 1000 \end{bmatrix}$. They are not feasible because they contain negative values for S_2 in the first equation and S_1 in the second. Notice the first equation involves only the decision vector A, so it must be on the ray generated by A in the constraint space. Similarly, the second solution is on the ray generated by B. These solutions are circled in the graph below. They are basic solutions because they are points of intersection of the defining equations, but are not feasible since they are outside of the rectangle. The first solution is the one circled on the vertical line where land is 150 acres (labeled U), and the second one is on the horizontal line where 1000 lbs. fertilizer is used (labeled V).



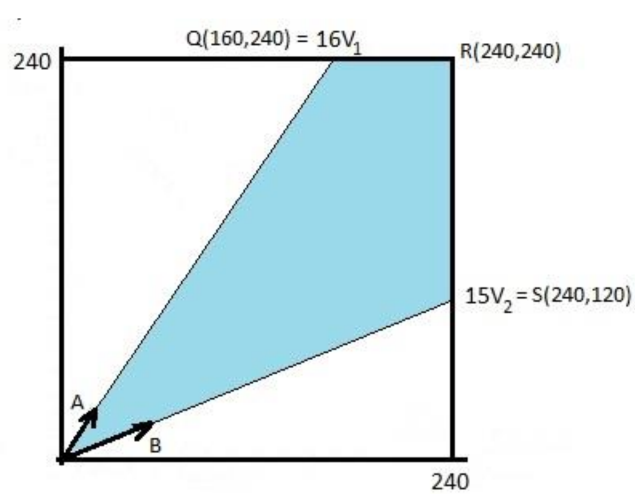
In the decision space, these non-feasible basic solutions are also circled (labeled E and F). Again, they are basic because they appear as the intersection of two of the defining lines but are infeasible since they lie outside the (shaded) feasible set.



3. Solve the following 2×2 problem (see exercise 3 from section 3.4) using the method of graphing in the constraint space:

$$\text{Maximize } P = x + 3y$$
$$\text{Subject to:}$$
$$10x + 16y \leq 240$$
$$15x + 8y \leq 240$$
$$x \geq 0, \qquad y \geq 0$$

Solution: In this problem, the rectangle is a square 240 units on a side, and the two decision vectors are $A = \begin{bmatrix} 10 \\ 15 \end{bmatrix}$ and $B = \begin{bmatrix} 16 \\ 8 \end{bmatrix}$. The region representing the feasible solutions is shaded:



The following table gives the four basic feasible solutions, as well as the value of the objective function at these solutions. For example, to determine the coordinates of Q, note that it lies on the ray determined by A, so x is a basic variable. Also, since the ray meets the rectangle on the top edge, we need to add a multiple of U_1 to obtain R from Q., so S_1 is the other basic variable. We set the nonbasic variables, y and S_2, equal to zero and solve:

$$x\begin{bmatrix}10\\15\end{bmatrix}+0\begin{bmatrix}16\\8\end{bmatrix}+S_1\begin{bmatrix}1\\0\end{bmatrix}+0\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}240\\240\end{bmatrix}$$

Or

$$x\begin{bmatrix}10\\15\end{bmatrix}+S_1\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}240\\240\end{bmatrix}$$

In matrix form:

$$\begin{bmatrix}10 & 1\\15 & 0\end{bmatrix}\begin{bmatrix}x\\S_1\end{bmatrix}=\begin{bmatrix}240\\240\end{bmatrix}$$

$$\begin{bmatrix}x\\S_1\end{bmatrix}=\frac{1}{-15}\begin{bmatrix}0 & -1\\-15 & 10\end{bmatrix}\begin{bmatrix}240\\240\end{bmatrix}=\begin{bmatrix}16\\80\end{bmatrix}$$

Thus. Q corresponds to the basic solution $(x, y, S_1, S_2) = (16,0,80,0)$.
The other rows are found in a similar manner:

Basic feasible solution	(x, y, S_1, S_2)	$P = x + 3y$
P	$(0,0,240,240)$	0
Q	$(16,0,80,0)$	16
R	$(12, 7.5, 0,0)$	34.5
S	$(0,15,0,120)$	45

The optimal solution is at point S:

$$x = 0$$
$$y = 15$$
$$P = 45\ (maximized)$$
$$S_1 = 0$$
$$S_2 = 120$$

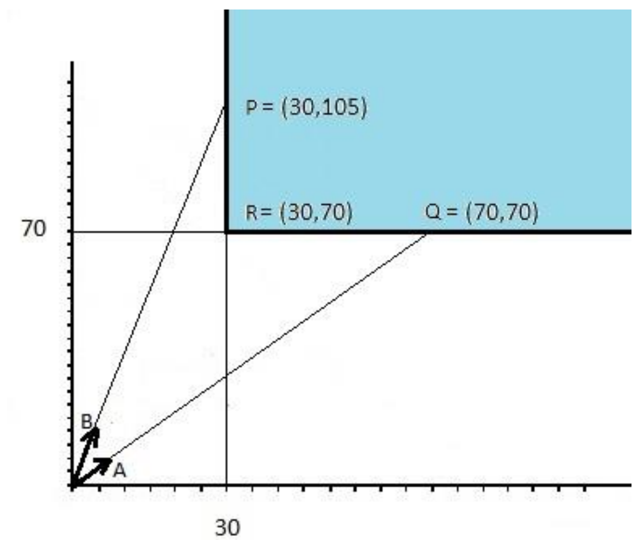
4. Solve the 2×2 problem (see exercise 4 from section 3.4) using the method of graphing in the constraint space. Notice that it is a minimization problem. What adjustments must be made in algorithm 3.4 to handle minimization problems?

Minimize $C = 20x + 12y$
Subject to:
$$5x + 2y \geq 30$$
$$5x + 7y \geq 70$$
$$x \geq 0, \qquad y \geq 0$$

Solution: We adjust the algorithm slightly as follows. The rectangle in the constraint space is no longer bounded, nor does it include the origin because of the greater then type constraints. Nevertheless, the basic solutions are still located where the decision rays cross the edges of the rectangle (or their extensions beyond the rectangle – see diagram below), and the feasible solutions must still lie between the decision rays. Let the decision vectors which determine the decision rays be $A = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$. By the way surplus variables S_1 and S_2 are defined, the basic vector equation has the slightly different form (with negative coefficients corresponding to the surplus variables):

$$xA + yB - S_1U_1 - S_2U_2 = R = \begin{bmatrix} 30 \\ 70 \end{bmatrix}$$

Here is a graph and accompanying table of basic feasible solutions:



Basic feasible point	Basic feasible solution	$C = 20x + 12y$
P	$0A + 15B - 0U_1 - 35U_2 = \begin{bmatrix} 30 \\ 70 \end{bmatrix} = R$	180
Q	$14A + 0B - 40U_1 - 0U_2 = R$	280
R	$2.8A + 8B - 0U_1 - 0U_2 = R$	152

To find the basic feasible solution in the middle column of this table, one must first determine which variables are basic, then solve the appropriate system (setting the nonbasic ones equal to 0), as we did in the previous exercise. For example, for point P, which lies on the ray determined by B and on the vertical line at 30, the basic variables are clearly y and S_2. Thus, we must solve:

$$yB - S_2U_2 = R$$

$$y\begin{bmatrix} 2 \\ 7 \end{bmatrix} - S_2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 30 \\ 70 \end{bmatrix}$$

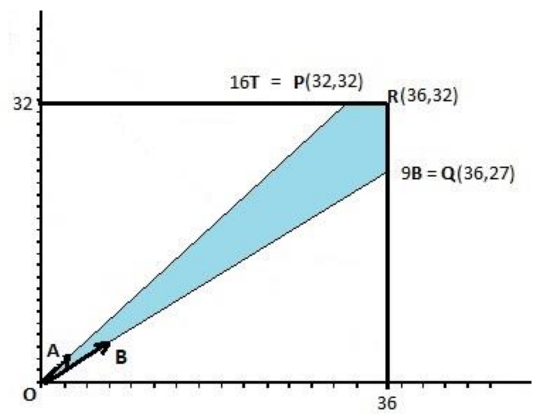
The reader can easily solve this to obtain $y = 15,\ S_2 = 35$.
The other rows are found similarly. For Q, the basic variables are x and S_1, while for R, they are x and y.

Finally, by considering the values of the objective function, we see the optimal solution is at point R:

$$x = \frac{14}{5} = 2.8$$
$$y = 8$$
$$C = 152\ (minimized)$$
$$S_1 = 0$$
$$S_2 = 0$$

5. Solve the following 2×2 problem (see exercise 6 from section 3.4) using the method of graphing in the constraint space. Bruce Jax is responsible for constructing end tables and kitchen tables for the *White Room Woodshop*. Each end table uses 2 square yards of $\frac{3}{4}$ inch oak boards and takes two hours to complete. Each kitchen table uses 4 square yards of oak boards and takes 3 hours to complete. This week he has available 36 square yards of oak boards and 32 hours of time. Other resources are unlimited. How many of each item should he make if he is paid $70 for each end table and $100 for each kitchen table?

Solution: The decision vectors are $A = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ for an end table and $B = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ for a kitchen table. The graph in the constraint space is:



The table of basic feasible solutions:

Point	Basic feasible solution	$P = 70x + 100y$
O	$0A + 0B + 36U_1 + 32U_2 = R = \begin{bmatrix} 36 \\ 32 \end{bmatrix}$	$0
P	$16A + 0B + 4U_1 + 0U_2 = R$	$1120
Q	$0A + 9B + 0U_1 + 5U_2 = R$	$900
R	$10A + 4B + 0U_1 + 0U_2 = R$	$1100

The optimal solution is at point P:

$$x = 16\ end\ tables$$
$$y = 0\ kitchen\ tables$$
$$P = \$1{,}120\ (maximized)$$
$$S_1 = 4\ (leftover\ yards\ wood)$$
$$S_2 = 0\ (leftover\ hours\ time)$$

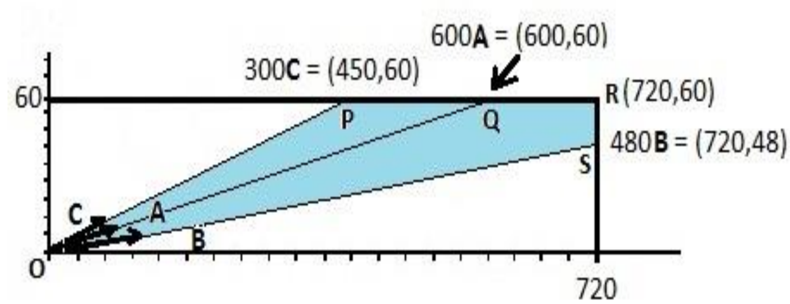
6. The Glass House is a shop the produces three types of specialty drinking glasses. The *Runaway* is a 12 oz. water tumbler, the *Reunion* is a 16 oz. beer glass, and the *Experience* is an elegant stemmed champagne flute glass. A case of Runaway glasses takes 1 hour on the molding machine, 0.1 hour to pack, and generates a profit of $40. A case of Reunion glasses takes 1.5 hours on the molding machine, 0.1 hours to pack, and generates a profit of $60. A case of Experience glasses takes 1.5 hours on the molding machine, 0.2 hours to pack, and generates a profit of$70. Each week, there are 720 hours of time available on the molding machine, and 60 hours available to pack. How many cases of each type should they manufacture in order to maximize profit?

Music Homage to the British progressive rock band Gentle Giant. Their 1973 album was entitled In a Glass House, and included songs entitled 'The Runaway', 'A Reunion', and 'Experience'. (Website: https://gentlegiantmusic.com/GG/Gentle_Giant)

Solution: Let x be the number of cases of Runaway, y the number of cases of Reunion, and z the number of cases of Experience. Then:

$$\text{Maximize Profit } P = 40x + 60y + 70z$$
$$\text{Subject to:}$$
$$x + 1.5y + 1.5z \leq 720 \ (hours\ molding\ machine\ time)$$
$$0.1x + 0{,}1y + 0.2z \leq 60 \ (hours\ packing\ time)$$
$$x \geq 0, \qquad y \geq 0, \qquad z \geq 0$$

The decision vectors are $A = \begin{bmatrix} 1 \\ 0.1 \end{bmatrix}$, $B = \begin{bmatrix} 1.5 \\ 0.1 \end{bmatrix}$, and $\begin{bmatrix} 1.5 \\ 0.2 \end{bmatrix}$.
The relevant graph in the Constraint space is:



The ray through A meets the constraint rectangle at point Q, the ray through B at point S, and the ray through C at point P. R is the corner of the rectangle given by $\begin{bmatrix} 720 \\ 60 \end{bmatrix}$. The fundamental vector equation is then:

$$xA + yB + zC + S_1U_1 + S_2U_2 = R$$

In the table below, we find the basic feasible solutions to this equation. Recall that in order for the basic solution to be feasible, the basis must be chosen so that the vector R lies between the basis vectors. So we must choose one vector from the three vectors A, C, and U_2 'above' R and one from the two B, U_1 'below' R. That leads to six possible choices, so there are six basic feasible solutions:

Point	Basic Vectors	Basic feasible solution	$P = 40x + 60y + 70z$
O	U_1, U_2	$0A + 0B + 0C + 720U_1 + 60U_2 = R$	\$0
P	B, U_2	$0A + 480B + 0C + 0U_1 + 12U_2 = R$	\$28,800
Q	U_1, A	$600A + 0B + 0C + 120U_1 + 0U_2 = R$	\$24,000
R	B, A	$360A + 240B + 0C + 0U_1 + 0U_2 = R$	\$28,800
S	U_1, C	$0A + 0B + 300C + 270U_1 + 0U_2 = R$	\$21,000
R	B, C	$0A + 360B + 120C + 0U_1 + 0U_2 = R$	\$30,000

Observe that two rows of this table correspond to the same point R in the graph. This is because in both of these solutions, both U_1 and U_2 are nonbasic vectors, whence S_1 *and* S_2 are both 0, which means no leftover molding time or packing time – this is what the point R entails. (Recall that, for a $2 \times n$ problem, with $n > 2$, we observed in the text that the same point in the constraint space corresponds to a multiplicity of points in the decision space – the mapping between them fails to be one-to-one.) Thus. the optimal solution is the last row:

$$x = 0 \text{ cases Runaway glasses}$$
$$y = 360 \text{ cases Reunion glasses}$$
$$z = 120 \text{ cases Experience glasses}$$
$$P = \$30{,}000 \text{ (maximized)}$$
$$S_1 = 0 \text{ leftover hours molding time}$$
$$S_2 = 0 \text{ leftover hours packing time}$$

7. The Spooky Boogie Costume Salon makes and sells four different Halloween costumes: the witch, the ghost, the goblin, and the werewolf. Each witch costume uses 3 yards material and takes 2 hours to sew. Each ghost costume uses 2 yards of material and takes 1 hour to sew. Each goblin costume uses 2 yards of material and takes 3 hours to sew. Each werewolf costume uses 2 yards of material and takes 4 hours to sew. The profits for each costume are as follows: \$10 for the witch, \$8 for the ghost, \$12 for the goblin, and \$16 for the werewolf. If they have 600 yards of material and 510 sewing hours available before the holiday, how many of each costume should they make in order to maximize profit, assuming they can sell everything they make?

Music homage to Gentle Giant continues. 'Spooky Boogie' is a song from their 1978 album Giant for a Day (Website: https://gentlegiantmusic.com/GG/Gentle_Giant)

Solution: Let w be the number of witch costumes, x the number of ghost costumes, y the number of goblin costumes, and z the number of werewolf costumes. Then

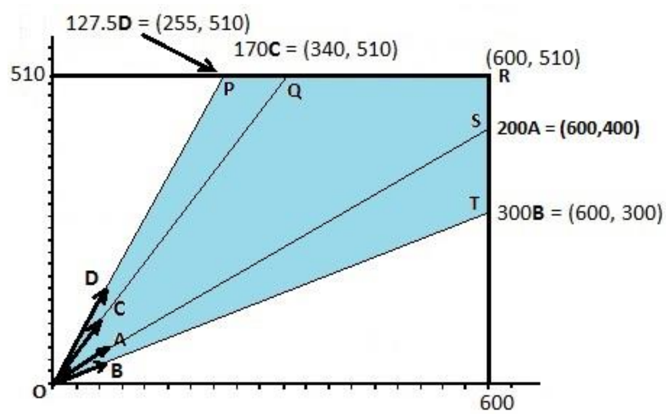
Maximize Profit $P = 10w + 8x + 12y + 16z$

Subject to:

$$3w + 2x + 2y + 2z \leq 600 \text{ (yards marerial)}$$
$$2w + x + 3y + 4z \leq 510 \text{ (hours for sewing)}$$
$$w \geq 0, \quad x \geq 0, \quad y \geq 0, \quad z \geq 0$$

(Also, for a realistic model, w, x, y, z should be integers.)

We now have four decision vectors $A = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ (witch), $B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ (ghost), $C = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ (goblin), and $D = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ (werewolf). Let $R = \begin{bmatrix} 600 \\ 510 \end{bmatrix}$, the corner of the constraint rectangle. This leads to the graph:



In this graph, the ray through A meets the edge of the rectangle at the point S, the ray through B at T, The ray through C at Q, and the ray through D at P. The basic vector equation is

$$wA + xB + yC + zD + S_1U_1 + S_2U_2 = R = \begin{bmatrix} 600 \\ 510 \end{bmatrix}$$

This time there are three vectors U_2, C, D 'above' R and three vectors U_1, A, B 'below' R, leading to 9 basic feasible solutions, as indicated in the table below:

Basic vectors	Point	Basic feasible solution (w, x, y, z, S_1, S_2)	Profit $P = 10w + 8x + 12y + 16z$
U_1, U_2	O	$(0, 0, 0, 0, 600, 510)$	$0
U_1, C	Q	$(0, 0, 170, 0, 260, 0)$	$2.040
U_1, D	P	$\left(0,0,0,\frac{255}{2}, 345, 0\right)$	$2,040
A, U_2	S	$(200, 0, 0, 0, 0, 110)$	$2,000
A, C	R	$(156, 0, 66, 0, 0, 0)$	$2,352
A, D	R	$\left(\frac{345}{2}, 0, 0, \frac{165}{4}, 0, 0\right)$	$2,385
B, U_2	T	$(0, 300, 0, 0, 0, 210)$	$2,400
B, C	R	$(0, 195, 105, 0, 0, 0)$	$2,820
B, D	R	$(0, 230, 0, 70, 0, 0)$	$2,960

Again, all the basic solutions with $S_1 = 0 = S_2$ correspond to the point R in the graph, so it appears several times in the table. The optimal solution is the last row:

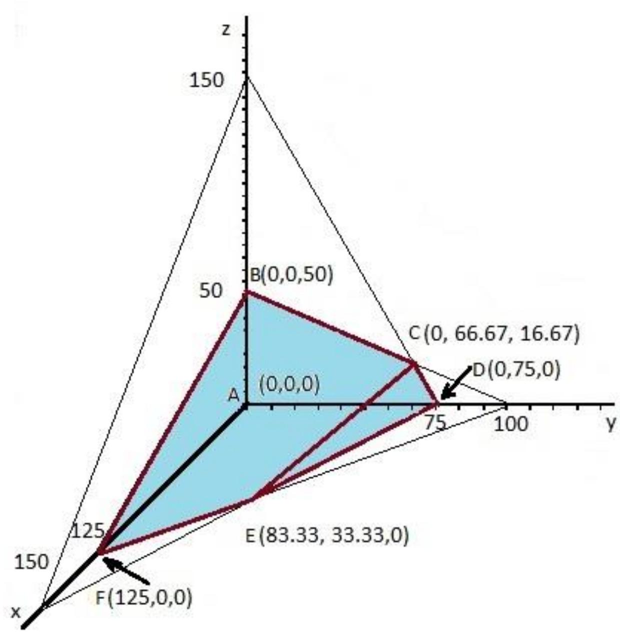
$$w = 0\ witch\ costumes$$
$$x = 230\ ghost\ costumes$$
$$y = 0\ goblin\ costumes$$
$$z = 70\ werewolf\ costumes$$
$$P = \$2,960\ (maximized)$$
$$S_1 = 0\ (leftover\ yards\ mateial)$$
$$S_2 = 0\ (leftover\ hours\ sewing\ time)$$

8. Consider the three-crop farming problem discussed in this section of the text. Solve the problem by graphing in the decision space. (Remark: This problem is for readers who have basic experience making plots in space. Since it is a 2×3 problem, the decision space is three dimensions, and each inequality determines a half-space with a plane as a boundary.)

Solution: Recall the set-up of the problem is: Let x be the number of bags asparagus seed, y the number of bags of bean seed, and z the number of bags of corn seed. Then

$$\text{Maximize } P = 1500x + 2000y + 3700z$$
$$\text{Subject to:}$$
$$x + 2y + z \leq 150 \ (acres\ land)$$
$$8x + 10y + 20z \leq 1000 \ (lbs.fertilizer)$$
$$x \geq 0, \qquad y \geq 0, \qquad z \geq 0$$

Here is a graph of the three-dimensional feasible set in the decision space:



There are six corners of the simplex which is the shaded wedge-shaped region in the first octant of space. One corner is the origin $A = (0,0,0)$. Points B, D, F lie on the coordinate axes, so they are the intercepts of the relevant planes on these axes. Thus, $B = (0,0,50)$ is the z intercept of the fertilizer plane. Similarly, $D = (0,75,0)$ is the y intercept of the land plane and $F = (125,0,0)$ is the x intercept of the fertilizer plane. It remains to find the coordinates of the corners C, E. Notice that C lies in the yz-plane, so its x coordinate is 0. The y, z coordinates come from the point of intersection of the two lines which are where the constraint planes meet the yz coordinate plane, so we solve the system (obtained by setting $x = 0$ in the constraint planes):

$$2y + z = 150$$
$$10y + 20z = 1000$$

The reader can easily find the solution, which is $\left(\frac{200}{3}, \frac{50}{3}\right)$. Thus, the coordinates of C are $\left(0, \frac{200}{3}, \frac{50}{3}\right)$. Similarly, to find the coordinates of E, we set $z = 0$ in the constraint equations, because E lies in the xy plane. Thus, we solve:

$$x + 2y = 150$$
$$8x + 10y = 1000$$

The solution is $\left(\frac{250}{3}, \frac{100}{3}\right)$, so the coordinates of E are $\left(\frac{250}{3}, \frac{100}{3}, 0\right)$.

We can now evaluate the objective function at all the corners:

corner	(x, y, z)	$P = 1500x + 2000y + 3700z$
A	$(0,0,0)$	$\$0$
B	$(0,0,50)$	$\$185{,}000$
C	$\left(0, \frac{200}{3}, \frac{50}{3}\right)$	$\$195{,}000$
D	$(0,75,0)$	$\$150{,}000$
E	$\left(\frac{250}{3}, \frac{100}{3}, 0\right)$	$\$191{,}666.67$
F	$(125,0,0)$	$\$187{,}500$

The optimal solution is at point C:

$$x = 0 \ bags\ asparagus\ seed$$
$$y = \frac{200}{3} = 66.67 \ bags\ bean\ seed$$
$$z = \frac{50}{0} = 16.67 \ bags\ corn\ seed$$
$$P = \$195{,}000 \ (maximized)$$
$$S_1 = 0 \ leftover\ acres\ land$$
$$S_2 = 0 \ leftover\ lbs.fertilizer$$

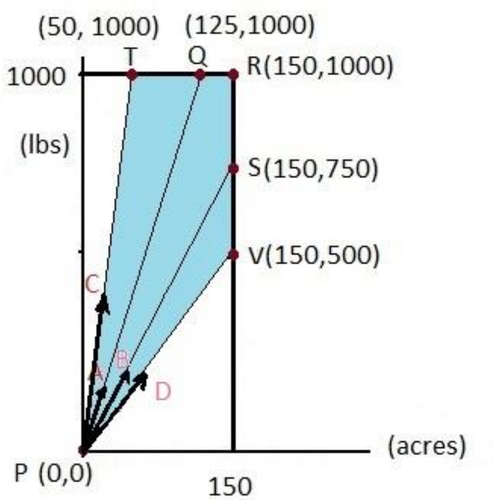
This agrees with the solution we found in the text by graphing in the constraint space.

9. Consider the three-crop farming problem discussed in this section of the text. In addition to asparagus, beans, and corn, now the farmer is considering a fourth possible crop – eggplant. Each bag of eggplant seeds is enough to plant 3 acres and requires 10 lbs. fertilizer. Each bag of eggplant seed will generate $2,800 profit. There are no other changes to the problem – the other three crops still have the same resource requirements and generate the same profit. With the addition of this new crop, re-solve the problem.

Solution: The set-up is: Let x be the number of bags asparagus seed, y the number of bags of bean seed, z the number of bags of corn seed, and w the number of bags of eggplant seed. Then

$$\text{Maximize } P = 1500x + 2000y + 3700z + 2800w$$
$$\text{Subject to:}$$
$$x + 2y + z + 3w \leq 150 \ (acres\ land)$$
$$8x + 10y + 20z + 10w \leq 1000 \ (lbs.fertilizer)$$
$$x \geq 0, \quad y \geq 0, \quad z \geq 0, \quad w \geq 0$$

The problem is now 2×4, so the decision space is four-dimensional. Thus, we eschew the method of graphing in the decision space and use graphing in the constraint space. With this approach, we need add only one new decision vector for eggplant, which we denote by $D = \binom{3}{10}$. The result is:



The ray through the new decision vector D meets the side of the rectangle in a new point V. Observe that in the table of the basic feasible solutions for just the three-crop problem (Table 3.9, page 143), every basic feasible solution corresponds to the solution of the four-crop problem with have $w = 0$, since D is a non-basic vector. Thus, those three rows in table 3.9 will be identical with a row in the table for the four-crop problem (with "$0D$" added to the fundamental vector equation.) In addition, there is a new row corresponding to the new basic feasible solution V, but there will also be some new rows corresponding to the point R. These new rows, in which w is a basic variable in each one) are added to the bottom of the table:

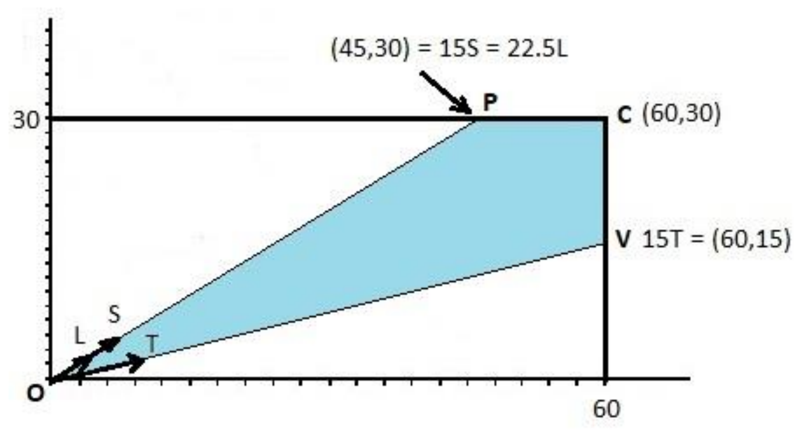
Basic vectors	Point	Basic solution (x, y, z, w, S_1, S_2)	$P = 1500x + 2000y + 3700z + 2800w$
A, B	R	$\left(\frac{250}{3}, \frac{100}{3}, 0,0,0,0\right)$	\$191,667
A, U_1	Q	$(135,0,0,0,25,0)$	\$187,500
B, C	R	$\left(0, \frac{200}{3}, \frac{50}{3}, 0,0,0\right)$	\$195,000
C, U_1	T	$(0,0,50,0,100,0)$	\$185,000
B, U_2	S	$(0,75,0,0,250)$	\$150,000
U_1, U_2	P	$(0,0,0,0,150,1000)$	\$0
D, U_2	V	$(0,0,0,50,0,500)$	\$140,000
D, A	R	$\left(\frac{750}{7}, 0,0, \frac{100}{7}, 0,0\right)$	\$200,714.28
D, C	R	$(0,0,30,40,0,0)$	\$223,000

The optimal solution is in the last row:

$$x = 0 \; bags \; asparagus \; seed$$
$$y = 0 \; bags \; bean \; seed$$
$$z = 30 \; bags \; corn \; seed$$
$$w = 40 \; bags \; eggplant \; seed$$
$$P = \$223{,}000 \; (maximized)$$
$$S_1 = 0 \; leftover \; acres \; land$$
$$S_2 = 0 \; leftover \; lbs. fertilizer$$

10. Consider the following variation on Angie and Kiki's lemonade stand. Suppose Kiki's idea was to include a third type of lemonade instead of a third ingredient? Thus, the resources are just lemons and sugar (no limes), They have 60 lemons and 30 Tbsp. sugar available. In addition to sweet lemonade (3 lemons and 2 Tbsp. sugar per glass) and tart lemonade (4 lemons and 1 Tbsp. sugar per glass), they also make a Lite lemonade (2 lemons and $\frac{4}{3}$ Tbsp. sugar per glass.) The prices are \$1.25 for a glass of sweet, \$1.50 for a glass of tart, and \$1.00 for a glass of Lite. How many glasses of each should they make in order to maximize their revenue? Use the method of graphing in the constraint space,

Solution: Let x be the number of glasses of sweet, y the number of glasses of tart, and z the number of glasses of lite lemonade. Then we have decision vectors $S = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ for sweet lemonade, $T = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ for tart lemonade, and $L = \begin{bmatrix} 2 \\ \frac{4}{3} \end{bmatrix}$ for lite lemonade. Let the capacity vector be $C = \begin{bmatrix} 60 \\ 30 \end{bmatrix}$. Note that S and L are parallel vectors (that is, each is a scalar multiple of the other), so they generate the same ray in the constraint space. The graph in the constraint space is:



Let P be the point where the ray through S meets the top edge of the rectangle, and let V be the point where the ray through T meets the right edge of the rectangle. Here is the table of basic, feasible solutions to the fundamental vector equation

$$xS + yT + zL + S_1U_1 + S_2U_2 = C:$$

Basic vectors	Point	Basic feasible solution (x, y, z, S_1, S_2)	Revenue $R = 1.35x + 1.5y + z$
U_1, U_2	O	$(0,0,0,60,30)$	$\$0$
U_1, S	P	$(15,0,0,15,0)$	$\$18.75$
U_1, L	P	$(0,0,22.5,15,0)$	$\$22.50$
T, U_2	V	$(0,15,0,0,15)$	$\$22.50$
T, S	C	$(12,6,0,0,0)$	$\$24.00$
T, L	C	$(0,6,18,0,0)$	$\$27.00$

Notice that two different basic feasible solutions correspond to the point P (because the vectors S, L are parallel. (In some of the above exercises, we saw that the corner of the rectangle (C in this case) might correspond to multiple basic feasible solutions, but in case the reader was wondering, this exercise shows there is nothing special about the corner of the rectangle in that regard – other points might also correspond to multiple basic feasible solutions.)

The optimal solution is in the last row of the table:

$$x = 0 \; glssses \; sweet \; lemonade$$
$$y = 6 \; glasses \; tart \; lemonade$$
$$z = 18 \; glasses \; lite \; lemonade$$
$$R = \$27.00 \; (maximized)$$
$$S_1 = 0 \; leftover \; lemons$$
$$S_2 = 0 \; leftover \; tbsp. sugar$$

11. *Touch of Grey Art Supplies, Ltd*., produces and sells artists' supplies, including acrylic paint. In any acrylic paint, there are two basic ingredients: pigment (which is what gives the paint its color), and a synthetic resin binder (which forms a base to hold particles of pigment.) Touch of Grey has five different recipes for red paint of various hues, values, intensities. The amounts of pigment and binder in one tube of each of the five colors, as well as the profit generated, is indicated in the table below. There are other ingredients in the paint, such as fillers and dyes, but thee are in unlimited supply and do not affect the ability of the company to make a profit.

	Cadmium Red	Napthol Crimson	King Crimson	Venetian Red	Providence Red
Pigment (g)	50	40	40	40	50
Binder (g)	80	100	60	50	60
Profit ($)	4	3	5	4	2

Suppose the company has available 10 kg. of pigment and 15 kg. binder. The problem is to determine how many tubes of each color they should mix if they want to maximize profit, assuming they sell everything they make.

Music references to the Grateful Dead and King Crimson. 'Touch of Grey' was a 1987 song by the Grateful Dead from their album In the Dark. The 1974 album from King Crimson was entitled Red, and contained an improvised song called 'Providence'.

(Websites: https://en.wikipedia.org/wiki/Touch_of_Grey and https://en.wikipedia.com/wiki/Red_(King_Crimson_album)

a) Why can we conclude, before we even solve the problem, that there will be at most two different colors of paint mixed at the optimal point?

Solution: This is a 2×5 problem. Thus, a basis of the constraint space consists of exactly two (linearly independent) vectors. The basic vectors correspond to the variables with a nonzero coefficient value at the optimal point. Therefore, there are exactly two variables variables with a nonzero coefficient in the basic solution. So, at most two of these variables can be decision variables (they could be slack variables, so it may not be exactly two…) Thus, at most two colors are mixed.

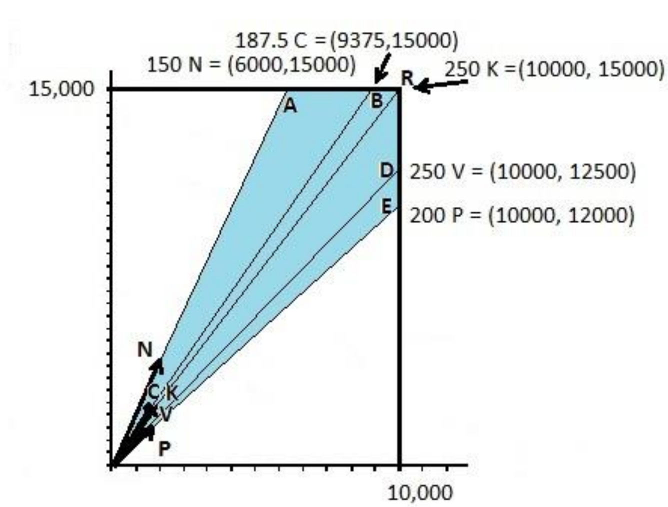
b) Solve the problem by graphing in the constraint space.

Solution: Let v be the number of tubes of Cadmium Red, w the number of tubes of Napthol Crimson, x the number of tubes of King Crimson, y the number of tubes of Venetian Red, and z the number of tubes of Providence Red. Then (converting kilograms to grams):

$$\text{Maximize Profit } P = 4v + 3w + 5x + 4y + 2z$$
$$\text{Subject to:}$$
$$50v + 40w + 40x + 40y + 50z \leq 10{,}000\ (g.pigment)$$
$$80v + 100w + 60x + 50y + 60z \leq 15{,}000\ \text{(g. binder)}$$
$$v \geq 0, \quad w \geq 0, \quad x \geq 0, \quad y \geq 0, \quad z \geq 0$$

The five decision vectors (one for each color) and the Capacity vector R are:

$$C = \begin{bmatrix} 50 \\ 80 \end{bmatrix}, \quad N = \begin{bmatrix} 40 \\ 100 \end{bmatrix}, \quad K = \begin{bmatrix} 40 \\ 60 \end{bmatrix}, \quad V = \begin{bmatrix} 40 \\ 50 \end{bmatrix}, \quad P = \begin{bmatrix} 50 \\ 60 \end{bmatrix}, \quad R = \begin{bmatrix} 10{,}000 \\ 15{,}000 \end{bmatrix}.$$



The graph in the constraint space is above. The points A, B, R, D, E are where the rays through N, C, K, V, P meet the edges of the rectangle. Note that the ray through K passes exactly through the corner R of the rectangle. That means the problem is degenerate – mixing only King Crimson entails running out of both resources at the same time. It also means that when selecting basic vectors, the vector K can be considered to be either "above" R or "below" R. This leads to fifteen possible basic feasible solutions to the fundamental vector equation $vC + wN + xK + yV + zP = R$, as indicated in the following table (but because of the degeneracy, not all fifteen are distinct):

Basic Vectors	Point	Basic feasible solution $(v, w, x, y, z, S_1, S_2)$	$P = 4v + 3w + 5x + 4y + 2z$
U_1, U_2	O	$(0,0,0,0,0,10000, 15000)$	$\$0$
U_1, N	A	$(0,150,0,0,0, 4000, 0)$	$\$450$
U_1, C	B	$(187.5, 0,0,0,0, 625)$	$\$750$
U_1, K	R	$(0,0, 250, 0,0,0,0)$	$\$1{,}250$
P, U_2	E	$(0,0,0,0, 200, 0, 3000)$	$\$400$
P, N	R	$\left(0, \frac{750}{13}, 0,0, \frac{2000}{13}, 0,0\right)$	$\$480.77$
P, C	R	$(150, 0,0,0, 50, 0,0)$	$\$700$
P, K	R	$(0,0, 250, 0,0,0,0)$	$\$1{,}250$
V, U_2	D	$(0,0,0\ 250, 0\ 0\ , 2500)$	$\$1{,}000$
V, N	R	$(0, 50, 0, 200, 0,0,0)$	$\$950$
V, C	R	$\left(\frac{1000}{7}, 0,0, \frac{500}{7}, 0,0,0\right)$	$\$857.14$
V, K	R	$(0,0, 250, 0,0,0,0)$	$\$1{,}250$
K, U_2	R	$(0,0, 250, 0,0,0,0)$	$\$1{,}250$
K, N	R	$(0,0, 250, 0,0,0,0)$	$\$1{,}250$
K, C	R	$(0,0, 250, 0,0,0,0)$	$\$1{,}250$

The optimal solution is:

$$v = 0\ tubes\ Cadmium\ Red$$
$$w = 0\ tubes\ Napthol\ Red$$
$$x = 250\ tubes\ King\ Crimson$$
$$y = 0\ tubes\ Venetian\ Red$$
$$z = 0\ tubes\ Providence\ Red$$
$$P = \$1{,}250\ (maximized)$$
$$S_1 = 0\ leftover\ g.pigment$$
$$S_2 = 0\ leftover\ g.binder$$

c) In real life, the company would probably mix some of each color instead of just one or two. Speculate on how our model might be modified to make it more realistic.

Solution: As noted in part (a), at most two colors are mixed because there are only two constraints. Adding constraints could increase the number of different colors mixed in an optimal solution. A simple way to add constraints is to take them from consumer demand. For example, if the company can always sell at least 20 tubes of Cadmium Red, and at least 30 tubes of Providence Red, we simply annex the constraints $v \geq 20$ and $z \geq 30$. Now, with four constraints, it becomes a 4×5 problem, and a basic feasible solution would then allow for up to 4 different basic decision variables with nonzero coefficients (including v and z, of course!), and so up to four different colors mixed in an optimal solution.

To make sure they mix some of each color, they might instead simply add a blanket constraint to make at least 10 tubes of each color (thereby adding five constraints to the problem.)

Of course, as soon as there are more than 3 constraints (and more than two decision variables), we cannot solve it by either graphing technique in this chapter. We need more powerful methods (developed in Chapter 5) to handle problems of such dimensions.

Chapter 4. Sensitivity Analysis and Duality

Section 4.1 Introduction to Sensitivity Analysis

No exercises or new music references in this section.

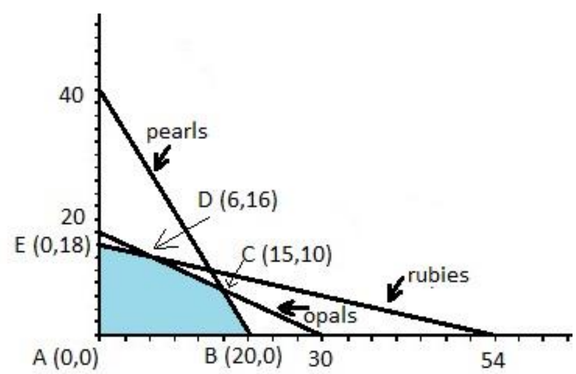
Section 4.2 Changes in the Objective Coefficients

Solutions to Exercises. The exercises below appeared previously in the exercises to section 3.4, where they were solved by graphing in the decision space.

1. Joni is putting her designing skills to work at the *Court and Sparkle Jewelry Emporium*. She makes two signature design bracelet models. Each Dawntreader bracelet uses 1 ruby, 6 pearls, and 10 opals. Each Hejira bracelet uses 3 rubies, 3 pearls, and 15 opals. She has 54 rubies, 120 pearls, and 300 opals to work with. If either model results in a profit of $1,800 for Joni, how many of each type should she make in order to maximize her profit?

a) Suppose the profit for a Dawntreader bracelet increases to $2,400. Find the new optimal data.

Solution: The feasible set does not change, so it is only necessary to evaluate the new objective function at the corner points. For convenience we recall the solution to the original problem: x is the number of Dawntreader bracelets, and y is the number of Hejira bracelets. The feasible set is:



Evaluating the new objective function at the corners:

(x, y)	$P = 2400x + 1800y$
A (0,0)	$0
B (20,0)	$48,000
C (15,10)	$54,000
D (6,16)	$43,200
E (0,18)	$32,400

Thus, the optimal corner remains the same as in the original problem:

$(x, y) = (15,10)$ with 9 leftover rubies and no other leftover gems
Maximum possible profit is $\$54{,}000$

b) Find the stable range for the profit of a Dawntrader bracelet.

Solution: Let b be the profit for a Dawntreader bracelet, so that the objective function is $P = bx + 1800y$. The slopes of the level objective lines are then $-\frac{b}{1800}$. The slopes of the constraint lines are $-\frac{2}{3}$ for opals, and -2 for pearls. Since the optimal point is the intersection of these two lines, the stable range would be all values of b satisfying the inequalities (making the slope of the level objective lines between these two values):

$$-2 \leq -\frac{b}{1800} \leq -\frac{2}{3}$$

$$-3600 \leq -b \leq -1200$$

$$1200 \leq b \leq 3600$$

Thus, the stable range for profit for a Dawntreader bracelet is $1200 \leq b \leq 3600$. (Note that 2400 is within that range, explaining why in part (a) the optimal corner did not change.

c) Find the stable range for the profit of a Hejira bracelet.

Solution: Let b be the profit for a Hejira bracelet, so that the objective function is $P = 1800x + by$. The slopes of the level objective lines are then $-\frac{1800}{b}$. As in part (b), we solve the inequalities:

$$-2 \leq -\frac{1800}{b} \leq -\frac{2}{3}$$

$$\frac{2}{3} \leq \frac{1800}{b} \leq 2$$

$$\frac{1}{2} \leq \frac{b}{1800} \leq \frac{3}{2}$$

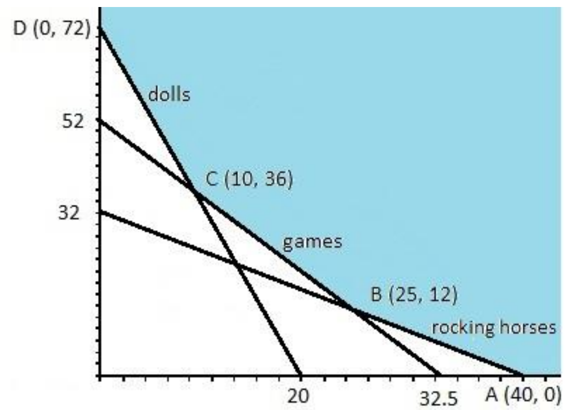
$$900 \leq b \leq 2700$$

Thus, the stable range for profit of a Hejira bracelet is $900 \leq b \leq 2700$.

2. *Toys in the Attic, Inc*. operates two workshops to build toys for needy children. Mr. Tyler's shop can produce 36 *Angel* dolls, 16 *Kings and Queens* board games, and 16 *Back in the Saddle* rocking horses each day it operates. Mr. Perry's shop can produce 10 *Angel* dolls, 10 *Kings and Queens* board games, and 20 *Back in the Saddle* rocking horses each day it operates. It costs \$144 to operate Mr. Tyler's shop for one day and \$166 to operate Mr. Perry's shop for one day. Suppose the company receives an order from *Kids Dream On* Charity Foundation for at least 720 *Angel* dolls, at least 520 *Kings and Queens* board games, and at least 640 *Back in the Saddle* rocking horses. How many days should they operate each shop in order to fill the order at least possible cost?

a) Suppose Mr. Perry can streamline operation sin his shop so that the daily cost comes down to \$120 per day. Find the new optimal data.

Solution: Again, we evaluate the new objective function at the old corner points. For convenience, we copy the feasible set graph from the solution to the original problem:



Evaluating the new objective function:

(x, y)	$C = 144x + 120y$
$A\ (40,0)$	\$5,760
$B\ (25,12)$	\$5,040
$C\ (10,36)$	\$5,760
$D\ (0,72)$	\$8,640

Thus, the optimal corner does not change. $(x, y) = (25,12)$ with 300 surplus Angel dolls. Minimum cost is \$5,040.

b) Find the stable range of the cost of daily operations for both of the shops.

Solution: The binding constraints are the ones for Back in the Saddle rocking horses and Kings and Queens board games. Their slopes are $-\frac{4}{5}$ and $-\frac{8}{5}$, respectively. Thus, if the cost of operating Mr. Tyler's shop for a day is c, then the objective function is $C = cx + 166y$, with slope $-\frac{c}{166}$. To find the stable range for this cost, we must solve

$$-\frac{8}{5} \leq -\frac{c}{166} \leq -\frac{4}{5}$$

$$\frac{4}{5} \leq \frac{c}{166} \leq \frac{8}{5}$$

$$\frac{664}{5} \leq c \leq \frac{1328}{5}.$$

Thus, the stable range is $\frac{664}{5} \leq c \leq \frac{1328}{5}$, or $\$132.80 \leq c \leq \$265.60$.

Similarly, if c is the cost of running Mr. Perry's shop for a day, the objective function becomes $C = 144x + cy$, with slope $-\frac{144}{c}$. So we must solve:

$$-\frac{8}{5} \leq -\frac{144}{c} \leq -\frac{4}{5}$$

$$\frac{4}{5} \leq \frac{144}{c} \leq \frac{8}{5}$$

$$\frac{5}{8} \leq \frac{c}{144} \leq \frac{5}{4}$$

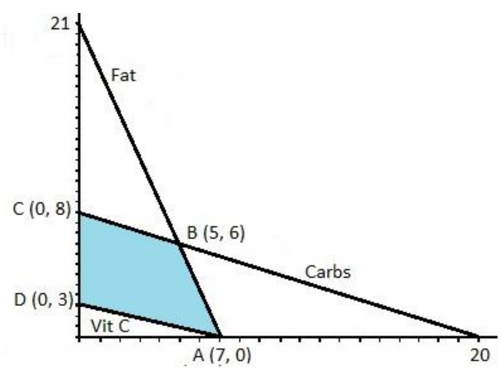
$$90 \leq c \leq 180$$

Thus, the stable range is $90 \leq c \leq 180$. Note that 120 is within this range, which explains why the optimal corner did not change in part (a).

3. *Kerry's Kennel* is mixing two commercial brands of dog food for its canine guests. A bag of *Dog's Life Canine Cuisine* contains 3 lbs. fat, 2 lbs. carbohydrates, 5 lbs. protein, and 3 oz. vitamin C. A giant size bag of *Way of Life Healthy Mix* contains 1 lb. fat, 5 lbs. carbohydrates, 10 lbs. protein, and 7 oz. vitamin C. The requirements for a weeks' supply of food for the kennel are that there should be at most 21 lbs. fat, at mist 40 lbs. carbohydrates, and at least 21 oz. vitamin C. How many bags of each type of food should be mixed in order to design a diet that maximizes protein?

a) Suppose the recipe for Dog's Life Cane Cuisine changes so that the protein content of each bag increases to 12 lbs. Find the new optimal data.

Solution: Here is the feasible set we obtained in the original solution to exercise 9, Section 3.4:



We evaluate the new objective function at the corners:

(x, y)	$P = 12x + 10y$
A (7,0)	84
B (5,6)	120
C (0,8)	80
D (0,3)	30

The optimal corner has not changed: $(x, y) = (5,6)$, with $36\ oz$. Surplus vitamin C. The total protein is 120 lbs. (maximized)

b) Find the stable range for the protein in a bag of Dog's Life Canine Cuisine.

Solution: Let the amount of protein in a bag be p pounds. Then the objective function is $P = px + 12y$, with slope $-\frac{p}{10}$. The slopes of the binding constraints are -3 for the fat constraint and $-\frac{2}{5}$ for the carbohydrate constraint. Thus, we must solve:

$$-3 \leq -\frac{p}{10} \leq -\frac{2}{5}$$

$$\frac{2}{5} \leq \frac{p}{10} \leq 3$$

$$4 \leq p \leq 30$$

Note that 12 is in this range, explaining why the optimal corner did not change in part (a).

c) Find the stable range for the protein in a bag of Way of Life Healthy Mix.

Solution: Let the amount of protein in a bag be p pounds. Then the objective function is $P = 5x + py$, with slope $-\frac{5}{p}$. Thus we solve the inequalities coming from the slopes of the binding constraints, which are the same as in part (b):

$$-3 \leq -\frac{5}{p} \leq -\frac{2}{5}$$

$$\frac{2}{5} \leq \frac{5}{p} \leq 3$$

$$\frac{1}{3} \leq \frac{p}{5} \leq \frac{5}{2}$$

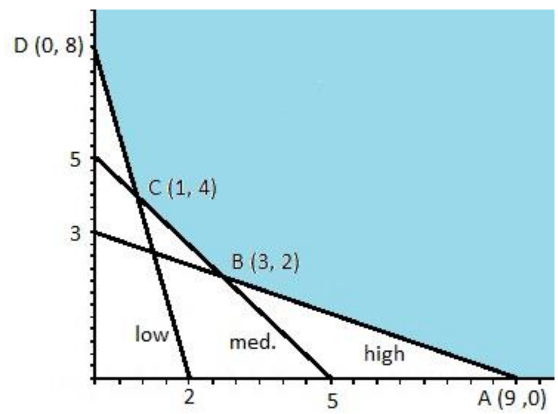
$$\frac{5}{3} \leq p \leq \frac{25}{2}$$

Thus, the stable range is $\frac{5}{3} \leq p \leq \frac{25}{2}$, or $1.667 \leq p \leq 12.5$ lbs. protein.

4. The *Poseidon's Wake* petroleum company operates two refineries. The *Cadence Refinery* can produce 40 units of low grade oil, 10 units medium grade oil, and 10 units high grade oil in a single day. (Each unit is 1000 barrels.) The *Cascade Refinery* can produce 10 units low grade oil, 10 units medium grade oil, and 30 units high grade oil in a single day. They receive an order from the Mars Triangle Oil Retailers for at least 80 units low grade oil, at least 50 units medium grade oil, and at least 90 units high grade oil. If it costs Poseidon's Wake \$1,800 to operate the Cadence refinery for a day, and \$2,000 to operate the Cascade Refinery for a day, how many days should they operate each refinery to fill the order at least cost?

a) Suppose that the Cadence refinery daily cost can be reduced to \$1200. Find the new optimal data.

Solution: Here is the feasible set from exercise 10, Section 3.4:



Evaluating the new objective function at the corners:

(x, y)	$C = 1200x + 2000y$
$A\ (9,0)$	$10,800
$B(3,2)$	$7,600
$C\ (1,4)$	$9,200
$D\ (0,8)$	$16,000

The optimal corner did not change. The solution is $(x, y) = (3,2)$ with 60 units surplus low grade oil. The minimum cost is $7,600.

b) Find the stable range for the daily cost of operating each refinery.

Solution: First consider the Cadence refinery, and let the daily cost of operating it be c. The objective function is then $C = cx + 2000y$, with slope $-\frac{c}{2000}$. The binding constraints have slopes -1 for the medium grade oil constraint, and $-\frac{1}{3}$ for the high grade oil constraint. Then

$$-1 \le -\frac{c}{2000} \le -\frac{1}{3}$$

$$\frac{1}{3} \le \frac{c}{2000} \le 1$$

$$\frac{2000}{3} \le c \le 2000$$

Thus, the stable range for the daily cost of operating the Cadence refinery is $\$666.67 \le c \le \$2000$. Note that 1200 is in this range, explaining why the optimal corner did not change in part (a).

Now let c be the daily cost of operating the Cascade refinery. The objective function is $C = 1800x + cy$, with slope $-\frac{1800}{c}$. Thus

$$-1 \le -\frac{1800}{c} \le -\frac{1}{3}$$

$$1 \le \frac{c}{1800} \le 3$$

$$1800 \le c \le 5400.$$

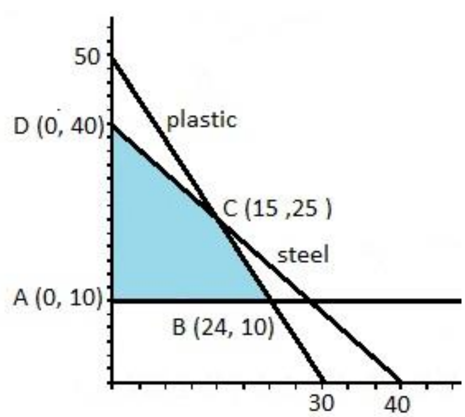
c) Suppose that both operating costs are cut in half. Explain in a short sentence or two why this has no effect on which corner is optimal.

Solution: If both operating costs are cut in half, the slopes of the level objective lines do not change from the original problem (because these slopes are determined by the ratios of the operating costs, so if both costs are cut in half they still have the same ratio.) Therefore, the same corner will be optimal (but of course the total cost of operating the refineries at that corner would be cut in half as well.)

5. The *Jefferson Plastic Fantastic Assembly Corporation* manufactures gadgets and widgets for airplanes and starships. Each case of gadgets uses 2 kg. steel and 5 kg. plastic. Each case of widgets uses 2 kg. steel and 3 kg. plastic. The profit for a case of gadgets is $360 and the profit for a case of widgets is $200. Suppose they have 80 kg. steel available and 150 kg. plastic available on a daily basis, and can sell everything they manufacture. How many cases of each should they manufacture if they are obligated to produce at least 10 cases of widgets per day? The objective is to maximize daily profit.

Find the stable range for the profit of a case of gadgets and for the profit of a case of widgets.

Solution: Here is the feasible set from the solution to exercise 13, Section 3.4:



The optimal point was point B at $(24,10)$. Observe that means the binding constraints are the plastic constraint and the contractual obligation to manufacture at least 10 cases widgets. The slopes of these constraints are $-\frac{5}{3}$ (the plastic constraint) and 0 (the contractual constraint). Thinking about the level objective lines, it is clear that in order for the point B to remain optimal, all that is required is that these level objective lines be steeper than the plastic line (i.e., have a more negative slope.) If we let the profit for a case of gadgets be denoted p, then the objective function is $P = px + 200y$, with slope $-\frac{p}{200}$. Thus:

$$-\frac{p}{200} \leq -\frac{5}{3}$$

$$\frac{5}{3} \leq \frac{p}{200}$$

$$\frac{1000}{3} \leq p$$

Thus, the stable range for the profit for a case of gadgets is the interval $\left(\frac{1000}{3}, \infty\right)$.

Now suppose the profit for a case of widgets is denoted p. Then the objective function becomes: $C = 360x + py$, with slope $-\frac{360}{p}$. Again, we need only satisfy the condition that the level lines be steeper than the plastic line, so:

$$-\frac{360}{p} \leq -\frac{5}{3}$$

$$\frac{5}{3} \leq \frac{360}{p}$$

$$\frac{p}{360} \leq \frac{3}{5}$$

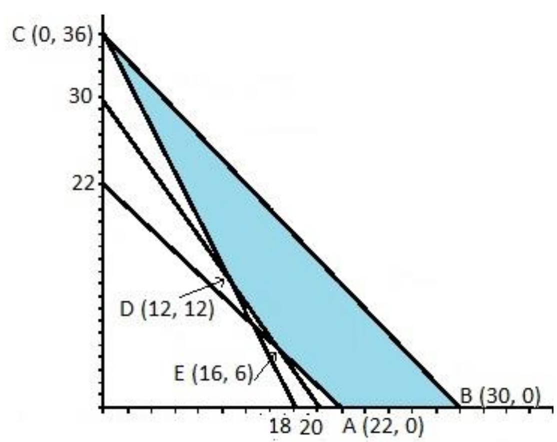
$$p \leq 216$$

Thus, the stable range is $(-\infty, 216]$ in interval notation.

6. As a result of a federal discrimination lawsuit, the town of Yankee, New York is required to build a low-income housing project. The outcome of the lawsuit specifies that Yankee should build enough units to be able to house at least 44 adults and at least 72 children. They must also meet a separate requirement to be able to house at least 120 people altogether. They have available up to 54,000 square feet on which to build. Each townhouse requires 1,800 square feet and can house 6 people (2 adults and 4 children.) Each apartment requires 1,500 square feet and can house 4 people (2 adults and 2 children.) Each townhouse costs $100,000 to build and each apartment costs $80,000 to build. How many of each type of housing unit should they build in order to minimize the total cost?

Find the stable range of the cost of building a townhouse and for the cost of building an apartment.

Solution: The feasible set below is copied from the solution to exercise 17, Section 3.4, where x is the number of townhouses and y is the number of apartments:



We saw that the optimal point is E located at $(16,6)$. The binding constraints for this point are the requirement to house at least 44 children, with slope -1, and the requirement for housing at least 120

people altogether, with slope $-\frac{3}{2}$. Denote the cost for a townhouse by c. The objective function is then $C = cx + 80000y$, with slope $-\frac{c}{80000}$. Thus

$$-\frac{3}{2} \leq -\frac{c}{80000} \leq -1$$

$$1 \leq \frac{c}{80000} \leq \frac{3}{2}$$

$$80000 \leq c \leq 120000$$

So, the stable range for the cost of a townhouse is $[80000,\ 120000]$ in interval notation.

Now denote the cost for an apartment by c. This time the objective function becomes $C = 100000x + cy$, with slope $-\frac{100000}{c}$. Thus

$$-\frac{3}{2} \leq -\frac{100000}{c} \leq -1$$

$$\frac{2}{3} \leq \frac{c}{100000} \leq 1$$

$$\frac{200000}{3} \leq c \leq 100000$$

To the nearest penny, $\frac{200000}{3}$ is \$66,666.67, so the stable range for the cost of building an apartment is $[66{,}666.67, 100{,}000]$ in interval notation.

Section 4.3 Marginal Values associated with Constraints

Solutions to exercises: The exercises below appeared previously in the exercises to section 3.4, where they were solved by graphing in the decision space.

1. Joni is putting her designing skills to work at the *Court and Sparkle Jewelry Emporium*. She makes two signature design bracelet models. Each Dawntreader bracelet uses 1 ruby, 6 pearls, and 10 opals. Each Hejira bracelet uses 3 rubies, 3 pearls, and 15 opals. She has 54 rubies, 120 pearls, and 300 opals to work with. If either model results in a profit of $1,800 for Joni, how many of each type should she make in order to maximize her profit?

a) Explain why the marginal value of a ruby is 0.

Solution: In the optimal solution (Exercise 12, Section 3.4), we have $S_1 = 9$ leftover rubies. Therefore, obtaining additional rubies does no lead to an increase in profit – just to an increase in leftover rubies. Alternately, we have noted that any constraint which is non-binding must have a marginal value of 0 by complementary slackness.

b) Find the marginal value of a pearl by increasing the pearl supply to 121, finding the coordinates of the modified optimal point, and computing the net increase in profit.

Solution: The binding constraints are the pearl and opal constraints, so the solution is the intersection of those lines:

$$6x + 3y = 120$$
$$10x + 15y = 300$$

We replace the 120 pearls with 121 and solve the resulting system:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{60}\begin{bmatrix} 15 & -3 \\ -10 & 6 \end{bmatrix}\begin{bmatrix} 121 \\ 300 \end{bmatrix} = \begin{bmatrix} \frac{61}{4} \\ \frac{59}{6} \end{bmatrix}$$

The profit at this point is

$$P = 1800\left(\frac{61}{4}\right) + 1800\left(\frac{59}{6}\right) = 45{,}150$$

Since the original profit with 120 pearls was $45,000, we see the marginal value of a pearl is $150.

c) Find the optimal value of an opal using the same approach as in part (b). Increase the opal supply by 1, find the coordinates of the modified optimal point, and compute the net increase in profit.

Solution: We replace 300 opals by 301 in the above system, and solve:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{60}\begin{bmatrix} 15 & -3 \\ -10 & 6 \end{bmatrix}\begin{bmatrix} 120 \\ 301 \end{bmatrix} = \begin{bmatrix} \frac{299}{20} \\ \frac{101}{10} \end{bmatrix}$$

The profit at this point is

$$P = 1800\left(\frac{299}{20}\right) + 1800\left(\frac{101}{10}\right) = 45090$$

Thus, the marginal value of an opal is $\Delta P = 45090 - 45000 = \$90$.

d). Instead, find the marginal values of both pearls and opals simultaneously using Proposition 4.4.

Solution: Using the proposition, we compute:

$$FA^{-1} = [1800 \quad 1800]\left(\frac{1}{60}\begin{bmatrix} 15 & -3 \\ -10 & 6 \end{bmatrix}\right) = [150 \quad 90],$$

which agrees with our answers in parts (b) and (c).

2. *Kerry's Kennel* is mixing two commercial brands of dog food for its canine guests. A bag of *Dog's Life Canine Cuisine* contains 3 lbs. fat, 2 lbs. carbohydrates, 5 lbs. protein, and 3 oz. vitamin C. A giant size bag of *Way of Life Healthy Mix* contains 1 lb. fat, 5 lbs. carbohydrates, 10 lbs. protein, and 7 oz. vitamin C. The requirements for a weeks' supply of food for the kennel are that there should be at most 21 lbs. fat, at mist 40 lbs. carbohydrates, and at least 21 oz. vitamin C. How many bags of each type of food should be mixed in order to design a diet that maximizes protein?

Compute the marginal value associated with each constraint. For each one, interpret exactly what it means using the correct units.

Solution: From the solution to exercise 9, Section 3.4, we know the optimal point is $(x, y) = (5,6)$ at the intersection of the lines for the binding constraints, which are the fat constraint and the carbohydrate constraint. The corresponding system is

$$\begin{aligned} 3x + y &= 21 \\ 2x + 5y &= 40 \end{aligned}$$

Thus, to compute the marginal values of these two constraints, we use Proposition 4.4.

$$FA^{-1} = [5 \quad 10]\left(\frac{1}{13}\begin{bmatrix} 5 & -1 \\ -2 & 3 \end{bmatrix}\right) = \begin{bmatrix} \frac{5}{13} & \frac{25}{13} \end{bmatrix}$$

Thus, the marginal value of fat is $\frac{5}{13}$ lbs. protein/lbs. fat. It means that for each additional pound of fat we allow into the diet, the we can also obtain $\frac{5}{13}$ lbs. protein. Similarly, the marginal value of carbohydrates is $\frac{25}{13}$ lbs. protein/lbs. carbohydrate. IT means for each additional lb. of carbohydrate we allow into the diet, we'll also obtain $\frac{15}{13}$ lb. additional protein. The last constraint was for ounce of vitamin C, but this constraint is not binding, so it's marginal value is 0.

3. The *Jefferson Plastic Fantastic Assembly Corporation* manufactures gadgets and widgets for airplanes and starships. Each case of gadgets uses 2 kg. steel and 5 kg. plastic. Each case of widgets uses 2 kg. steel and 3 kg. plastic. The profit for a case of gadgets is \$360 and the profit for a case of widgets is \$200. Suppose they have 80 kg. steel available and 150 kg. plastic available on a daily basis, and can sell

everything they manufacture. How many cases of each should they manufacture if they are obligated to produce at least 10 cases of widgets per day? The objective is to maximize daily profit.

Compute the marginal value associated with each constraint.

Solution: Recall from exercise 13, Section 3.4, the set-up: x is the number of cases gadgets and y the number of cases widgets.

$$\text{Maximize Profit } P = 360x + 200y$$
$$\text{Subject to:}$$
$$2x + 2y \leq 80\ (kg.steel)$$
$$5x + 3y \leq 150\ (kg.plastic)$$
$$y \geq 10\ (contractual\ obligation)$$
$$x \geq 0, \qquad y \geq 0$$

From the solution we obtained there, the optimal point is $(x, y) = (24,10)$, where we have leftover steel but no leftover plastic. Since $S_1 > 0$ (leftover steel), the marginal value of steel is $M_1 = 0$. The optimal point is the intersection of the lines for the binding constraints:

$$5x + 3y = 150$$
$$y = 10$$

Thus, by Proposition 4.4,

$$[M_2 \quad M_3] = FA^{-1} = [360 \quad 200]\left(\frac{1}{5}\begin{bmatrix}1 & -3\\ 0 & 5\end{bmatrix}\right) = [72 \quad -16]$$

So $M_2 = \$72$/lb. plastic, and $M_3 = -\$16$ /case widgets. As mentioned in the text, the marginal value M_3 is negative because the constraint points the "wrong way" for a maximization problem – it is a 'greater than' type constraint. One interprets this as meaning that for each additional case of widgets we are contractually obligated to produce, there is a corresponding loss of $16 to profit. Alternately, for each case we could lower the constraint, we would gain $16 profit.

4. *Toys in the Attic, Inc*. operates two workshops to build toys for needy children. Mr. Tyler's shop can produce 36 *Angel* dolls, 16 *Kings and Queens* board games, and 16 *Back in the Saddle* rocking horses each day it operates. Mr. Perry's shop can produce 10 *Angel* dolls, 10 *Kings and Queens* board games, and 20 *Back in the Saddle* rocking horses each day it operates. It costs $144 to operate Mr. Tyler's shop for one day and $166 to operate Mr. Perry's shop for one day. Suppose the company receives an order from *Kids Dream On* Charity Foundation for at least 720 *Angel* dolls, at least 520 *Kings and Queens* board games, and at least 640 *Back in the Saddle* rocking horses. How many days should they operate each shop in order to fill the order at least possible cost?

Compute the marginal value associated with each constraint. For each one, interpret exactly what it means using the correct units.

Solution: In exercise 13, Section 3.1 we set this problem up. x is the number of days to operate Mr. Tyler's shop, and y the number of days to operate Mr. Perry's shop. Then

$$\text{Minimize Cost } C = 144x + 166y$$

$$\text{Subject to:}$$
$$36x + 10y \geq 720\ (Angel\ Dolls)$$
$$16x + 10y \geq 520\ (Kings\ and\ Queens\ board\ games)$$
$$16x + 20y \geq 640\ (Back\ in\ the\ Saddle\ Rocking\ Horses)$$
$$x \geq 0, \qquad y \geq 0$$

From the solution to exercise 8, Section 3.4, we know the optimal point is $(x, y) = (25,10)$, and at this point there are surplus Angel Dolls produced, but no surplus rocking horses or board games. Thus, $M_1 = 0$, the marginal value of an Angel Doll, because that constraint is not binding. For the marginal values of the binding constraints, use Proposition 4.4.

$$[M_2 \quad M_3] = FA^{-1} = [144 \quad 166]\left(\frac{1}{160}\begin{bmatrix} 20 & -10 \\ -16 & 16 \end{bmatrix}\right) = \begin{bmatrix} \frac{7}{5} & \frac{38}{5} \end{bmatrix}$$

Thus, the marginal value of a board game is $M_2 = \frac{7}{5} = \$1.40\ per\ game$, and the marginal value of a rocking horse is $M_3 = \frac{38}{5} = \$7.60\ per\ rocking\ horse$. The interpretation is that is we could lower the minimum number of required games by one, we could save \$1.40, and if we could lower the minimum number of required rocking horses by one, we could save \$7.60.

5. The *Poseidon's Wake* petroleum company operates two refineries. The *Cadence Refinery* can produce 40 units of low grade oil, 10 units medium grade oil, and 10 units high grade oil in a single day. (Each unit is 1000 barrels.) The *Cascade Refinery* can produce 10 units low grade oil, 10 units medium grade oil, and 30 units high grade oil in a single day. They receive an order from the Mars Triangle Oil Retailers for at least 80 units low grade oil, at least 50 units medium grade oil, and at least 90 units high grade oil. If it costs Poseidon's Wake \$1,800 to operate the Cadence refinery for a day, and \$2,000 to operate the Cascade Refinery for a day, how many days should they operate each refinery to fill the order at least cost?

Compute the marginal value associated with each constraint.

Solution: Recall the set-up: x is the number of days to operate the Cadence Refinery, and y is the number of days to operate the Cascade Refinery. Then

$$\text{Minimize Cost } C = 1800x + 2000y$$
$$\text{Subject to:}$$
$$40x + 10y \geq 80\ (units\ low\ grade\ oil)$$
$$10x + 10y \geq 50\ (units\ medium\ grade\ oil)$$
$$10x + 30y \geq 90\ (units\ high\ grade\ oil)$$
$$x \geq 0, \qquad y \geq 0$$

In the solution to exercise 10, Section 3.4, we saw that the optimal point was $(x, y) = (3,2)$, and there was surplus low grade oil produced at this point. Thus, the marginal value M_1 of low grade oil is 0. The system corresponding to the optimal corner is determined by the binding constraints:

$$10x + 10y = 50$$
$$10x + 30y = 90$$

Thus, by Proposition 4.4, the marginal values for medium and high grade oil are:

$$[M_2 \quad M_3] = FA^{-1} = [1800 \quad 2000]\left(\frac{1}{200}\begin{bmatrix} 30 & -10 \\ -10 & 10 \end{bmatrix}\right) = [170 \quad 10]$$

Thus, $M_2 = \$170$ per unit medium grade oil, and $M_3 = \$10$ per unit of high grade oil.

6. Mr. Cooder, a farmer in the Purple Valley, has at most 400 acres to devote to two crops: rye and barley. Each acre of rye yields $100 profit per week, while each acre of barley yields $80 profit per week. Due to local demand, Mr. Cooder must plant at least 100 acres of barley. The federal government provides a subsidy to grow these crops in the form of tax credits. They credit Mr. Cooder 4 units for each acre of rye and 2 units for each acre of barley. The exact value of a 'unit' of tax credit is not important. Mr. Cooder has decided that he needs at least 600 units of tax credits in order to be able to afford his loan payments on a new harvester. How many acres of each crop should he plant in order to maximize his profit?

Compute the marginal value associated with each constraint.

Solution: The set-up: x is the number of acres of Rye, and y is the number of acres of barley. Then

Maximize Weekly Profit $P = 100x + 80y$

Subject to:

$$x + y \leq 400 \textit{ acres land}$$
$$y \geq 100 \textit{ acres barley}$$
$$4x + 2y \geq 600 \textit{ units tax credit}$$
$$x \geq 0, \qquad y \geq 0$$

In the solution to exercise 14, Section 3.4, we saw that the optimal point was $(x, y) = (300,100)$, and at this point, there were surplus tax credits, so $S_3 > 0$. It follows that the marginal value $M_3 = 0$. For the other two marginal values, use Proposition 4.4:

$$[M_1 \quad M_2] = FA^{-1} = [100 \quad 80]\left(\frac{1}{1}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\right) = [100 \quad -20]$$

Thus, $M_1 = \$100 \textit{ per acre}$ (each additional acre would generate an additional $100 profit.) M_2 is negative, because the constraint for barley demand is a greater than type constraint. $M_2 = -\$20$ per acre means that each increase in barley demand of one acre results in a $20 loss in weekly profits.

7. *Joanne's Antique Restoration Emporium* refurbishes Victorian furniture. Each dining room set requires 16 hours stripping and sanding time, 4 hours refinishing time, and 4 hours in the upholstery shop. Each bedroom set requires 36 hours stripping and sanding time, 4 hours refinishing time, and 2 hours in the upholstery shop. Each month, they have 720 person-hours in the stripping and sanding shop, 100 person-hours in the refinishing shop, and 80 person-hours in the upholstery shop. Each dining room set generates $600 profit and each bedroom set generates $900 profit. How many of each type of furniture set should Joanne accept to work on each month in order to maximize her profit?

Compute the marginal value associated with each constraint.

Solution: Here is the set-up: x is the number of dining room sets per month, and y is the number of bedroom sets per month. Then

$$\text{Maximize Profit } P = 600x + 900y$$
$$\text{Subject to:}$$
$$16x + 36y \leq 720 \; hours\ stripping\ and\ sanding\ time$$
$$4x + 4y \leq 10 \; hours\ refinishing\ time$$
$$4x + 2y \leq 80 \; hours\ for\ upholstery\ shop$$
$$x \geq 0, \qquad y \geq 0$$

In the solution to this problem in exercise 15, Section 3.4, we saw that the optimal point is $(x, y) = (9,16)$, and there were leftover hours in the upholstery shop at this corner. Thus, $M_3 = 0$. For the other marginal values, we use Proposition 4.4:

$$[M_1 \quad M_2] = FA^{-1} = [600 \quad 900]\left(\frac{1}{-80}\begin{bmatrix} 4 & -36 \\ -4 & 16 \end{bmatrix}\right) = [15 \quad 90]$$

Thus, $M_1 = \$15$ per hour stripping and sanding time, $M_2 = \$90$ per hour refinishing time.

8. As we saw in Section 4.2, when an objective function coefficient is changes to a value within its stable range, the coordinates of the optimal point do not change, because the feasible set is unchanged. However, because of the modified objective coefficient, the value of the objective function changes at the optimal point. Also, because of the modified coefficient, the marginal values can also change. Consider Angie and Kiki's lemonade stand problem. We know the stable range for the cost of a glass of sweet lemonade is the interval $[1.125, \quad 3]$

a) Suppose the price for a glass of sweet lemonade is raised from \$1.25 to \$1.40. Compute the new optimal revenue. Also, compute the new marginal values for lemons and sugar. Use the matrix multiplication FA^{-1} of Proposition 4.4 at the end of the section.

Solution: The new price \$1.40 is within the stable range, so the optimal corner remains $(x, y) = (12,6)$. The new maximum revenue changes to $1.4(12) + 1.5(6) = \$28.50$. Since we have leftover limes at this corner, the marginal value for limes is $M_2 = 0$. For the others:

$$[M_1 \quad M_3] = FA^{-1} = [1.4 \quad 1.5]\left(\frac{1}{-5}\begin{bmatrix} 1 & -4 \\ -2 & 3 \end{bmatrix}\right) = [0.32 \quad 0.22]$$

Thus, $M_1 = \$0.32$ per lemon and $M_3 = \$0.22$ per tablespoon sugar.

b) Same question as part (a) if the price for a glass of sweet lemonade is lowered from \$1.25 to \$1.15.

Solution: The new price is within the stable range. Therefore, the optimal corner does not change. The maximum revenue is now $1.15(12) + 1.5(6) = \$22.80$. As in part (a), there are leftover limes so $M_2 = 0$. The other marginal values are:

$$[M_1 \quad M_3] = FA^{-1} = [1.15 \quad 1.5]\left(\frac{1}{-5}\begin{bmatrix} 1 & -4 \\ -2 & 3 \end{bmatrix}\right) = [0.37 \quad 0.02]$$

Thus, $M_1 = \$0.37$ per lemon and $M_3 = \$0.02$ per tablespoon sugar.

9. Consider the Court and Sparkle Jewelry Emporium problem (See exercise 1 above.) Suppose the profit for a Dawntreader bracelet increases to $2,400. Compute the new marginal values for the three resources.

Solution: One may verify that the corner $(x, y) = (15,10)$ is still optimal (so that $2,400 is within the stable range for the price of a Dawntreader.) Thus, the new maximum profit is $P = 2400(15) + 1800(10) = \$54{,}000$. Since we have rubies leftover, $M_1 = 0$. For the others, use Proposition 4.4:

$$[M_2 \quad M_3] = FA^{-1} = [2400 \quad 1800]\left(\frac{1}{60}\begin{bmatrix} 15 & -3 \\ -10 & 6 \end{bmatrix}\right) = [300 \quad 60]$$

Thus, $M_2 = \$300$ per pearl, and $M_3 = \$60$ per opal.

Section 4.4 Other Changes – Drawbacks of the Graphical Methods

Solutions to exercises: The exercises below appeared previously in the exercises to section 3.4, where they were solved by graphing in the decision space.

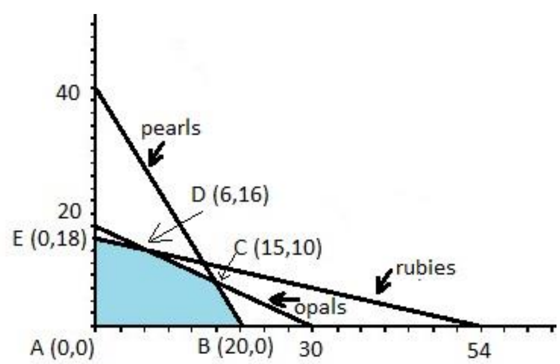
1. Joni is putting her designing skills to work at the *Court and Sparkle Jewelry Emporium*. She makes two signature design bracelet models. Each Dawntreader bracelet uses 1 ruby, 6 pearls, and 10 opals. Each Hejira bracelet uses 3 rubies, 3 pearls, and 15 opals. She has 54 rubies, 120 pearls, and 300 opals to work with. If either model results in a profit of $1,800 for Joni, how many of each type should she make in order to maximize her profit?

a) Compute the stable range of each resource.

Solution: Recall the set-up: x is the number of Dawntreader bracelets and y is the number of Hejira bracelets. Then

$$\text{Maximize Profit } P = 1800x + 1800y$$
$$\text{Subject to:}$$
$$x + 3y \leq 54\, rubies$$
$$6x + 3y \leq 120\, pearls$$
$$10x + 15y \leq 300\, opals$$
$$x \geq 0, \qquad y \geq 0$$

From our solution to exercise 7, Section 3.4, we found the feasible set below:



We know the optimal corner is C, located at $(x, y) = (15{,}10)$, with leftover rubies at this point.

Consider rubies first. We have surplus rubies at the optimal point. If we increase the supply of rubies by any amount, the ruby line moves farther out, but the optimal point is unchanged. If we decrease the supply of rubies, the ruby line moves closer to point C, and we will have fewer leftover rubies. Until we reduce by the 9 surplus rubies that we have (to 45 rubies). At that point, all three constraints are concurrent at point C. From here, further reduction in the ruby supply entails a change of basic variables. Thus, the stable range for the ruby supply is $[45, \infty)$.

Next consider pearls. Decreasing the supply of pearls moves the pearl line closer to the origin, and the coordinates of the optimal point C change by moving up the segment CD towards D. If we reduce the supply of pearls just enough to make all three lines concurrent at D, the problem degenerates and further reduction in pearl supply entails a change of basic variables. To find out when the lines are concurrent, we replace the pearl line with a parallel one with an unknown right side, and substitute in the point D.

$$6x + 3y = d$$
$$6(6) + 3(16) = d$$
$$84 = d$$

So, when the pearl supply is 84, the constraints are concurrent at D.
Similarly, increasing the supply of pearls moves the pearl line farther out and the coordinates of point C change as it moves down the segment of the opal line towards the intercept at $(30,0)$. Thus, when the line passes through (30,0), the problem degenerates and we've reached the end of the stable range as further increases in pearl supply entail a change of basic variables. To see when this happens, substitute the point (30,0) into the line above:

$$6x + 3y = d$$
$$6(30) + 3(0) = d$$
$$180 = d$$

It follows that the stable range for the pearl supply is $[84, 180]$.

Finally, consider opals. Decreasing the supply moves the opal line closer to the origin, and the point C moves down the pearl line towards point B. The problem degenerates when this point is reached and further decreases entail a change of basic variable. To see when this happens, replace the opal line with a parallel one with an unknown right side and plug in the point B:

$$10x + 15y = d$$
$$10(20) + 15(0) = d$$
$$200 = d$$

Similarly, increasing the opal supply moves the point C up the pearl line towards the point of intersection of the pearl line with the ruby line, and the problem degenerates when this point is reached. We need to know the coordinates of this point (which is a basic solution but not a feasible one), so we solve:

$$x + 3y = 54$$
$$6x + 3y = 120$$

To obtain the point of intersection $(x, y) = (13.2, \quad 13.6)$
Thus,

$$10x + 15y = d$$
$$10(13.2) + 15(13.6) = d$$
$$336 = d$$

It follows that the stable range for the opal supply is $[200, 336]$

b) Find the revised optimal data if the number of pearls is increased to 150.

Solution: Since 150 is within the stable range for pearls, the revised optimal solution still has the same basic variables. That is, the optimal corner is still the intersection of the pearl line and the opal line:

$$6x + 3y = 150$$
$$10x + 15y = 300$$

The solution is $(22.5, \quad 5)$. The number of rubies used is $x + 3y = 37.5$, leaving 16.5 leftover rubies; and the profit is $1800(22.5) + 1800(5) = \$49{,}500$. The complete revised optimal data is:

$$x = 22.5 \; Daawntreader \; bracelets$$
$$y = 5 \; Hejira \; bracelets$$
$$P = \$49{,}500 \; (maximized)$$
$$S_1 = 16.5 \; leftover \; rubies$$
$$S_2 = 0 \; leftover \; pearls$$
$$S_3 = 0 \; leftover \; opals$$

c) Find the revised optimal data if the number of opals is increased to 330.

Solution: As in part (b), the change is within the stable range, so the revised optimal corner is again the intersection of the pearl and the opal lines.

$$6x + 3y = 120$$
$$10x + 15y = 330$$

The solution is $(13.5, \quad 13)$. The number of rubies used is $x + 3y = 52.5$, leaving 1.5 surplus rubies. The revised profit is $P = 1800(13.5) + 1800(13) = \$47{,}700$. The complete revised optimal data is:

$$x = 13.5 \; Daawntreader \; bracelets$$
$$y = 13 \; Hejira \; bracelets$$
$$P = \$47{,}700 \; (maximized)$$
$$S_1 = 1.5 \; leftover \; rubies$$
$$S_2 = 0 \; leftover \; pearls$$
$$S_3 = 0 \; leftover \; opals$$

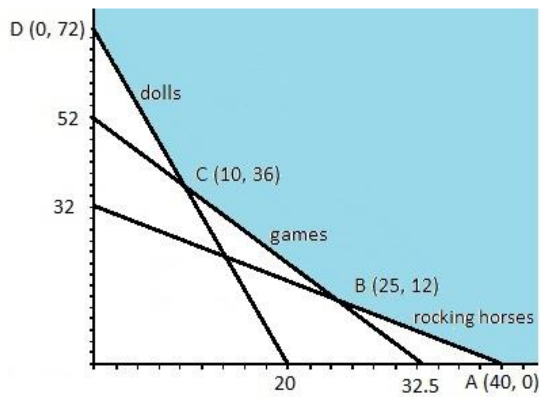
2. *Toys in the Attic, Inc*. operates two workshops to build toys for needy children. Mr. Tyler's shop can produce 36 *Angel* dolls, 16 *Kings and Queens* board games, and 16 *Back in the Saddle* rocking horses each day it operates. Mr. Perry's shop can produce 10 *Angel* dolls, 10 *Kings and Queens* board games, and 20 *Back in the Saddle* rocking horses each day it operates. It costs \$144 to operate Mr. Tyler's shop for one day and \$166 to operate Mr. Perry's shop for one day. Suppose the company receives an order from *Kids Dream On* Charity Foundation for at least 720 *Angel* dolls, at least 520 *Kings and Queens* board games, and at least 640 *Back in the Saddle* rocking horses. How many days should they operate each shop in order to fill the order at least possible cost?

a) Find the stable range for each constraint.

Solution: Recall the set-up of the problem: x is the number of days to operate Mr. Tyler's shop, and y is the number of days to operate Mr. Perry's shop. Then

$$\text{Minimize Cost } C = 144x + 166y$$
Subject to:
$$36x + 10y \geq 720 \ Angel\ Dolls$$
$$16x + 10y \geq 520 \ Kings\ and\ Queens\ Board\ Games$$
$$16x + 20y \geq 640 \ Back\ in\ the\ Saddle\ rocking\ horses$$
$$x \geq 0, \qquad y \geq 0$$

We copy the feasible set from our solution to exercise 8, Section 3.4:



From our previous solution, we know the optimal corner is $B = (25{,}12)$, and there is a surplus of 300 Angel dolls at this corner.

Consider Angel Dolls first. Lowering the required minimum amount from 720 would only increase the number of surplus dolls. Raising the required minimum moves the doll line out until all three lines are concurrent at B. At this point we will reach the end of the stable range because the problem degenerates and any further increase in minimum games would entail a change of basic variables. We know this will happen when we raise the minimum number of dolls to exactly compensate for the 300 surplus dolls we have now; that is, the stable range is $(-\infty, 120]$ in interval notation.

Next, consider Kings and Queens board games. Increasing the minimum number of board games from 520 will move the board game line out, and the point B moves down along the rocking horse line towards point A. When it reaches that point, the problem degenerates and further change entails a different set of basic variables, so that's the end of the stable range. For this to happen, we need:

$$16x + 10y = d$$
$$16(40) + 10(0) = d$$
$$640 = d$$

On the other hand, decreasing the required number of games will move the game line towards the origin, so B moves up the rocking horse line until it reaches the point where the rocking horse and doll lines cross. At this point, the problem again degenerates. We need to know the coordinates of this point (which is a basic but not a feasible solution.)

$$36x + 10y = 720$$
$$16x + 20y = 640$$

The solution is $\left(\frac{100}{7}, \frac{144}{7}\right)$. Thus, for the problem to degenerate at this point, we need the game line to also go through it:

$$16\left(\frac{100}{7}\right) + 10\left(\frac{144}{7}\right) = \frac{3040}{7} \approx 434.29.$$

So, the stable range for Kings and Queens board games is the interval $\left[\frac{3040}{7}, \quad 640\right]$.

Finally, consider Back in the Saddle rocking horses. Increasing the required number beyond 640 moves the rocking horse line out from the origin, as the point B moves up the game line towards C. The lines are concurrent at this point and problem degenerates. For the rocking horse line to pass through C, we need:

$$16(10) + 20(36) = d$$
$$880 = d$$

On the other hand, decreasing the required number of rocking horses moves the rocking horse line towards the origin, and B moves down the game line towards the x intercept at $(32.5,0)$. Here is the other end of the stable range as the problem again degenerates. For the rocking horse line to pass through this intercept we need:

$$16(32.5) + 20(0) = d$$
$$520 = d$$

Thus, the stable range for Back in the Saddle rocking horses is $[520 \quad 880]$.

b) Suppose the request for at least 640 rocking horses was lowered to at least 608 rocking horses. Find the revised optimal data.

Solution: Since 608 is within the stable range, the revised solution has the same basic variables. Thus, the optimal corner is still the intersection of the board game and rocking horse lines:

$$16x + 10y = 520$$
$$16x + 20y = 608$$

The solution is $(27, \quad 8.8)$. The cost at this point is $C = 144(27) + 166(8.8) = \$5{,}348.80$. The number of Angel dolls produced is $36(27) + 10(8.8) = 1{,}060$, which is a surplus of 340 dolls. The other constraints are binding, so the complete revised optimal solution is:

$$x = 27 \; days\ to\ operate\ Mr.Tyler's\ shop$$
$$y = 8.8 \; days\ to\ operate\ Mr.Perry's\ shop$$
$$C = \$5{,}348{,}80 \; (minimized)$$
$$S_1 = 340 \; surplus\ Angel\ dolls$$
$$S_2 = 0 \; surplus\ Kings\ and\ Queens\ board\ games$$
$$S_3 = 0 \; surplus\ Back\ in\ the\ Saddle\ rocking\ horses$$

3. Mr. Cooder, a farmer in the Purple Valley, has at most 400 acres to devote to two crops: rye and barley. Each acre of rye yields $100 profit per week, while each acre of barley yields $80 profit per week. Due to local demand, Mr. Cooder must plant at least 100 acres of barley. The federal government provides a subsidy to grow these crops in the form of tax credits. They credit Mr. Cooder 4 units for each acre of rye and 2 units for each acre of barley. The exact value of a 'unit' of tax credit is not important. Mr. Cooder has decided that he needs at least 600 units of tax credits in order to be able to afford his loan payments on a new harvester. How many acres of each crop should he plant in order to maximize his profit?

a) Find the stable range for land in acres.

Solution: Left for the reader. The solution is the interval $[200, \infty)$.

b) Suppose that Mr. Cooder can buy 50 more acres. Find the revised optimal data.

Solution: Since 450 is within the stable range, the revised solution must have the same basic variables. In the solution to this problem in exercise 14, Section 3.4, we saw that the constraint requiring a minimum number of acres of barley was binding. This means it is still binding, in the new solution, so the entire additional 50 acres should be devoted to rye:

$$x = 350 \text{ } acres\ rye$$
$$y = 100 \text{ } acres\ barley$$
$$P = \$43{,}000\ (maximized)$$
$$S_1 = 0\ leftover\ acres\ land$$
$$S_2 = 0\ surplus\ acres\ barley\ planted$$
$$S_3 = 1000\ surplus\ units\ tax\ credit$$

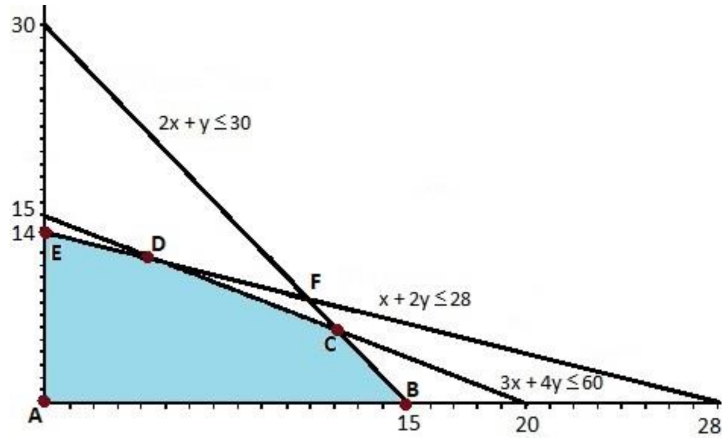
c) Suppose that one year he decides to let half of his farm to lie fallow (that is, plant no crops in half his land) so that the soil can regain nutrients. What is the revised optimal data in this case?

Solution: This amounts to lowering the available land to 200 acres, which is the endpoint of the stable range. Thus, the problem becomes degenerate. The revised solution is:

$$x = 100\ acres\ rye$$
$$y = 100\ acres\ barley$$
$$P = \$18{,}000\ (maximized)$$
$$S_1 = 0\ leftover\ acres\ land$$
$$S_2 = 0\ surplus\ acres\ barley\ planted$$
$$S_3 = 0\ surplus\ units\ tax\ credit$$

4. In the discussion of the lemonade stand problem in this section, we suggested that a stable range can be found for the coefficient which is the number of lemons used in a glass of sweet lemonade. Find this stable range.

Solution: Let a be the number of lemons used in a glass of sweet lemonade. Then the lemon constraint line has equation $ax + 4y = 60$, with slope $-\frac{a}{4}$. Note that this line passes through $(0,15)$ regardless of the value of a. Consider the graph of the feasible set:



By changing the value of a, the lemon line will change slope (but still pass through $(0,15)$). (Think of the lemon line as rotating or "pivoting" about its y intercept $(0,15)$.) As it rotates, the coordinates of the optimal point C will change as C moves up or down along the sugar line, specifically along the segment BF, where F is the point where the sugar and lime constraint lines meet. If a is chosen so that the intersection C is exactly B or F, the problem degenerates. The stable range we seek is between these values.

First, to pass through $B = (15,0)$, we must have

$$a(15) + 4(0) = 60$$
$$a = 4$$

Next, we need the coordinates of F:

$$x + 2y = 28$$
$$2x + y = 30$$

This has solution $F = \left(\frac{32}{3}, \frac{26}{3}\right)$. Now for the lemon constraint line to also pass through this point we must have:

$$a\left(\frac{32}{3}\right) + 4\left(\frac{26}{3}\right) = 60$$
$$32a + 104 = 180$$
$$a = \frac{19}{8} = 2.375$$

So, the stable range for the number of lemons used in the recipe for a glass of sweet lemonade is $[2.375,\ 4]$ lemons.

5. In the lemonade stand problem, suppose the girls decrease the amount of sugar used in a glass of sweet lemonade tea from 2 tablespoons to $\frac{5}{3}$ tablespoons. Find the revised optimal data.

Solution: With the change, the new equation for the sugar constraint is

$$\frac{5}{3}x + y = 30$$

Notice that this line has an x intercept of $(18, 0)$. Compare to the graph of the feasible set in exercise 4, and clearly with the changed line the problem has not degenerated, so this change is within the stable range of the amount of sugar in a glass of sweet lemonade. It follows that the point C is still the optimal corner, which is the intersection of the lemon and sugar constraint lines:

$$3x + 4y = 60$$
$$\frac{5}{3}x + y = 30$$

The solution is $\left(\frac{180}{11}, \quad \frac{30}{11}\right)$. The number of limes used at this point is $x + 2y = \frac{240}{11}$, so we have $\frac{68}{11}$ surplus limes. The complete revised optimal data:

$$x = \frac{180}{11} \approx 16.364 \; glasses \; sweet \; lemonade$$
$$y = \frac{30}{11} \approx 2.7273 \; glasses \; tart \; lemonade$$
$$R = \frac{270}{11} \approx \$24.55 \; (maximized)$$
$$S_1 = 0 \; leftover \; lemons$$
$$S_2 = \frac{68}{11} \approx 6.18 \; leftover \; limes$$
$$S_3 = 0 \; leftover \; Tbsp. sugar$$

Section 4.5 Duality

Solutions to exercises: For the following problems 1-6, find the dual problem following the method in Subsection 4.5.1.

1. Joni is putting her designing skills to work at the *Court and Sparkle Jewelry Emporium*. She makes two signature design bracelet models. Each Dawntreader bracelet uses 1 ruby, 6 pearls, and 10 opals. Each Hejira bracelet uses 3 rubies, 3 pearls, and 15 opals. She has 54 rubies, 120 pearls, and 300 opals to work with. If either model results in a profit of $1,800 for Joni, how many of each type should she make?

Solution: Letting u, v, w be the dual decision variables (and T_1, T_2 the surplus variables) in the dual, the dual problem is:

$$\text{Minimize } z = 54u + 120v + 300w$$
$$\text{Subject to:}$$
$$u + 6v + 10w \geq 1800$$
$$3u + 3v + 15w \geq 1800$$
$$u \geq 0, \quad v \geq 0, \quad w \geq 0$$

2. The *Poseidon's Wake* petroleum company operates two refineries. The *Cadence Refinery* can produce 40 units of low grade oil, 10 units medium grade oil, and 10 units high grade oil in a single day. (Each unit is 1000 barrels.) The *Cascade Refinery* can produce 10 units low grade oil, 10 units medium grade oil, and 30 units high grade oil in a single day. They receive an order from the Mars Triangle Oil Retailers for at least 80 units low grade oil, at least 50 units medium grade oil, and at least 90 units high grade oil. If it costs Poseidon's Wake $1,800 to operate the Cadence refinery for a day, and $2,000 to operate the Cascade Refinery for a day, how many days should they operate each refinery to fill the order at least cost?

Solution: Letting u, v, w be the dual decision variables (and T_1, T_2 the surplus variables) in the dual, the dual problem is:

$$\text{Maximize } z = 80u + 50v + 90w$$
$$\text{Subject to:}$$
$$40u + 10v + 10w \leq 1800$$
$$10u + 10v + 30w \leq 2000$$
$$u \geq 0, \quad v \geq 0, \quad w \geq 0$$

3. *Joanne's Antique Restoration Emporium* refurbishes Victorian furniture. Each dining room set requires 16 hours stripping and sanding time, 4 hours refinishing time, and 4 hours in the upholstery shop. Each bedroom set requires 36 hours stripping and sanding time, 4 hours refinishing time, and 2 hours in the upholstery shop. Each month, they have 720 person-hours in the stripping and sanding shop, 100 person-hours in the refinishing shop, and 80 person-hours in the upholstery shop. Each dining room set generates $600 profit and each bedroom set generates $900 profit. How many of each type of furniture set should Joanne accept to work on each month in order to maximize her profit?

Solution: Letting u, v, w be the dual decision variables (and T_1, T_2 the surplus variables) in the dual, the dual problem is:

$$\text{Minimize } z = 720u + 100v + 80w$$
Subject to:
$$16u + 4v + 4w \geq 600$$
$$36u + 4v + 2w \geq 900$$
$$u \geq 0, \qquad v \geq 0, \qquad w \geq 0$$

4. Bruce Jax is responsible for constructing end tables and kitchen tables for the *White Room Woodshop*. Each end table uses 2 square yards of $\frac{3}{4}$ inch oak boards and takes two hours to complete. Each kitchen table uses 4 square yards of oak boards and takes 3 hours to complete. This week he has available 36 square yards of oak boards and 32 hours of time. Other resources are unlimited. How many of each item should he make if he is paid $70 for each end table and $100 for each kitchen table?
Be sure to include the complete data set for the solution.

Solution: Letting u, v be the dual decision variables (and T_1, T_2 the surplus variables) in the dual, the dual problem is:

$$\text{Minimize } z = 36u + 32v$$
Subject to:
$$2u + 2v \geq 70$$
$$4u + 3v \geq 100$$
$$u \geq 0, \qquad v \geq 0$$

5. *The Glass House* is a shop the produces three types of specialty drinking glasses. The *Runaway* is a 12 oz. water tumbler, the *Reunion* is a 16 oz. beer glass, and the *Experience* is an elegant stemmed champagne flute glass. A case of Runaway glasses takes 1 hour on the molding machine, 0.1 hour to pack, and generates a profit of $40. A case of Reunion glasses takes 1.5 hours on the molding machine, 0.1 hours to pack, and generates a profit of $60. A case of Experience glasses takes 1.5 hours on the molding machine, 0.2 hours to pack, and generates a profit of$70. Each week, there are 720 hours of time available on the molding machine, and 60 hours available to pack. How many cases of each type should they manufacture in order to maximize profit?

Solution: Letting u, v be the dual decision variables (and T_1, T_2, T_3 the surplus variables) in the dual, the dual problem is:

$$\text{Minimize } z = 720u + 60v$$
Subject to:
$$u + 0.1v \geq 40$$
$$1.5u + 0.1v \geq 60$$
$$1.5u + 0.2v \geq 70$$
$$u \geq 0, \qquad v \geq 0$$

6. The *Spooky Boogie Costume Salon* makes and sells four different Halloween costumes: the witch, the ghost, the goblin, and the werewolf. Each witch costume uses 3 yards material and takes 3 hours to sew. Each ghost costume uses 2 yards of material and takes 1 hour to sew. Each goblin costume uses 2 yards of material and takes 3 hours to sew. Each werewolf costume uses 2 yards of material and takes 4 hours to sew. The profits for each costume are as follows: $10 for the witch, $8 for the ghost, $12 for the goblin, and $16 for the werewolf. If they have 600 yards of material and 510 sewing hours available

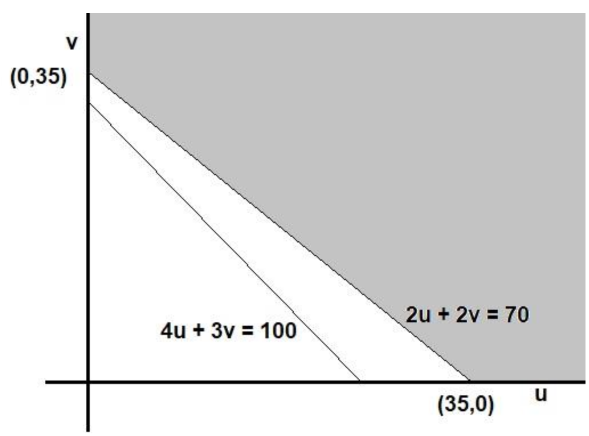
before the holiday, how many of each costume should they make in order to maximize profit, assuming they can sell everything they make?

Solution: Letting u, v be the dual decision variables (and $T_i, i = 1,2,3,4$ the surplus variables) in the dual, the dual problem is:

$$\text{Minimize } z = 600u + 510v$$
$$\text{Subject to:}$$
$$3u + 3v \geq 10$$
$$2u + v \geq 8$$
$$2u + 3v \geq 12$$
$$2u + 4v \geq 16$$
$$u \geq 0, \qquad v \geq 0$$

7. Observe that in exercise 4 above, the dual problem has just two decision variables. Solve the dual problem using the method of graphing in the decision space. Compare your answer to the answer to the primal problem (which was obtained in exercise 6, Section 3.4.) Verify that the principle of strong duality holds as well as complementary slackness.

Solution: See exercise 4 above for the set up of the dual problem. Here is the feasible set:



There are just two corners, and the optimal point is clearly at $(0,35)$ with value of the objective function as $z = 32(35) = 1120$. Thus, the solution to the dual is:

$$u = 0$$
$$v = 35$$
$$z = 1120\ (minimized)$$
$$T_1 = 0$$
$$T_2 = 5$$

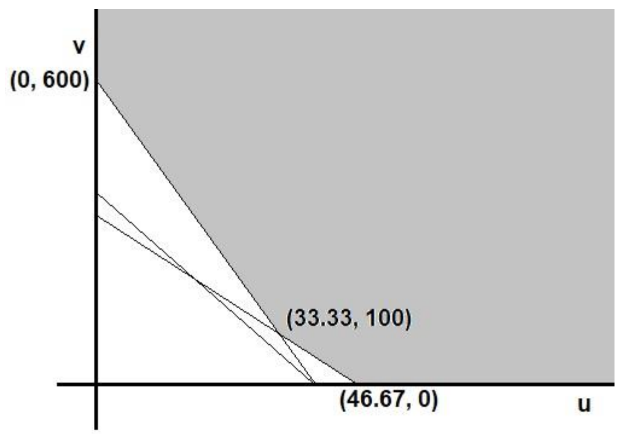
Comparing with the solution to the original problem (from section 3.4) we have:

$$x = 16\ end\ tables$$
$$y = 0\ kitchen\ tables$$
$$P = \$1120\ (maximized)$$
$$S_1 = 4\ leftover\ square\ yards\ wood$$
$$S_2 = 0\ leftover\ hours\ time$$

Because the minimum value of z equals the maximum value of P, we see strong duality holds. Also complementary slackness holds because $u = M_1$, the marginal value of wood, and we have $S_1 \cdot M_1 = 4 \cdot 0 = 0$. Similarly, $v = M_2$, the marginal value of time, and we have $S_2 \cdot M_2 = 0 \cdot 35 = 0$.

8. Observe that in exercise 5 above, the dual problem has just two decision variables. Solve the dual problem using the method of graphing in the decision space. In this problem, the primal problem has too many variables to solve by using the decision space. However, in Section 3.5 this problem was assigned to be solved using the method of graphing in the constraint space. Compare the solutions of the primal and the dual. Verify strong duality and complementary slackness.

Solution: See exercise 5 for the set up of the dual. Here is the feasible set:



Evaluate the objective function at the corners:

(u, v)	$z = 720u + 60v$
$(0,600)$	36,000
$\left(\frac{100}{3}, 100\right)$	30,000
$\left(\frac{140}{3}, 0\right)$	33,600

The optimal solution is:

$$u = \frac{100}{3} \approx 33.33$$
$$v = 100$$
$$z = 30{,}000\ (minimized)$$
$$T_1 = \frac{10}{3}$$
$$T_2 = 0$$
$$T_3 = 0$$

Compare to the original solution from section 3.5:

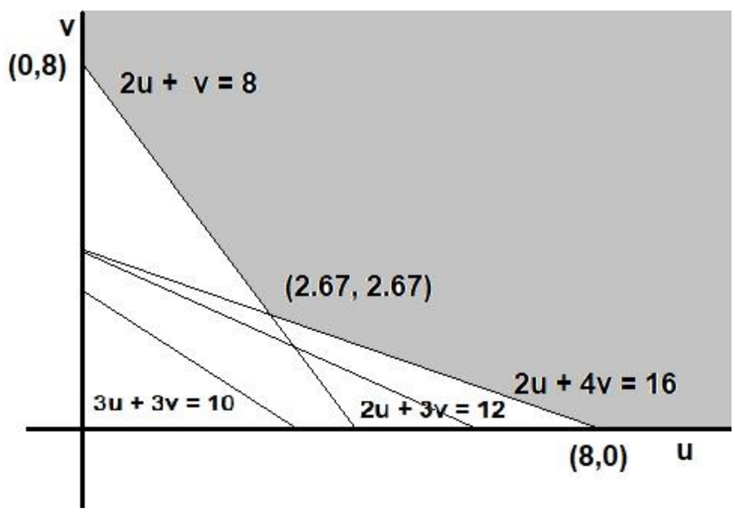
$x = 0$ cases Runaway glasses

$$y = 360 \text{ cases Reunion glasses}$$
$$z = 120 \text{ cases Experience glasses}$$
$$P = \$30{,}000 \text{ (maximized)}$$
$$S_1 = 0 \text{ leftover hours molding time}$$
$$S_2 = 0 \text{ leftover hours packing time}$$

Since the minimum value of P equals the maximum value of z, strong duality holds. Also complementary slackness holds since u is the marginal value of molding time and v is the marginal value of packing time. Both $S_i = 0$, so clearly $S_i M_i = 0$.

9. observe that in exercise 6 above, the dual problem has just two decision variables. Solve the dual problem using the method of graphing in the decision space. In this problem, the primal has too mny variables to solve by using the decision space. However, in Section 3.5, this problem was assigned to be solved using the method of graphing in the constraint space. Compare the solutions of the primal and the dual. Verify strong duality and complementary slackness.

Solution: See Exercise 6 for the set up of the dual problem. Here is the feasible set:



Evaluate the objective function at the corners:

(u, v)	$C = 600u + 510v$
$(0, 8)$	4,080
$\left(\frac{8}{3}, \frac{8}{3}\right)$	2,960
$(8, 0)$	4,800

So the optimal solution is:

$$u = \frac{8}{3} \approx 2.67$$
$$v = \frac{8}{3} \approx 2.67$$
$$C = 2{,}960 \ \ (minimized)$$

$$T_1 = 6$$
$$T_2 = 0$$
$$T_3 = \frac{4}{3}$$
$$T_4 = 0$$

Compare to the solution to the original problem from Section 3.5:

$w = 0$ witch costumes
$x = 230$ ghost costumes
$y = 0$ goblin costumes
$z = 70$ werewolf costumes
$P = \$2{,}960$ (maximized)
$S_1 = 0$ leftover yards material
$S_2 = 0$ leftover hours sewing time

Since the minimum value of C equals the maximum value of P, we see strong duality holds. Also, complementary slackness holds since $S_1 \cdot u = 0 = S_2 \cdot v$. (Alternately, working with the dual problem, since w, x, y, z, are the marginal values of the constraints of the dual, we see $w \cdot T_1 = 0, x \cdot T_2 = 0, y \cdot T_3 = 0$, and $z \cdot T_4 = 0$.)

10. In Section 3.4, the following problem was assigned in the exercises:

Maximize $P = x + 3y$
Subject to:
$$10x + 16y \leq 240$$
$$15x + 8y \leq 240$$
$$x \geq 0, \qquad y \geq 0$$

Find the dual problem and solve it. Compare the solutions of the primal and the dual – verify strong duality and complementary slackness.

Solution: The dual problem is:

Minimize $C = 240u + 240v$
Subject to:
$$10u + 15v \geq 1$$
$$16u + 8v \geq 3$$
$$u \geq 0, \quad v \geq 0$$

Here is the final solution (details of obtaining it are left for the reader)

$$u = \frac{3}{16} = 0.1875$$
$$v = 0$$
$C = 45$ (minimized)

$$T_1 = \frac{6}{7} = 0.875$$
$$T_2 = 0$$

We leave it to the reader to verify strong duality and complementary slackness.

11. Same directions as Exercise 10 for this problem:

$$\text{Minimize } C = 20x + 12y$$
$$\text{Subject to}$$
$$5x + 2y \geq 30$$
$$5x + 7y \geq 70$$
$$x \geq 0, \qquad y \geq 0$$

Solution: The dual problem is:

$$\text{Maximize } P = 30u + 70v$$
$$\text{Subject to:}$$
$$5u + 5v \leq 20$$
$$2u + 7v \leq 12$$
$$u \geq 0, \quad v \geq 0$$

Here is the final solution (details of obtaining it left for the reader):

$$u = 3.2$$
$$v = 0$$
$$P = 152 \text{ (maximized)}$$
$$T_1 = 0$$
$$T_2 = 0$$

We leave it to the reader to verify strong duality and complementary slackness.

12. In this exercise we explore the dual problem in case there are mixed constraints. Consider the following problem:

$$\text{Maximize } R = 120x + 200y$$
$$\text{Subject to}$$
$$10x + 6y \leq 120$$
$$10x + 4y \geq 100$$
$$x \geq 0, \ \ y \geq 0$$

a) Solve the problem by graphing in the decision space. (Your answer should include the values x, y, R, and the slack variables).

Solution: We leave it to the reader to plot the feasible set and evaluate the objective function at the corners. Here is the optimal result:

$$x = 6$$
$$y = 10$$
$$R = 2{,}720 \text{ (maximized)}$$
$$S_1 = 0$$
$$S_2 = 0$$

b) Use matrix multiplication FA^{-1} to find the marginal values for the two constraints.

Solution:

$$[M_1, M_2] = FA^{-1} = [120 \quad 200]\left(\frac{1}{-20}\begin{bmatrix} 4 & -6 \\ -10 & 10 \end{bmatrix}\right) = [76 \quad -64]$$

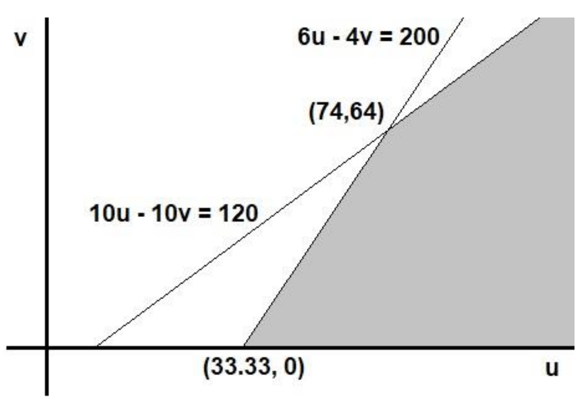
c) Our construction of the dual is based on having a problem in standard form. For a maximization problem, this means all the constraints should be of the less than type. Convert the second constraint to this canonical form by multiplying through by -1. Then construct the condensed tableau as usual, and write down the dual problem.

Solution: The second constraint becomes $-10x - 4y \leq -100$. The dual problem is:

$$\text{Minimize } z = 120u - 100v$$
$$\text{Subject to:}$$
$$10u - 10v \geq 120$$
$$6u - 4v \geq 200$$
$$u \geq 0, \quad v \geq 0$$

d) Solve the dual problem you obtained in part (c) by graphing in the decision space. Verify that strong duality holds. Do the optimal values of the dual agree with the marginal values you computed in part (b)?

Solution: Here is the feasible set:



Evaluate the objective function at the two corners:

(u, v)	$C = 120u - 100v$
$\left(\frac{100}{3}, 0\right)$	4000
$(74, 64)$	2720

The optimal solution is:

$$u = 76$$
$$v = 64$$
$$C = 2{,}720 \text{ (minimized)}$$
$$T_1 = 0$$
$$T_2 = 0$$

This optimal value agrees with the marginal values, except for a sign. The M_i come out negative for a constraint pointing the “wrong” way. But as decision variables in the dual, they cannot be negative. (This discrepancy was caused by the step of multiplying through the inequality by -1.)

13. Following the steps of part (c) of exercise 12, find the dual problem to the following problem: Mr. Cooder, a farmer in the Purple Valley, has at most 400 acres to devote to two crops: rye and barley. Each acre of rye yields $100 profit per week, while each acre of barley yields $80 profit per week. Due to local demand, Mr. Cooder must plant at least 100 acres of barley. The federal government provides a subsidy to grow these crops in the form of tax credits. They credit Mr. Cooder 4 units for each acre of rye and 2 units for each acre of barley. The exact value of a ‘unit’ of tax credit is not important. Mr. Cooder has decided that he needs at least 600 units of tax credits in order to be able to afford his loan payments on a new harvester. How many acres of each crop should he plant in order to maximize his profit?

Solution: Recall the set-up of the problem from our previous solution: Let x be the number of acres of rye and y the number of acres of barley. Then

Maximize weekly profit $P = 100x + 80y$

Subject to:

$x + y \leq 400$ (acres land)

$y \geq 100$ (acres barley demand)

$4x + 2y \geq 600$ (units of tax credit)

$x \geq 0, \quad y \geq 0$

Converting the inequalities:

Maximize weekly profit $P = 100x + 80y$

Subject to:

$x + y \leq 400$ (acres land)

$-y \leq -100$ (acres barley demand)

$-4x - 2y \leq -600$ (units of tax credit)

$x \geq 0, \quad y \geq 0$

Thus, the dual problem is:

$$\text{Minimize } C = 400u - 100v - 600w$$
$$\text{Subject to:}$$
$$u - 4w \geq 100$$
$$u - v - 2w \geq 80$$
$$u \geq 0, \;\; v \geq 0, \;\; w \geq 0$$

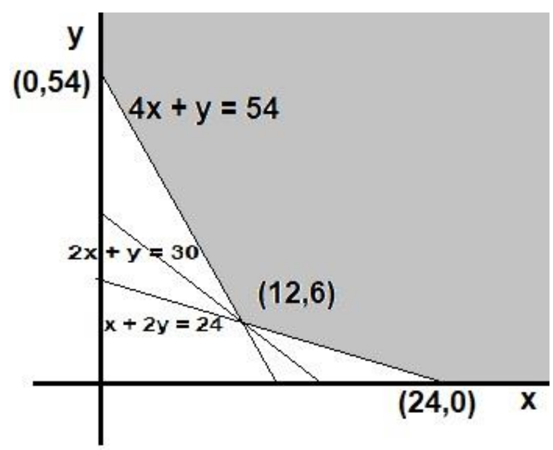
14. Consider the following variation of the O'Casek used car problem from Section 3.2 in the text:

$$\text{Minimize } C = 6x + 4y$$
$$\text{Subject to:}$$
$$x + 2y \geq 24 \; (SUV's)$$
$$2x + y \geq 30 \text{ (Sedans)}$$
$$4x + y \geq 54 \text{ (compact cars)}$$
$$x \geq 0, \quad y \geq 0$$

In this variation, the only difference from the original is that we are requiring at least 54 compact cars to be shipped, instead of at least 40.

a) Find the optimal solution and verify that the problem is degenerate.

Solution: The problem is degenerate because all three of the constraint lines pass through the point $(12,6)$.



Evaluate the objective function at the corners:

(x, y)	$C = 6x + 4y$
$(\mathbf{0}, \mathbf{54})$	\$216
$(\mathbf{12}, \mathbf{6})$	\$96
$(\mathbf{24}, \mathbf{0})$	\$133

The optimal solution is:

$$x = 12$$
$$y = 6$$

$$C = 96 \text{ (minimized)}$$
$$S_1 = 0$$
$$S_2 = 0$$
$$S_3 = 0$$

b) Compute the marginal values associated with each constraint. Verify that complementary slackness holds, and that for one constraint, both S_i and M_i are simultaneously zero.

Solution: One can check that $M_1 = \frac{10}{7}, M_2 = 0,$ and $M_3 = \frac{10}{7}$. Thus, $M_2 = 0 = S_2$, and complementary slackness holds.

Chapter 5. The Simplex Algorithm

Section 5.1 Standard Form Maximization Problems

No exercises or new music references in this section.

Section 5.2 Phase II Pivoting

Music references in the text.

Example 5.6, about trail mix alludes to several bands. Quicksilver Messenger Service was a San Francisco based band who released an album in 1969 entitled Happy Trails, and a 1970 song entitled 'Fresh Air'. Fleetwood Max is a pun on the band name Fleetwood Mac, and Rodeohead is a pun on the band name Radiohead, who in 2000 released an album entitled Kid A. (Websites: https://en.wikipedia.org/wiki/Quicksilver_Messenger_Service , https://www.fleetwoodmac.com , https://www.radiohead.com)

Solutions to exercises:

In exercises 1-10, simplex tableaux are shown. For each tableau, determine whether or not it corresponds to a feasible point. If the answer is no, explain why not. If the answer is yes, then write down the complete set of data for that corner point by reading off the tableau; that is, the values of all the variables, including the marginal values of each constraint. Finally, determine if the tableau represents an optimal point. If not, determine the next pivot row and pivot column.

1 a)

	x	y	S_1	S_2	S_3	C
S_1	0	-2	1	-1	0	6
x	1	$\frac{1}{2}$	0	2	0	21
S_3	0	3	0	$-\frac{3}{2}$	1	15
P	0	-5	0	-3	0	64

Solution: Yes, it represents a feasible point. The current data is:

$$x = 21$$
$$y = 0$$
$$P = 64$$
$$S_1 = 0, \quad M_1 = 0$$
$$S_2 = 0, \quad M_2 = -3$$
$$S_3 = 15, \quad M_3 = 0$$

The point is not optimal, since there is a negative entry in the bottom row. The next pivot column is the y column, and the pivot row is S_3.

b) Interpret the meaning of all the entries in the y column.

Solution: If we bring one unit of y into solution, this has the effect of increasing the value of S_1 by 2 units, decreasing the value of x by $^1/_2$ a unit, decreasing the value of S_3 by 3 units, and increasing the value of the objective function by 5 units.

2.

	x	y	S_1	S_2	S_3	S_4	Capacity
x	1	0	$\frac{4}{3}$	0	0	0	$\frac{100}{3}$
S_2	0	0	$\frac{1}{3}$	1	0	$\frac{1}{3}$	25
S_3	0	0	$-\frac{2}{3}$	0	1	$\frac{11}{3}$	22
y	0	1	$\frac{5}{3}$	0	0	$\frac{7}{3}$	$\frac{77}{3}$
z	0	0	$-\frac{4}{3}$	0	0	$-\frac{7}{3}$	$\frac{400}{3}$

Solution: This is a feasible point. The current data is:

$$x = \frac{100}{3}$$
$$y = \frac{77}{3}$$
$$z = \frac{400}{3}$$
$$S_1 = 0, \quad M_1 = -\frac{4}{3}$$
$$S_2 = 25, \quad M_2 = 0$$
$$S_3 = 22, \quad M_3 = 0$$
$$S_4 = 0, \quad M_4 = -\frac{7}{3}$$

It's not an optimal point since there are negative entries in the bottom row. The next pivot column is S_4 and the next pivot row is S_3.

3.

	x	y	S_1	S_2	C
y	-2	1	4	0	125
S_2	5	0	9	1	-66
P	-1	0	-3	0	1875

Solution: This is not a feasible point, since $S_2 = -66 < 0$.

4.

	x	y	S_1	S_2	Capacity
y	0	1	-4	6	70
x	1	0	8	-1	45
z	0	0	12	9	1450

Solution: This is both a feasible and an optimal point. The optimal data is:

$$x = 45$$
$$y = 70$$
$$z = 1450 \text{ (maximized)}$$
$$S_1 = 0, \quad M_1 = 12$$
$$S_2 = 0, \quad M_2 = 9$$

5.

	x_1	x_2	x_3	x_4	x_5	S_1	S_2	C
S_2	$\frac{1}{4}$	1	0	0	12	$-\frac{3}{4}$	1	100
x_3	$\frac{3}{4}$	-2	1	3	3	$\frac{11}{4}$	0	23
P	5	-2	0	6	-3	7	0	42

Solution: This is a feasible but not an optimal point. The next pivot column is x_5 and the next pivot row is x_3. The current data is:

$$x_1 = 0$$
$$x_2 = 0$$
$$x_3 = 23$$
$$x_4 = 0$$
$$x_5 = 0$$
$$P = 42$$
$$S_1 = 0, \quad M_1 = 7$$
$$S_2 = 100, \quad M_2 = 0$$

6.

	w	x	y	z	S_1	S_2	S_3	C
w	1	−1	0	0	$\frac{3}{5}$	$\frac{6}{5}$	4	200
y	0	2	1	0	$-\frac{2}{5}$	1	$\frac{4}{5}$	270
z	0	1	0	1	$\frac{1}{5}$	−2	$\frac{3}{5}$	35
P	0	5	0	0	4	7	6	2018

Solution: This is a feasible and an optimal point. The optimal data is:

$$x = 200$$
$$x = 0$$
$$y = 270$$
$$z = 35$$
$$P = 2018 \text{ (maximized)}$$
$$S_1 = 0, \quad M_1 = 4$$
$$S_2 = 0, \quad M_2 = 7$$
$$S_3 = 0, \quad M_3 = 6$$

7. a)

	x_1	x_2	x_3	x_4	x_5	S_1	S_2	S_3	$Capacity$
S_1	$-\frac{1}{2}$	8	0	0	1	1	5	10	160
x_4	3	0	0	1	2	0	7	−3	125
x_3	6	−1	1	0	4	0	$\frac{11}{2}$	5	90
P	−5	−9	0	0	−1	0	$\frac{3}{2}$	−9	1000

Solution: This is a feasible but not an optimal point. The next pivot column is x_2 (following Bland's rule), and the next pivot row is S_1. The current data is:

$$x_1 = 0$$
$$x_2 = 0$$
$$x_3 = 90$$
$$x_4 = 125$$

$$x_5 = 0$$
$$P = 1000$$
$$S_1 = 160, \quad M_1 = 0$$
$$S_2 = 0, \quad M_1 = \frac{3}{2}$$
$$S_3 = 0, \quad M_3 = -9$$

b) Interpret the meaning of the entries in the x_2 column.

Solution: If we bring one unit of x_2 into solution, then S_1 decreases by 8 units, x_4 is unaffected, but x_3 increases by 1 unit, and P increases by 9 units.

8.

	x	y	z	S_1	S_2	S_3	$Capacity$
z	0	$\frac{3}{17}$	1	$-\frac{2}{17}$	0	$\frac{31}{17}$	$-\frac{150}{17}$
x	1	$\frac{11}{17}$	0	$\frac{1}{17}$	0	$\frac{5}{17}$	$\frac{203}{17}$
S_2	0	$\frac{20}{17}$	0	$\frac{3}{17}$	1	$\frac{14}{17}$	$\frac{100}{17}$
w	0	$\frac{10}{17}$	0	$-\frac{6}{17}$	0	$-\frac{2}{17}$	$\frac{800}{17}$

Solution: This is not a feasible point since $z = -\frac{150}{17} < 0$.

9.

	p	q	r	t	w	S_1	S_2	C
w	$\frac{3}{2}$	0	$\frac{1}{2}$	2	1	8	$\frac{7}{2}$	18
q	0	1	$-\frac{1}{2}$	5	0	$\frac{15}{2}$	$\frac{11}{2}$	45
z	-4	0	-1	-8	0	6	5	88

Solution: This is a feasible point, but not optimal. The next pivot column is t, and the next pivot row is q according to Bland's rule (q precedes w in the list of variables.) The current data is:

$$p = 0$$
$$q = 45$$
$$r = 0$$
$$t = 0$$
$$w = 18$$
$$z = 88$$
$$S_1 = 0, \quad M_1 = 6$$
$$S_2 = 0, \quad M_2 = 5$$

10.

	x_1	x_2	x_3	x_4	x_5	S_1	S_2	S_3	S_4	C
x_3	0	-3	1	12	0	0	1	4	4	77
S_1	0	1	0	-8	0	1	-2	5	3	150
x_5	0	2	0	6	1	0	1	6	-2	225
x_1	1	5	0	6	0	0	2	10	0	101
P	0	60	0	38	0	0	50	65	42	166,280

Solution: This is a feasible and an optimal point. The optimal data is:

$$x_1 = 101$$
$$x_2 = \mathbf{0}$$
$$x_3 = \mathbf{77}$$
$$x_4 = \mathbf{0}$$
$$x_5 = \mathbf{225}$$
$$P = 166{,}280 \text{ (maximized)}$$
$$S_1 = \mathbf{150}, \quad \boldsymbol{M_1} = \mathbf{0}$$
$$S_2 = \mathbf{0}, \quad \boldsymbol{M_2} = \mathbf{50}$$
$$S_3 = \mathbf{0}, \quad \boldsymbol{M_3} = \mathbf{65}$$
$$S_4 = \mathbf{0}, \quad \boldsymbol{M_4} = \mathbf{42}$$

11. Using the Simplex Algorithm, solve the problem which was set up in exercise 11 from Section 3.1, which we reproduce here for convenience: Bruce Jax is responsible for constructing end tables and kitchen tables for the *White Room Workshop*. Each end table uses 2 square yards of $\frac{3}{4}$-inch oak boards and takes two hours to complete. Each kitchen table uses 4 square yards of oak boards and takes three hours to complete. This week he has available 36 square yards of oak boards and 32 hours of time. Other resources are unlimited. How many of each item should he make if he is paid $70 for each end table and $100 for each kitchen table? Be sure to include the complete data set for the solution.

Solution: Recall the set-up from exercise 11, Section 3.1: Let x be the number of end tables and y the number of kitchen tables. Then we must

$$\text{Maximize Pay } P = 70x + 100y$$
$$\text{Subject to:}$$
$$2x + 4y \le 36 \text{ (square yards oak board)}$$
$$2x + 3x \le 32 \text{ (hours)}$$
$$x \ge 0, \quad y \ge 0$$

The Simplex Algorithm yields (we shade in the pivot rows and columns, as in the textbook):

	x	y	S_1	S_2	$Capacity$
S_1	2	4	1	0	36
S_2	2	3	0	1	32
P	-70	-100	0	0	0

	x	y	S_1	S_2	$Capacity$
y	$\frac{2}{4}$	1	$\frac{1}{4}$	0	9
S_2	$\frac{2}{4}$	0	$-\frac{3}{4}$	1	5
P	-20	0	25	0	900

	x	y	S_1	S_2	$Capacity$
y	0	1	1	-1	4
x	1	0	$-\frac{3}{2}$	2	10
P	0	0	-5	40	1100

	x	y	S_1	S_2	$Capacity$
S_1	0	1	1	-1	4
x	1	$\frac{3}{2}$	0	$\frac{1}{2}$	16
P	0	5	0	35	1120

Thus, the optimal data is:

$x = 16$ end tables
$y = 0$ kitchen tables
$P = \$1{,}120$ (maximized)
$S_1 = 4$ (square yards leftover oak), $M_1 = 0$
$S_2 = 0$ (hours leftover time), $M_2 = \$35$ (per hour)

12. Using the Simplex Algorithm, solve the following problem (see exercise 12 of Section 3.1). Joni is putting her design skills to work at the *Court and Sparkle Jewelry Emporium*. She makes two signature bracelet designs. Each *Dawntreader* design uses 1 ruby, 6 pearls, and 10 opals. Each Hejira design uses 3 rubies, 3 pearls, and 15 opals. She has 54 rubies, 120 pearls, and 300 opals to work with. If either model yields a profit of \$1800, how many of each type of bracelet should she make in order to maximize her profit? Be sure to include the complete data set for the solution.

Solution: Recall the set-up of the problem: Let x be the number of Dawntreader bracelets, and y the number of Hejira bracelets.

Maximize $P = 1800x + 1800y$
Subject to:
$x + 3y \leq 54$ (rubies)

$$6x + 3y \leq 120 \text{ (pearls)}$$
$$10x + 15y \leq 300 \text{ (opals)}$$
$$x \geq 0, \quad y \geq 0$$

The simplex algorithm yields:

	x	y	S_1	S_2	S_3	$Capacity$
S_1	1	3	1	0	0	54
S_2	6	3	0	1	0	120
S_3	10	15	0	0	1	300
P	-1800	-1800	0	0	0	0

	x	y	S_1	S_2	S_3	$Capacity$
S_1	0	$\frac{5}{2}$	1	$-\frac{1}{6}$	0	34
x	1	$\frac{1}{2}$	0	$\frac{1}{6}$	0	20
S_3	0	10	0	$-\frac{10}{6}$	1	100
P	0	-900	0	300	0	36000

	x	y	S_1	S_2	S_3	$Capacity$
S_1	0	0	1	$\frac{1}{4}$	$-\frac{1}{4}$	9
x	1	0	0	$\frac{1}{4}$	$-\frac{1}{20}$	15
y	0	1	0	$-\frac{1}{6}$	$\frac{1}{10}$	10
P	0	0	0	150	90	45000

The optimal data is:

$x = 15$ Dawntreader bracelets
$y = 10$ Hejira bracelets
$P = \$45{,}000$ (maximized)
$S_1 = 9$ (leftover rubies), $M_1 = \$0$/ruby
$S_2 = 0$ (leftover pearls), $M_2 = \$150$/pearl
$S_3 = 0$ (leftover opals), $M_3 = \$90$/opal

13. Solve the following problem by the simplex method (see exercise 15 of Section 3.1): *Joanne's Antique Restoration Emporium* refurbishes Victorian furniture. Each dining room set requires 16 hours stripping and sanding time, 4 hours refinishing time, and 4 hours in the upholstery shop. Each bedroom

set requires 36 hours stripping and sanding time, 4 hours refinishing time, and 2 hours in the upholstery shop. Each month, they have available 720 person-hours in the stripping and sanding shop, 100 person-hours in the refinishing shop, and 80 person-hours in the upholstery shop. Each dining room set generates $600 profit and each bedroom set generates $900 profit. How many of each type of furniture set should Joanne accept to work on each month in order to maximize her profit?

Solution: Recall the set-up of the problem: Let x be the number of dining room sets per month, and y the number of bedroom sets per month. Then

Maximize $P = 600x + 900y$
Subject to:
$16x + 36y \leq 720$ (hours stripping and sanding)
$4x + 4y \leq 100$ (hours refinishing)
$4x + 2y \leq 80$ (hours for upholstery shop)
$x \geq 0, \quad y \geq 0$

The simplex algorithm yields:

	x	y	S_1	S_2	S_3	$Capacity$
S_1	16	36	1	0	0	720
S_2	4	4	0	1	0	100
S_3	4	2	0	0	1	80
P	-600	-900	0	0	0	0

	x	y	S_1	S_2	S_3	$Capacity$
y	$\frac{4}{9}$	1	$\frac{1}{36}$	0	0	20
S_2	$\frac{20}{9}$	0	$-\frac{4}{36}$	1	0	20
S_3	$\frac{28}{9}$	0	$-\frac{2}{36}$	0	1	40
P	-200	0	25	0	0	18000

	x	y	S_1	S_2	S_3	$Capacity$
y	0	1	$\frac{1}{20}$	$-\frac{4}{20}$	0	16
x	1	0	$-\frac{1}{20}$	$\frac{9}{20}$	0	9
S_3	0	0	$-\frac{2}{20}$	$-\frac{28}{20}$	1	12
P	0	0	15	90	0	19800

The optimal data:

$$x = 9 \text{ dining room sets}$$
$$y = 16 \text{ bedroom sets}$$
$$P = \$19{,}800 \text{ (maximized)}$$
$$S_1 = 0 \text{ leftover hours stripping and sanding time,} \quad M_1 = \$15/\text{hr}$$
$$S_2 = 0 \text{ leftover hours refinishing time,} \quad M_2 = \$90/\text{hr}$$
$$S_3 = 12 \text{ leftover hours upholstery time,} \quad M_3 = 0$$

14. Solve the following problem (see exercise 19 from Section 3.1): *Green's Heavy Metal Foundry* mixes three different alloys composed of copper, zinc, and iron. Each 100 lb. unit of Alloy I consists of 50 lbs. copper, 50 lbs. zinc, and no iron. Each 100 lb. unit of Alloy II consists of 30 lbs. copper, 30 lbs. zinc, and 40 lbs. iron. Each 100 lb. unit of Alloy III consists of 50 lbs. copper, 20 lbs. zinc, and 30 lbs. iron. Each unit of Alloy I generates \$100 profit, each unit of Alloy II generates \$80 profit, and each unit of Alloy III generates \$40 profit. There are 12,000 lbs. copper, 10,000 lbs. zinc, and 12,000 lbs. iron available. The foundry has hired Gary Giante, an outside consultant from *Moritmore, Weathers, and Smith, Ltd*., to help them maximize their profit. Fortunately, Mr. Giante knows the simplex algorithm.

Solution: Recall the set-up: Let x be the number of (100 lb.) units of Alloy I, y the number of units of Alloy II, and z the number of units of Alloy III. Then

$$\text{Maximize profit } P = 100x + 80y + 40z$$
$$\text{Subject to:}$$
$$50x + 30y + 50z \leq 12000 \text{ (lbs. copper)}$$
$$50x + 30y + 20z \leq 10{,}000 \text{ (lbs. zinc)}$$
$$40y + 30z \leq 12{,}000 \text{ (lbs. iron)}$$
$$x \geq 0, \;\; y \geq 0, \;\; z \geq 0$$

The Simplex Algorithm yields:

	x	y	z	S_1	S_2	S_3	$Capacity$
S_1	50	30	50	1	0	0	12000
S_2	50	30	20	0	1	0	10000
S_3	0	40	30	0	0	1	12000
P	-100	-80	-40	0	0	0	0

	x	y	z	S_1	S_2	S_3	$Capacity$
S_1	0	0	30	1	-1	0	2000
x	1	$\frac{3}{5}$	$\frac{2}{5}$	0	$\frac{1}{50}$	0	200
S_3	0	40	30	0	0	1	12000
P	0	-20	0	0	2	0	20000

	x	y	z	S_1	S_2	S_3	$Capacity$
S_1	0	0	30	1	-1	0	2000
x	1	0	$-\frac{1}{20}$	0	$\frac{1}{50}$	$-\frac{3}{200}$	20
y	0	1	$\frac{15}{20}$	0	0	$\frac{5}{200}$	300
P	0	0	15	0	2	$\frac{1}{2}$	26000

The optimal data:

$x = 20$ units Alloy I
$y = 300$ units Alloy II
$z = 0$ units Alloy III
$P = \$26,000$ (maximized)
$S_1 = 2000$ lbs. leftover copper, $\quad M_1 = 0$
$S_2 = 0$ lbs. leftover zinc, $\quad M_2 = \$2$/lb. zinc
$S_3 = 0$ lbs. leftover iron, $\quad M_3 = \$.50$/ lb. iron

15. Solve the following 2×2 problem (see exercise 3 from Section 3.4) using the simplex method:

Maximize $P = x + 3y$
Subject to:
$10x + 16y \leq 240$
$15x + 8y \leq 240$
$x \geq 0, \quad y \geq 0$

Solution:

	x	y	S_1	S_2	$Capacity$
S_1	10	16	1	0	240
S_2	15	8	0	1	240
P	-1	-3	0	0	0

	x	y	S_1	S_2	$Capacity$
y	$\frac{5}{2}$	1	$\frac{1}{16}$	0	15
S_2	10	0	$-\frac{1}{2}$	1	120
P	$\frac{7}{8}$	0	$\frac{3}{16}$	0	45

The optimal data:

$$x = 0$$
$$y = 15$$
$$P = 45 \text{ (maximized)}$$
$$S_1 = 0, \quad M_1 = \frac{3}{16}$$
$$S_2 = 120, \quad M_2 = 0$$

16. Solve the following problem with the simplex method (see exercise 6 from Section 3.4): *The Glass House* is a shop that produces three types of specialty drinking glasses. The *Runaway* is a 12. Oz. water tumbler, the *Reunion* is a 16 oz. beer glass, and the *Experience* is an elegant stemmed champagne flute glass. A case of Runaway glasses takes 1 hour on the molding machine, 0.1 hour to pack, and generates a profit of $40. A case of Reunion glasses takes 1.5 hours on the molding machine, 0.1 hour to pack, and generates a profit of $60. A case of Experience glasses takes 1.5 hours on the molding machine, 0.2 hour to pack, and generates a profit of $70. Each week, there are 720 hours of tie available on the molding machines, and 60 hours available to pack. How many cases of each type should they manufacture in order to maximize profit?

Solution: Recall the set-up: Let x be the number of cases of Runaway, y the number of cases of Reunion, and z the number of cases of Experience. Then

$$\text{Maximize Profit } P = 40x + 60y + 70z$$
$$\text{Subject to:}$$
$$x + 1.5y + 1.5z \leq 720 \text{ (hours on the molding machines)}$$
$$0.1x + 0.1y + 0.2z \leq 60 \text{ (hours packing time)}$$
$$x \geq 0, \quad y \geq 0, \quad z \geq 0$$

The Simplex Algorithm:

	x	y	z	S_1	S_2	$Capacity$
S_1	1	$\frac{3}{2}$	$\frac{3}{2}$	1	0	720
S_2	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{2}{10}$	0	1	60
P	-40	-60	-70	0	0	0

	x	y	z	S_1	S_2	$Capacity$
S_1	$\frac{1}{4}$	$\frac{3}{4}$	0	1	$-\frac{15}{2}$	270
z	$\frac{1}{2}$	$\frac{1}{2}$	1	0	5	300
P	-5	-25	0	0	350	21000

	x	y	z	S_1	S_2	$Capacity$
y	$\frac{1}{3}$	1	0	$\frac{4}{3}$	-10	360
z	$\frac{1}{3}$	0	1	$-\frac{2}{3}$	10	120
P	$\frac{10}{3}$	0	0	$\frac{100}{3}$	100	30000

The optimal data:

$x = 0$ cases Runaway glasses
$y = 360$ cases Reunion glasses
$z = 120$ cases Experience glasses
$P = \$30{,}000$ (maximized)
$S_1 = 0$ leftover hours molding time, $M_1 = \frac{100}{3} = \$33.33/\text{hr.}$
$S_2 = 0$ leftover hours packing time, $M_2 = \$100/\text{hr.}$

17. Solve the following with the simplex method (see exercise 7 from Section 3.5): The *Spooky Boogie Costume Salon* makes and sells four different Halloween costumes: the witch, the ghost, the goblin, and the werewolf. Each witch costume uses 3 yards material and takes 2 hours to sew. Each ghost costume uses 2 yards of material and takes 1 hours to sew. Each goblin costume uses 2 yards material and takes 3 hours to sew. Each werewolf costume uses 2 yards of material and takes 4 hours to sew. The profits for each costume are as follows: \$10 for the witch, \$8 for the ghost, \$12 for the goblin, and \$16 for the werewolf. If they have 600 yards of material and 510 sewing hours available before the holiday, how many of each costume should they make in order to maximize profit, assuming they sell everything they make?

Solution: Recall the set-up: Let w be the number of witch costumes, x the number o ghost costumes, y the number of goblin costumes, and z the number of werewolf costumes. Then

Maximize Profit $P = 10w + 8x + 12y + 16z$
Subject to:
$3w + 2x + 2y + 2z \leq 600$ (yards material)
$2w + x + 3y + 4z \leq 510$ (hours for sewing)
$w \geq 0,\ \ x \geq 0,\ \ y \geq 0,\ \ z \geq 0$

The Simplex Algorithm yields:

	w	x	y	z	S_1	S_2	$Capacity$
S_1	3	2	2	2	1	0	600
S_2	2	1	3	4	0	1	510
P	-10	-8	-12	-16	0	0	0

	w	x	y	z	S_1	S_2	$Capacity$
S_1	2	$\frac{3}{2}$	$\frac{1}{2}$	0	1	$-\frac{1}{2}$	345
z	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{4}$	1	0	$\frac{1}{4}$	$\frac{255}{2}$
P	-2	-4	0	0	0	4	2040

	w	x	y	z	S_1	S_2	$Capacity$
x	$\frac{4}{3}$	1	$\frac{1}{3}$	0	$\frac{2}{3}$	$-\frac{1}{3}$	230
z	$\frac{1}{6}$	0	$\frac{2}{3}$	1	$-\frac{1}{6}$	$\frac{1}{3}$	70
P	$\frac{10}{3}$	0	$\frac{4}{3}$	0	$\frac{8}{3}$	$\frac{8}{3}$	2960

The optimal data:

$w = 0$ witch costumes
$x = 230$ ghost costumes
$y = 0$ goblin costumes
$z = 70$ were wolf costumes
$P = \$2{,}960$ (maximized)
$S_1 = 0$ (leftover yards material), $M_1 = \$\frac{8}{3} = \$2.67/\text{yard}$
$S_2 = 0$ (leftover hours sewing time), $M_2 = \frac{8}{3} = \$2.67/\text{hr.}$

18. Consider the following variation on Angie and Kiki's lemonade stand (exercise 10 from Section 3.5). Suppose Kiki's idea was to include a third type of lemonade instead of a third ingredient? Thus. the resources are just lemons and sugar (no limes.) They have 60 lemons and 30 Tbsp. sugar available. In addition to the sweet lemonade (3 lemons and 2 Tbsp. sugar per glass), and the tart lemonade (4 lemons and 1 Tbsp. sugar per glass), they also make a Lite lemonade (2 lemons and $\frac{4}{3}$ Tbsp. sugar per glass.) The

prices are \$1.25 for a glass of sweet, \$1.50 for a glass of tart, and \$1.00 for a glass of Lite. How many glasses of each should they make in order to maximize their revenue? Use the Simplex Algorithm.

Solution: Recall the set-up: Let x be the number of glasses sweet, y the number of glasses of tart, and z the number of glasses of Lite. Then

$$\text{Maximize Revenue } R = 1.25x + 1.5y + z$$
$$\text{Subject to:}$$
$$3x + 4y + 2z \leq 60 \text{ (lemons)}$$
$$2x + y + \tfrac{4}{3}z \leq 30 \text{ (Tbsp. sugar)}$$
$$x \geq 0, \quad y \geq 0, \quad z \geq 0$$

The Simplex Algorithm yields:

	x	y	z	S_1	S_2	*Capacity*
S_1	3	4	2	1	0	60
S_2	2	1	$\frac{4}{3}$	0	1	30
R	$-\frac{5}{4}$	$-\frac{3}{2}$	-1	0	0	0

	x	y	z	S_1	S_2	*Capacity*
y	$\frac{3}{4}$	1	$\frac{1}{2}$	$\frac{1}{4}$	0	15
S_2	$\frac{5}{4}$	0	$\frac{5}{6}$	$-\frac{1}{4}$	1	15
R	$-\frac{1}{8}$	0	$-\frac{1}{4}$	$\frac{3}{8}$	0	$\frac{45}{2}$

	x	y	z	S_1	S_2	*Capacity*
y	0	1	0	$\frac{2}{5}$	$-\frac{3}{5}$	6
z	$\frac{3}{2}$	0	1	$-\frac{3}{10}$	$\frac{6}{5}$	18
R	$\frac{1}{4}$	0	0	$\frac{3}{10}$	$\frac{3}{10}$	27

The optimal data:

$x = 0$ glasses sweet lemonade
$y = 6$ glasses tart lemonade
$z = 18$ glasses Lite lemonade
$R = \$27.00$ (maximized)

$$S_1 = 0 \text{ (leftover lemons)}, \quad M_1 = \$0.30/\text{lemon}$$
$$S_2 = 0 \text{ (leftover Tbsp. sugar)}, \quad M_2 = \$0.30/\text{Tbsp. sugar}$$

19. Solve problem 11b from Section 3.5 (The *Shades of Grey Art Supplies* problem) using the simplex algorithm.

Solution: Recall the set-up: Let v be the number of tubes of Cadmium Red, w the number of tubes of Napthol Crimson, x the number of tubes of King Crimson, y the number of tubes of Venetian Red, and z the number of tubes of Providence Red. Then

$$\text{Maximize profit } P = 4v + 3w + 5x + 4y + 2z$$
$$\text{Subject to:}$$
$$50v + 40w + 40x + 40y + 50z \leq 10000 \text{ (} g \text{ pigment)}$$
$$80v + 100w + 60x + 50y + 60z \leq 15000 \text{ (} g \text{ binder)}$$
$$v \geq 0, \quad w \geq 0, \quad x \geq 0, \quad y \geq 0, \quad z \geq 0$$

The simplex algorithm yields:

	v	w	x	y	z	S_1	S_2	*Capacity*
S_1	50	40	40	40	50	1	0	10000
S_2	80	100	60	50	60	0	1	15000
P	-4	-3	-5	-4	-2	0	0	0

	v	w	x	y	z	S_1	S_2	*Capacity*
x	$\frac{5}{4}$	1	1	1	$\frac{5}{4}$	$\frac{1}{40}$	0	250
S_2	5	40	0	-10	-15	$-\frac{3}{2}$	1	0
P	$\frac{9}{4}$	2	0	1	$\frac{17}{4}$	$\frac{1}{8}$	0	1250

The optimal data:

$$v = 0 \text{ tubes Cadmium Red}$$
$$w = 0 \text{ tubes Napthol Crimson}$$
$$x = 250 \text{ tubes King Crimson}$$
$$y = 0 \text{ tubes Venetian Red}$$
$$z = 0 \text{ tubes Providence Red}$$
$$P = \$1{,}250 \text{ (maximized)}$$
$$S_1 = 0 \text{ leftover g pigment}, \quad M_1 = \frac{1}{8} = \frac{\$0.125}{g}$$
$$S_2 = 0 \text{ leftover g binder}, \quad M_2 = 0$$

Notice that the basic variable S_2 has value 0, indicating that this problem is degenerate.

20. Consider the following linear programming problem:

$$\text{Maximize } z = 8x + 7y$$
$$\text{Subject to:}$$
$$x + y \leq 17$$
$$x + 2y \leq 32$$
$$5x \leq 12y$$
$$x \geq 0, \quad y \geq 0$$

a) Convert the third inequality to standard form, and set up the initial tableau:

Solution: The inequality reads $5x - 12y \leq 0$. The initial tableau:

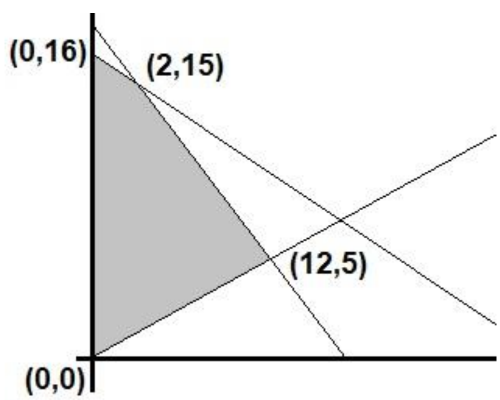
	x	y	S_1	S_2	S_3	$Capacity$
S_1	1	1	1	0	0	17
S_2	1	2	0	1	0	32
S_3	5	-12	0	0	1	0
z	-8	-7	0	0	0	0

b) Explain how you can tell this is a degenerate problem, by looking at the initial tableau.

Solution: The basic variable S_3 has value 0.

c) Graph the feasible region in the decision space. Explain why this is a degenerate problem, based on the graph.

Solution:



The problem is degenerate because three constraint lines are concurrent (at the origin.)

d) Solve it using the simplex algorithm.

Solution: After two pivots, we arrive at the final tableau:

	x	y	S_1	S_2	S_3	$Capacity$
y	0	1	$\frac{5}{17}$	0	$-\frac{1}{17}$	5
S_2	0	0	$-\frac{22}{17}$	1	$-\frac{63}{85}$	10
z	1	0	$\frac{12}{17}$	0	$\frac{1}{17}$	12
z	0	0	$\frac{131}{17}$	0	$\frac{1}{17}$	131

Optimal data:

$$x = 12$$
$$y = 5$$
$$z = 131 \text{ (maximized)}$$
$$S_1 = 0, \quad M_1 = \frac{131}{17}$$
$$S_2 = 10, \quad M_2 = 0$$
$$S_3 = 0, \quad M_3 = \frac{1}{17}$$

e) Solve by the method of graphing in the decision space. Your answer should agree with what you obtained in part (d).

Solution: Evaluate the objective function at the corners:

(x, y)	$z = 8x + 7y$
$(0, 0)$	0
$(0, 16)$	112
$(2, 15)$	121
(12,5)	131

The optimal data (minus the marginal values):

$$x = 12$$
$$y = 5$$
$$z = 131 \text{ (maximized)}$$
$$S_1 = 0$$
$$S_2 = 10$$
$$S_3 = 0$$

21. a) In example 5.7, in pivoting from the third to the fourth tableau, we mentioned that we could have still arrived at the optimal tableau (without any cycling occurring), had we mistakenly chosen the third row as the pivot row instead of the fourth. Verify this is true by performing the pivoting.
Solution: Pivoting had progressed to this point:

	x	y	S_1	S_2	S_3	S_4	C
y	0	1	$\frac{3}{25}$	$-\frac{1}{25}$	0	0	6
x	1	0	$-\frac{5}{25}$	$\frac{10}{25}$	0	0	10
S_3	0	0	$-\frac{10}{25}$	$-\frac{5}{25}$	1	0	0
S_4	0	0	$\frac{2}{25}$	$-\frac{9}{25}$	0	1	2
P	0	0	$-\frac{3}{25}$	$\frac{26}{25}$	0	0	54

Choosing the third row (incorrectly) as the pivot row, we obtain:

	x	y	S_1	S_2	S_3	S_4	C
y	0	1	0	$-\frac{1}{10}$	$\frac{3}{10}$	0	6
x	1	0	0	$\frac{5}{10}$	$-\frac{5}{10}$	0	10
S_1	0	0	1	$\frac{5}{10}$	$-\frac{25}{10}$	0	0
S_4	0	0	0	$-\frac{4}{10}$	$\frac{2}{10}$	1	2
P	0	0	0	$\frac{11}{10}$	$-\frac{3}{10}$	0	54

	x	y	S_1	S_2	S_3	S_4	C
y	0	1	0	$\frac{1}{2}$	0	$-\frac{3}{2}$	3
x	1	0	0	$-\frac{1}{2}$	0	$\frac{5}{2}$	15
S_1	0	0	1	$-\frac{9}{2}$	0	$\frac{25}{2}$	25
S_3	0	0	0	-2	1	5	10
P	0	0	0	$\frac{1}{2}$	0	$\frac{3}{2}$	57

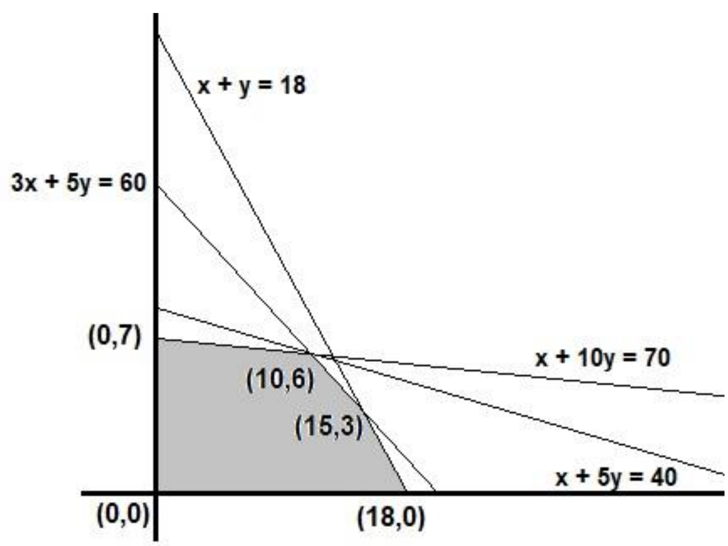
This matches the optimal tableau found in the text (bottom of page 206), except for the order of the rows (basic variables).

b) Solve the problem in example 5.7 by the method of graphing in the decision space, to verify that we obtained the correct answer by the simplex algorithm. Also, explain why, based on the graph, the problem is degenerate.

Solution: The set-up of Example 5.7 from the text is:

$$\text{Maximize } P = 3x + 4y$$
$$\text{Subject to:}$$
$$x + 10y \leq 70$$
$$3x + 5y \leq 60$$
$$x + 5y \leq 40$$
$$x + y \leq 18$$
$$x \geq 0, \quad y \geq 0$$

The feasible set:



Evaluate the objective function at the corners:

(x, y)	$P = 3x + 4y$
$(0, 0)$	0
$(0, 7)$	28
$(10, 6)$	54
$(15, 3)$	57
$(18, 0)$	54

The optimal point is (15,3) which agrees with the solution obtained via the simplex algorithm.

22. Consider the linear programming problem:

$$\text{Maximize } P = 2x + 2y$$
$$\text{Subject to:}$$
$$5x + 6y \leq 120$$
$$5x + 18y \leq 180$$
$$x - 2y \leq 40$$
$$x \geq 0, y \text{ unrestricted}$$

a) Solve it by the simplex algorithm, using the technique described in the paragraph at the end of the section.

Solution: Let $y = u - v$, where $u, v \geq 0$. Then rewrite the setup as:

$$\text{Maximize } P = 2x + 2u - 2v$$
$$\text{Subject to:}$$
$$5x + 6u - 6v \leq 120$$
$$5x + 18u - 18v \leq 180$$
$$x - 2u + 2v \leq 40$$
$$x \geq 0, \quad u \geq 0, \quad v \geq 0$$

The Simplex Algorithm yields:

	x	u	v	S_1	S_2	S_3	$Capacity$
S_1	5	6	−6	1	0	0	120
S_2	5	18	−18	0	1	0	180
S_3	1	−2	2	0	0	1	40
P	−2	−2	2	0	0	0	0

	x	u	v	S_1	S_2	S_3	$Capacity$
x	1	$\frac{6}{5}$	$-\frac{6}{5}$	$\frac{1}{5}$	0	0	24
S_2	0	12	−12	−1	1	0	60
S_3	0	$-\frac{16}{5}$	$\frac{16}{5}$	$-\frac{1}{5}$	0	1	16
P	0	$\frac{2}{5}$	$-\frac{2}{5}$	$\frac{2}{5}$	0	0	48

	x	u	v	S_1	S_2	S_3	$Capacity$
x	1	0	0	$\frac{1}{8}$	0	$\frac{3}{8}$	30
S_2	0	0	0	$-\frac{7}{4}$	1	$\frac{15}{4}$	120
v	0	-1	1	$-\frac{1}{16}$	0	$\frac{5}{16}$	5
P	0	0	0	$\frac{3}{8}$	0	$\frac{1}{8}$	50

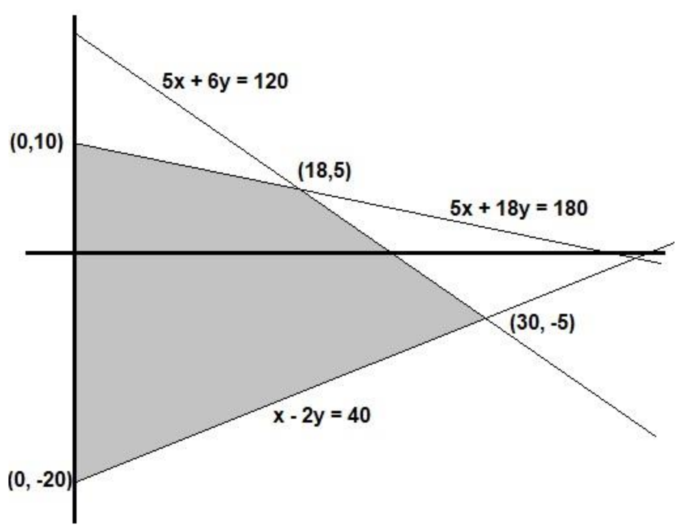
The optimal data is:

$$x = 30$$
$$u = 0$$
$$v = 5$$
$$P = 50 \text{ (maximized)}$$
$$S_1 = 0, \quad M_1 = \frac{3}{8}$$
$$S_2 = 120, \quad M_2 = 0$$
$$S_3 = 0, \quad M_3 = \frac{1}{8}$$

Therefore, since we defined $y = u - v$, the solution to the original problem is $(x, y) = (30, -5), P = 50$, and the rest of the optimal data is as above.

b) Solve it by graphing in the decision space and check that your answer agrees with what you got in part (a).

Solution: The feasible set:



Evaluate the objective function at the corners:

(x, y)	$P = 2x + 2y$
$(0, -20)$	-40
$(0, 10)$	20
$(18, 5)$	46
$(30, -5)$	50

Clearly, the optimal solution is $(x, y) = (30, -5), P = 50$, which agrees with part (a). The reader may check that the rest of the optimal data (slack variable values and marginal values) also agree with part (a).

23. Consider problem 13 from Section 3.4: The *Jefferson Plastic Fantastic Assembly Corporation* manufactures gadgets and widgets for airplanes and starships. Each case of gadgets uses 2 kg. steel and 5 kg. plastic. Each case of widgets uses 2 kg. steel and 3 kg. plastic. The profit for a case of gadgets is \$360 and the profit for a case of widgets is \$200. Suppose they have 80 kg. steel available and 150 kg. plastic available on a daily basis, and can sell everything they manufacture. How many cases of each should they manufacture if they are obligated to produce at least 10 cases of widgets per day? The objective is to maximize daily profit.

a) Let x be the number of cases of gadgets and y the number of cases of widgets. Write down the set-ip of the problem.

Solution: The set-up:

Maximize $P = 360x + 200y$
Subject to:
$2x + 2y \leq 80$ (kg steel)
$5x + 3y \leq 150$ (kg plastic)
$y \geq 10$ (contractual obligation)
$x \geq 0, \quad y \geq 0$

b) Notice that the obligation to produce at least 10 cases of widgets converts to an inequality of the form $y \geq 10$, so this problem has mixed constraints. That is, it is not a standard form maximization problem. As stated in the text, once Phase I pivoting is learned, problems like this can be solved using the full Simplex Algorithm, with both phases. However, here, taking out cue from the paragraph at the end of the section, we can solve this problem with just phase II pivoting by a substitution. Simply define a new variable $z = y - 10$. In the objective function, and in all of the constraints, rewrite y in terms of z. You may also find it convenient to make a substitution in the objective variable P so that the objective function has no constant term. Note that the constraint $y \geq 10$ reduces to the non-negativity constraint $z \geq 0$. In particular, not only is the new problem now a standard form maximization problem, but it has one fewer structural constraint than the original problem! Solve the resulting problem via Algorithm 5.1 (phase II pivoting), and thereby obtain a solution to the original problem as well.

Solution: Rewriting in terms of $z = y - 10$, we have:

$$\text{Maximize } P = 360x + 200z + 2000$$
$$\text{Subject to:}$$
$$2x + 2z \leq 60 \text{ (steel)}$$
$$5x + 3z \leq 120 \text{ (plastic)}$$
$$x \geq 0, \quad z \geq 0$$

Furthermore, (following the suggestion), if we define $Q = P - 2000 = 360x + 200z$, then Q and P differ by a constant, so when one is maximized, so is the other. So, we can maximize Q, using algorithm 5.1:

	$\boldsymbol{x}$	$\boldsymbol{z}$	$\boldsymbol{S_1}$	$\boldsymbol{S_2}$	$\boldsymbol{Capacity}$
$\boldsymbol{S_1}$	2	2	1	0	60
$\boldsymbol{S_2}$	5	3	0	1	120
$\boldsymbol{Q}$	-360	-200	0	0	0

	$\boldsymbol{x}$	$\boldsymbol{z}$	$\boldsymbol{S_1}$	$\boldsymbol{S_2}$	$\boldsymbol{Capacity}$
$\boldsymbol{S_1}$	0	$\frac{4}{5}$	1	$-\frac{2}{5}$	12
$\boldsymbol{x}$	1	$\frac{3}{5}$	0	$\frac{1}{5}$	24
$\boldsymbol{Q}$	0	16	0	72	8640

The optimal data:

$$x = 24 \text{ cases gadgets}$$
$$z = 0, \text{ so } y = 10 \text{ cases widgets}$$
$$Q = 8640\ (maximized), \text{ so } P = \$10{,}640$$
$$S_1 = 12 \text{ leftover kg steel}, \quad M_1 = 0$$
$$S_2 = 0 \text{ leftover kg plastic}, \quad M_2 = \$72/\text{kg}$$

This agrees with the previous solution (exercise 13 in Section 3.4.)

Section 5.3 Standard Form Minimization Problems

Solutions to exercises:

1. Perform the pivoting needed in the O'Casek's used car example and verify that the final tableau given above (in the text, page 218) is correct.

Solution: The simplex algorithm yields:

	u_1	u_2	u_3	T_1	T_2	$Capacity$
T_1	1	2	4	1	0	6
T_2	2	1	1	0	1	4
P	-24	-30	-40	0	0	0

	u_1	u_2	u_3	T_1	T_2	$Capacity$
u_3	$\frac{1}{4}$	$\frac{1}{2}$	1	$\frac{1}{4}$	0	$\frac{3}{2}$
T_2	$\frac{7}{4}$	$\frac{1}{2}$	0	$-\frac{1}{4}$	1	$\frac{5}{2}$
P	-14	-10	0	10	0	60

	u_1	u_2	u_3	T_1	T_2	$Capacity$
u_3	0	$\frac{3}{7}$	1	$\frac{2}{7}$	$-\frac{1}{7}$	$\frac{8}{7}$
u_1	1	$\frac{2}{7}$	0	$-\frac{1}{7}$	$\frac{4}{7}$	$\frac{10}{7}$
P	0	-6	0	8	8	80

	u_1	u_2	u_3	T_1	T_2	$Capacity$
u_2	0	1	$\frac{7}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{8}{3}$
u_1	1	0	$-\frac{2}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$
P	0	0	14	12	6	96

This agrees with the final tableau given in the text.

2. In the O'Casek used car example, (i) write out the set-up of the dual maximization problem in terms of inequalities; and (ii) write out the complete solution to this maximization problem by reading it off from the final tableau.

Solution: The set-up:

$$\text{Maximize } P = 24u_1 + 30u_2 + 40u_3$$
$$\text{Subject to:}$$
$$u_1 + 2u_2 + 4u_3 \le 6$$
$$2u_1 + u_2 + u_3 \le 4$$
$$u_1 \ge 0, \quad u_2 \ge 0, \quad u_3 \ge 0$$

The solution to this problem is given by the final tableau in the previous exercise:

$$u_1 = \frac{2}{3}$$
$$u_2 = \frac{8}{3}$$
$$u_3 = 0$$
$$P = 96 \text{ (maximized)}$$
$$T_1 = 0, \quad M_1 = 12$$
$$T_2 = 0, \quad M_2 = 6$$

3. Solve the minimization problem using the simplex algorithm (see exercise 8 from Section 3.1):

$$\text{Minimize } C = 20x + 12y$$
$$\text{Subject to}$$
$$5x + 2y \ge 30$$
$$5x + 7y \ge 70$$
$$x \ge 0, \qquad y \ge 0$$

Solution: Note that this minimization problem was solved graphically in exercise 4 from Section 3.4. You can compare the answer there to the one we now obtain via duality:

The set-up of the dual problem is

$$\text{Maximize } P = 30u + 70v$$
$$\text{Subject to:}$$
$$5u + 5v \le 20$$
$$2u + 7v \le 12$$
$$u \ge 0, \ \ v \ge 0$$

The simplex algorithm yields:

	$\boldsymbol{u}$	$\boldsymbol{v}$	$\boldsymbol{T_1}$	$\boldsymbol{T_2}$	$\boldsymbol{Capacity}$
$\boldsymbol{T_1}$	5	5	1	0	20
$\boldsymbol{T_2}$	2	7	0	1	12
$\boldsymbol{P}$	-30	-70	0	0	0

	u	v	T_1	T_2	$Capacity$
T_1	$\frac{25}{7}$	0	1	$-\frac{5}{7}$	$\frac{80}{7}$
v	$\frac{2}{7}$	1	0	$\frac{1}{7}$	$\frac{12}{7}$
P	-10	0	0	10	120

	u	v	T_1	T_2	$Capacity$
u	1	0	$\frac{7}{25}$	$-\frac{1}{5}$	$\frac{16}{5}$
v	0	1	$-\frac{2}{25}$	$\frac{1}{5}$	$\frac{4}{5}$
P	0	0	$\frac{14}{5}$	8	152

The solution to the original minimization is therefore:

$$x = \frac{14}{5}$$
$$y = 8$$
$$C = 152 \text{ (minimized)}$$
$$S_1 = 0, \quad M_1(= u) = \frac{16}{5}$$
$$S_2 = 0, \quad M_2(= v) = \frac{4}{5}$$

This agrees with the answer obtained in exercise 4 of Section 3.4.

4. Solve the *Toys in the Attic* problem (exercise 13 from Section 3.1) using the simplex algorithm.

Solution: Recall the set-up: Let x be the number of days to operate Mr. Tyler's shop, and y the number of days to operate Mr. Perry's shop.

Minimize cost $C = 144x + 166y$
Subject to:
$36x + 10y \geq 720$ (Angel dolls)
$16x + 10y \geq 520$ (Kings and Queens board games)
$16x + 20y \geq 640$ (Back in the Saddle Rocking Horses)
$x \geq 0, \quad y \geq 0$

The dual problem:

$$\text{Maximize } P = 720u + 520v + 640w$$
$$\text{Subject to:}$$
$$36u + 16v + 16w \le 144$$
$$10u + 10v + 20w \le 166$$
$$u \ge 0, \quad v \ge 0, \quad w \ge 0$$

The simplex algorithm yields:

	u	v	w	T_1	T_2	$Capacity$
T_1	36	16	16	1	0	144
T_2	10	10	20	0	1	166
P	−720	−520	−640	0	0	0

After three pivots, we arrive at the optimal tableau:

	u	v	w	T_1	T_2	$Capacity$
v	$\frac{7}{2}$	1	0	$\frac{1}{8}$	$-\frac{1}{10}$	$\frac{7}{5}$
w	$-\frac{5}{4}$	0	1	$-\frac{1}{16}$	$\frac{1}{10}$	$\frac{38}{5}$
P	300	0	0	25	12	5592

Thus, the solution to the original (primal) problem is:

$x = 25$ days to run Mr. Tyler's shop
$y = 12$ days to run Mr. Perry's shop
$C = \$5{,}592$ (minimized)
$S_1 = 300$ surplus Angel dolls, $M_1 = u = 0$
$S_2 = 0$ surplus Kings and Queens board games, $M_2 = v = \frac{7}{5} = \$1.40$/board game
$S_3 = 0$ surplus Back in the Saddle rocking horses, $M_3 = w = \frac{38}{5} = \$7.60$/rocking horse

This agrees with our previous solution.

5. Solve the *Poseidon's Wake Petroleum Company* problem (exercise 10 from Section 3.2) using the simple algorithm.

Solution: Recall the Set-up: Let x be the number of days to operate the Cadence refinery, and y the number of days to operate the Cascade refinery. Then

$$\text{Minimize Cost } C = 1800x + 2000y$$
$$\text{Subject to:}$$
$$40x + 10y \ge 80 \text{ (units low grade oil)}$$

$$10x + 10y \geq 50 \text{ (units medium grade oil)}$$
$$10x + 30y \geq 90 \text{ (units high grade oil)}$$
$$x \geq 0, \quad y \geq 0$$

The dual problem is:

$$\text{Maximize } P = 80u + 50v + 90w$$
$$\text{Subject to:}$$
$$40u + 10v + 10w \leq 1800$$
$$10u + 10v + 30w \leq 2000$$
$$u \geq 0, \quad v \geq 0, \quad w \geq 0$$

	u	v	w	T_1	T_2	$Capacity$
T_1	40	10	10	1	0	1800
T_2	10	10	30	0	1	2000
P	-80	-50	-90	0	0	0

After 3 pivots, we arrive at the optimal tableau:

	u	v	w	T_1	T_2	$Capacity$
v	$\frac{11}{2}$	1	0	$\frac{3}{20}$	$-\frac{1}{20}$	170
w	$-\frac{3}{2}$	0	1	$-\frac{1}{20}$	$\frac{1}{20}$	10
P	60	0	0	3	2	9400

Thus, the solution to the original (primal) problem is:

$x = 3$ days to operate the Cadence refinery
$y = 2$ days to operate the Cascade refinery
$C = \$9{,}400$ (minimized)
$S_1 = 60$ surplus units low grade oil, $M_1 = u = 0$
$S_2 = 0$ surplus units medium grade oil, $M_2 = v = \$170/\text{unit}$
$S_3 = 0$ surplus units high grade oil, $M_3 = w = \$10/\text{unit}$

This agrees with our previous solution.

6. Solve part (a) of the *Topographic Starship Tour Company* problem (exercise 20 from Section 3.2) using the simplex algorithm.

Solution: Recall the set-up: Let x be the number of Relayers and y the number of Khatrus. Then

$$\text{Minimize Cost } C = 8000x + 11500y$$
Subject to:
$$6x + 10y \geq 58 \geq \text{(fist class tourists)}$$
$$10x + 4y \geq 40 \text{ (economy class tourists)}$$
$$600x + 400y \geq 3400 \text{ (kg luggage and supplies)}$$
$$x \geq 0, \quad y \geq 0$$

The dual problem:

$$\text{Maximize } P = 58u + 40v + 3400w$$

Subject to:

$$6u + 10v + 600w \leq 8000$$
$$10u + 4v + 400w \leq 11500$$
$$u \geq 0, \quad v \geq 0, \quad w \geq 0$$

The simplex algorithm yields:

	u	v	w	T_1	T_2	$Capacity$
T_1	6	10	600	1	0	8000
T_2	10	4	400	0	1	11500
P	-58	-40	-3400	0	0	0

After two pivots, we arrive at the optimal tableau:

	u	v	w	T_1	T_2	$Capacity$
w	0	$\frac{19}{900}$	1	$\frac{1}{360}$	$-\frac{1}{600}$	$\frac{55}{18}$
u	1	$-\frac{4}{9}$	0	$-\frac{1}{9}$	$\frac{1}{6}$	$\frac{9250}{9}$
P	0	6	0	3	4	70000

Thus, the solution to the (primal) problem is:

$x = 3$ Relayers
$y = 4$ Khatrus
$C = \$70,000$ (Minimized)

$S_1 = 0$ surplus spaces for first class tourists, $\quad M_1 = u = \frac{9250}{9} \approx \$1.027.78$/first class ticket
$S_2 = 6$ surplus spaces for economy class tourists, $\quad M_2 = v = 0$
$S_3 = 0$ surplus kg luggage/supplies, $\quad M_3 = w = \frac{55}{18} \approx \$36.00$/kg

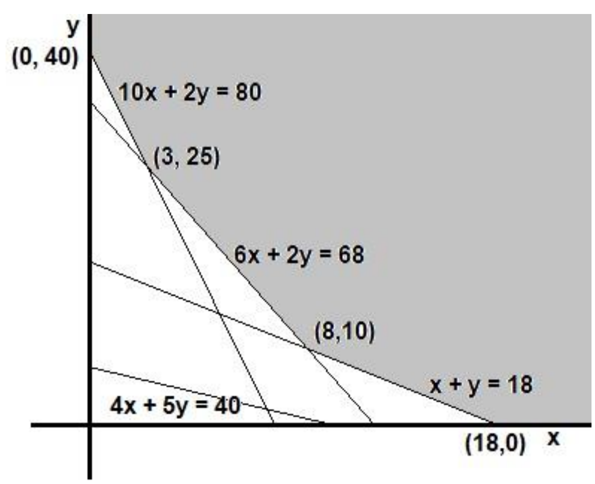
This agrees with our previous solution.

7. Consider the minimization problem:

$$\text{Minimize } C = 60x + 40y$$
Subject to:
$$10x + 2y \geq 80$$
$$6x + 2y \geq 68$$
$$4x + 5y \geq 40$$
$$x + y \geq 18$$
$$x \geq 0, \quad y \geq 0$$

a) Solve the problem via graphing in the decision space. One of the constraints has an unusual property – what is it?

Solution: The feasible set:



The third constraint, $4x + 5y \geq 40$, is superfluous. Evaluate the objective function at the corners:

(x, y)	$C = 60x + 40y$
$(0, 40)$	1600
$(3, 25)$	1180
$(8, 10)$	880
$(18, 0)$	1080

The optimal solution (except for the marginal values) is:

$$x = 8$$

$$y = 10$$
$$C = 880 \text{ (Minimized)}$$
$$S_1 = 20$$
$$S_2 = 0$$
$$S_3 = 42$$
$$S_4 = 0$$

b) Solve the problem via the simplex algorithm. Your solution should agree with what you got in part (a).

Solution: The dual problem is:

$$\text{Maximize } P = 80u + 68v + 40w + 18z$$
$$\text{Subject to:}$$
$$10u + 6v + 4w + z \leq 60$$
$$2u + 2v + 5w + z \leq 40$$
$$u \geq 0,\ \ v \geq 0,\ \ w \geq 0,\ \ z \geq 0$$

The simplex algorithm yields:

	u	v	w	z	T_1	T_2	$Capacity$
T_1	10	6	4	1	1	0	60
T_2	2	2	5	1	0	1	40
P	-80	-68	-40	-18	0	0	0

After three pivots, we arrive at the final tableau:

	u	v	w	z	T_1	T_2	$Capacity$
v	2	1	$-\frac{1}{4}$	0	$\frac{1}{4}$	$-\frac{1}{4}$	5
z	-2	0	$\frac{11}{2}$	1	$-\frac{1}{2}$	$\frac{3}{2}$	30
P	20	0	42	0	8	10	880

So, the solution to the primal problem is:

$$x = 8$$
$$y = 10$$
$$C = 880 \text{ (Minimized)}$$
$$S_1 = 20, \quad M_1 = u = 0$$
$$S_2 = 0, \quad M_2 = v = 5$$
$$S_3 = 42, \quad M_3 = w = 0$$
$$4 = 0, \quad M_4 = z = 30$$

This agrees with part (a).

8. A dietitian at a hospital must design a meal for a vitamin deficient patient. Each oz. of mashed potatoes has 1 unit protein, 6 units vitamins, and 2 units fat. Each oz. of broccoli has 2 units protein, 2 units vitamins, and 1 unit fat. Each oz. of chicken breast has 8 units protein, 1 unit vitamins, and 1 unit fat. Each oz. of corn has 3 units protein, 5 units vitamins, and 1 unit fat. The meal must consist of at least 60 units protein and at least 40 units vitamins. Also, it should contain at least 12 oz. food altogether. Design a diet that minimizes the fat content. Use the simplex algorithm.

Solution: Let w be the number of oz. of mashed potatoes, x the number of oz. of broccoli, y the number of oz. of chicken breast, and z the number of ounces of corn. Then

$$\text{Minimize fat } F = 2w + x + y + 2z$$
$$\text{Subject to:}$$
$$6w + 2x + 8y + 3z \geq 60 \text{ (units protein)}$$
$$6w + 2x + y + 5z \geq 40 \text{ (units vitamins)}$$
$$w + x + y + z \geq 12 \text{ (oz. food)}$$
$$w \geq 0, \quad x \geq 0, \quad y \geq 0, \quad z \geq 0$$

The dual problem:

$$\text{Maximize } P = 60r + 40s + 12t$$
$$\text{Subject to:}$$
$$6r + 6s + t \leq 2$$
$$2r + 2s + t \leq 1$$
$$8r + s + t \leq 1$$
$$3r + 5s + t \leq 2$$
$$r \geq 0, \quad s \geq 0, \quad t \geq 0$$

The simplex algorithm yields:

	r	s	t	T_1	T_2	T_3	T_4	*Capacity*
T_1	6	6	1	1	0	0	0	2
T_2	2	2	1	0	1	0	0	1
T_3	8	1	1	0	0	1	0	1
T_4	3	5	1	0	0	0	1	2
P	-60	-40	-12	0	0	0	0	0

After three pivots, we arrive at the final tableau:

	r	s	t	T_1	T_2	T_3	T_4	$Capacity$
s	0	1	0	$\frac{3}{14}$	$-\frac{1}{14}$	$-\frac{1}{7}$	0	$\frac{3}{14}$
t	0	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	0	0	$\frac{1}{2}$
r	1	0	0	$\frac{1}{28}$	$-\frac{5}{28}$	$\frac{1}{7}$	0	$\frac{1}{28}$
T_4	0	0	0	$-\frac{19}{28}$	$-\frac{17}{28}$	$\frac{2}{7}$	1	$\frac{9}{28}$
P	0	0	0	$\frac{33}{7}$	$\frac{31}{7}$	$\frac{20}{7}$	0	$\frac{117}{7}$

The solution to the primal problem is:

$$w = \frac{33}{7} \approx 4.71 \text{ oz. mashed potatoes}$$
$$x = \frac{31}{7} \approx 4.43 \text{ oz. broccoli}$$
$$y = \frac{20}{7} \approx 2.86 \text{ oz. chicken breast}$$
$$z = 0 \text{ oz. corn}$$
$$F = \frac{117}{7} \approx 16.71 \text{ units fat (Miniized)}$$
$$S_1 = 0 \text{ surplus unit protein}, \quad M_1 = r = \frac{1}{28}$$
$$S_2 = 0 \text{ surplus units vitamins}, \quad M_2 = s = \frac{3}{14}$$
$$S_3 = 0 \text{ surplus oz. food}, \quad M_3 = t = \frac{1}{2}$$

Section 5.4 Solving Linear Programming Problems with a Computer

Music Reference in the text: The example of the Tillerman Tea Company (Example 5.11, page 223) is homage to singer/songwriter Cat Stevens. His 1970 album was entitled Tea for the Tillerman, and included the song Wild World. He also released songs entitled Peace Train and Moonshadow, as well as an album in 1972 entitled Catch Bull at Four. (Website: https://catstevens.com/)

Solutions to exercises:

1. a) Use *Mathematica* to solve the problem which was set up in exercise 11 from Section 3.1, which we reproduce here for convenience: Bruce Jax is responsible for constructing end tables and kitchen tables for the *White Room Woodshop*. Each end table uses 2 square yards of $\frac{3}{4}$ inch oak boards and takes two hours to complete. Each kitchen table uses 4 square yards of oak boards and takes 3 hours to complete. This week he has available 36 square yards of oak boards and 32 hours of time. Other resources are unlimited. How many of each item should he make if he is paid \$70 for each end table and \$100 for each kitchen table?

Then use the **DualLinearProgramming** command in order to determine the marginal values of the resources.

Solution: Recall the set-up: Let x be the number of end tables and y the number of kitchen tables.

$$\text{Maximize pay } P = 70x + 100y$$
$$\text{Subject to:}$$
$$2x + 4y \leq 36 \text{ (square yards wood)}$$
$$2x + 3y \leq 32 \text{ (hours)}$$
$$x \geq 0, \quad y \geq 0$$

We must convert this to a minimization problem. One approach would be to take the dual problem. In the text we suggested instead negating the objective function. So we want to Minimize $-p = -70x - 100y$. The correct Mathematic command is

$$\textbf{LinearPrograming}[\{-70, -100\}, \{\{2, 4\}, \{2, 3\}\}, \{\{36, -1\}, \{32, -1\}\}]$$

Mathematica returns $\{16,0\}$. So $x = 16$ and $y = 0$. The pay is $70(16) + 100(0) = 1120$. To find leftover resources, $2(16) + 4(0) = 32$ yards wood, so $S_1 = 36 - 32 = 4$ yards leftover wood. Similarly, $S_2 = 32 - 32 = 0$ leftover hours.

To get the marginal values, we need the dual linear programming command:

$$\textbf{DualLinearPrograming}[\{-70, -100\}, \{\{2, 4\}, \{2, 3\}\}, \{\{36, -1\}, \{32, -1\}\}]$$

Mathematic returns $\{\{16,0\},\{0,-35\},\{0,5\},\{0,0\}\}$. Dropping the negatives for the marginal values, we obtain $M_1 = \$0$/yard wood, and $M_2 = \$35$/hr.

b) Use *Excel* to solve this problem. Obtain the complete solution, including the marginal values, from the Answer and Sensitivity Reports.

Solution: Follow the steps outlined in the text – you will of course obtain the same answer as in part (a).

2. Same directions as problem 1 (both parts a and b) for the following problem (see exercise 12 of Section 3.1): Joni is putting her designing skills to work at the *Court and Sparkle Jewelry Emporium*. She makes two signature design bracelet models. Each Dawntreader bracelet uses 1 ruby, 6 pearls, and 10 opals. Each Hejira bracelet uses 3 rubies, 3 pearls, and 15 opals. She has 54 rubies, 120 pearls, and 300 opals to work with. If either model results in a profit of $1,800, how many of each type of bracelet should she make in order to maximize her profit?

Solution: Recall the set-up: Let x be the number of Dawntreader bracelets, and y the number of Hejira bracelets.

$$\text{Maximize profit } P = 1800x + 1800y$$
$$\text{Subject to:}$$
$$x + 3y \le 54 \text{ (rubies)}$$
$$6x + 3y \le 120 \text{ (pearls)}$$
$$10x + 15y \le 300 \text{ (opals)}$$
$$x \ge 0, \quad y \ge 0$$

Again, to use Mathematica we must convert it to a minimization problem. The command is

DualLinearProgramming$[\{-\mathbf{1800},-\mathbf{1800}\},\{\{\mathbf{1},\mathbf{3}\},\{\mathbf{6},\mathbf{3}\},\{\mathbf{10},\mathbf{15}\}\},\{\{\mathbf{54},-\mathbf{1}\},\{\mathbf{120},-\mathbf{1}\},\{\mathbf{300},-\mathbf{1}\}\}]$

Mathematica returns: $\{\{15,10\},\{0,-150,-90\},\{0,0\},\{0,0\}\}$. Thus,

$$x = 15 \text{ Dawntreader bracelets}$$
$$y = 10 \text{ Hejira bracelets}$$
$$P = 1800(15) + 1800(10) = \$45{,}000 \text{ (Maximized)}$$
$$M_1 = \$0/\text{ruby}$$
$$M_2 = \$150/\text{pearl}$$
$$M_3 = \$90/\text{opal.}$$

Since $M_1 = 0$, we expect there to be leftover rubies, but since the other marginal values are nonzero, the corresponding slack variables must be 0 by complementary slackness, and so there are no leftover pearls or opals. Since $1(15) + 3(10) = 45$ rubies are used, we have

$$S_1 = 54 - 45 = 9 \text{ surplus rubies}$$
$$S_2 = 0 \text{ surplus pearls}$$
$$S_3 = 0 \text{ surplus opals}$$

We leave the reader to check that Excel gives the same answer.

3. Same directions as problem 1 (both parts a and b) for the following problem (see exercise 13 of Section 3.1): *Toys in the Attic, Inc*. operates two workshops to build toys for needy children. Mr. Tyler's shop can produce 36 *Angel* dolls, 16 *Kings and Queens* board games, and 16 *Back in the Saddle* rocking horses each day it operates. Mr. Perry's shop can produce 10 *Angel* dolls, 10 *Kings and Queens* board games, and 20 *Back in the Saddle* rocking horses each day it operates. It costs $144 to operate Mr. Tyler's shop for one day and $166 to operate Mr. Perry's shop for one day. Suppose the company receives an order from *Kids Dream On* Charity Foundation for at least 720 *Angel* dolls, at least 520 *Kings and Queens* board games, and at least 640 *Back in the Saddle* rocking horses. How many days should they operate each shop in order to fill the order at least possible cost?

Solution: Recall the set-up: Let x be the number of days to operate Mr. Tyler's shop, and y the number of days to operate Mr. Perry's shop.

Minimize cost $C = 144x + 166y$
Subject to:
$36x + 10y \geq 720$ (Angel dolls)
$16x + 10y \geq 520$ (Kings and Queens board games)
$16x + 20y \geq 640$ (Back in the Saddle rocking horses)
$x \geq 0, \quad y \geq 0$

Since this already is a minimization problem, we do not multiply through by -1. The command is

DualLinearProgramming$[\{\mathbf{144}, \mathbf{166}\}, \{\{\mathbf{36}, \mathbf{10}\}, \{\mathbf{16}, \mathbf{10}\}, \{\mathbf{16}, \mathbf{20}\}\}, \{\mathbf{720}, \mathbf{520}, \mathbf{640}\}]$

Mathematica returns $\left\{\{25,12\}, \left\{0, \frac{7}{5}, \frac{38}{5}\right\}, \{0,0\}, \{0,0\}\right\}$. Thus:

$x = 25$ days to operate Mr. Tyler's shop
$y = 12$ days to operate Mr. Perry's shop
$C = 144(25) + 166(12) = \$5{,}592$ (Minimized)

$$M_1 = 0$$
$$M_2 = \frac{7}{5}$$
$$M_3 = \frac{38}{5}$$

Again, by complementary slackness, we could only have possible leftover Angel dolls. To find the number: $36(25) + 10(12) = 1020$ Angel dolls are produced, so

$S_1 = 1020 - 720 = 300$ surplus Angel dolls
$S_2 = 0$ surplus Kings and Queens board games
$S_3 = 0$ surplus Back in the Saddle rocking horses

Excel gives the same answer.

4. Same directions as problem 1 (both parts a and b) for the following diet mix problem (Problem 9 of section 3.4): *Kerry's Kennel* is mixing two commercial brands of dog food for its canine guests. A bag of *Dog's Life Canine Cuisine* contains 3 lbs. fat, 2 lbs. carbohydrates, 5 lbs. protein, and 3 oz. vitamin C. A giant size bag of *Way of Life Healthy Mix* contains 1 lb. fat, 5 lbs. carbohydrates, 10 lbs. protein, and 7 oz. vitamin C. The requirements for a weeks' supply of food for the kennel are that there should be at most 21 lbs. fat, at mist 40 lbs. carbohydrates, and at least 21 oz. vitamin C. How many bags of each type of food should be mixed in order to design a diet that maximizes protein?

Solution: Recall the set-up: Let x be the number of bags of Dog's Life Canine Cuisine, and y the number of bags of Way of Life Healthy Mix. Then

$$\begin{gathered}\text{Maximize protein } P = 5x + 10y \\ \text{Subject to:} \\ 3x + y \leq 21 \text{ (lbs. fat)} \\ 2x + 5y \leq 40 \text{ (lbs. carbohydrates)} \\ 3x + 7y \geq 21 \text{ (oz. vitamin C)} \\ x \geq 0, \quad y \geq 0\end{gathered}$$

The problem has mixed constraints, leading to the following Mathematic command:

$$\textbf{DualLinearProgramming}[\{-\mathbf{5}, -\mathbf{10}\}, \{\{\mathbf{3}, \mathbf{1}\}, \{\mathbf{2}, \mathbf{5}\}, \{\mathbf{3}, \mathbf{7}\}\}, \{\{\mathbf{21}, -\mathbf{1}\}, \{\mathbf{40}, -\mathbf{1}\}, \{\mathbf{21}, \mathbf{1}\}\}]$$

Mathematica returns $\left\{\{5,6\}, \left\{-\frac{5}{13}, -\frac{25}{13}, 0\right\}, \{0,0\}, \{0,0\}\right\}$. Thus

$$\begin{gathered}x = 5 \text{ bags Dog's Life Canine Cuisine} \\ y = 6 \text{ bags Way of Life Healthy Mix} \\ P = 5(5) + 10(6) = 85 \\ M_1 = \frac{5}{13} \text{ lbs protein/lb fat} \\ M_2 = \frac{25}{13} \text{ lbs protein/lb carbohydrate} \\ M_3 = 0 \text{ lbs protein/oz. vitamin C}\end{gathered}$$

We know S_3 is the only slack variable that could be nonzero by complementary slackness. We have $3(5) + 7(6) = 57$, so:

$$\begin{gathered}S_1 = 0 \\ S_2 = 0 \\ S_3 = 57 - 21 = 36 \text{ surplus oz. vitamin C.}\end{gathered}$$

Excel gives the same answer.

5. Dame directions as problem 1 (both parts a and b) for the following scheduling problem (Problem 10 of Section 3.4.) The *Poseidon's Wake* petroleum company operates two refineries. The *Cadence Refinery* can produce 40 units of low grade oil, 10 units medium grade oil, and 10 units high grade oil in a single day. (Each unit is 1000 barrels.) The *Cascade Refinery* can produce 10 units low grade oil, 10 units medium grade oil, and 30 units high grade oil in a single day. They receive an order from the Mars

Triangle Oil Retailers for at least 80 units low grade oil, at least 50 units medium grade oil, and at least 90 units high grade oil. If it costs Poseidon's Wake $1,800 to operate the Cadence refinery for a day, and $2,000 to operate the Cascade Refinery for a day, how many days should they operate each refinery to fill the order at least cost?

Solution: Recall the set-up: Let x be the number of days to operate Cadence refinery, and y the number of days to operate Cascade refinery. Then

$$\begin{gathered}\text{Minimize Cost } C = 1800x + 2000y \\ \text{Subject to:} \\ 40x + 10y \geq 80 \text{ (units low grade oil)} \\ 10x + 10y \geq 50 \text{ (units medium grade oil)} \\ 10x + 30y \geq 90 \text{ (units high grade oil)} \\ x \geq 0, \quad y \geq 0\end{gathered}$$

This is a standard form minimization, so the Mathematica command is

DualLinearProgramming$[\{\mathbf{1800}, \mathbf{2000}\}, \{\{\mathbf{40}, \mathbf{10}\}, \{\mathbf{10}, \mathbf{10}\}, \{\mathbf{10}, \mathbf{30}\}\}, \{\mathbf{80}, \mathbf{50}, \mathbf{90}\}]$

Mathematica returns $\{\{3,2\}, \{0,170,10\}, \{0,0\}, \{0,0\}\}$. So the solution is

$$\begin{gathered}x = 3 \text{ days to run Cadence refinery} \\ y = 2 \text{ days to operate Cascade refinery} \\ C = 1800(3) + 2000(2) = \$9{,}400 \text{ (Minimized)} \\ M_1 = 0 \\ M_2 = 170 \\ M_3 = 10\end{gathered}$$

Again, by complementary slackness, only S_1 could b nonzero. We have $40(3) + 10(2) = 140$, so

$$\begin{gathered}S_1 = 140 - 80 = 60 \text{ surplus units low grade oil} \\ S_2 = 0 \\ S_3 = 0\end{gathered}$$

Excel gives the same answer.

6. Same directions as problem 1 (both parts a and b) for the following linear programming problem (Problem 13 of Section 3.4.) The *Jefferson Plastic Fantastic Assembly Corporation* manufactures gadgets and widgets for airplanes and starships. Each case of gadgets uses 2 kg. steel and 5 kg. plastic. Each case of widgets uses 2 kg. steel and 3 kg. plastic. The profit for a case of gadgets is $360 and the profit for a case of widgets is $200. Suppose they have 80 kg. steel available and 150 kg. plastic available on a daily basis, and can sell everything they manufacture. How many cases of each should they manufacture if they are obligated to produce at least 10 cases of widgets per day? The objective is to maximize daily profit.

Solution: Recall the set-up: Let x be the number of cases of gadgets produced per day and y the number of cases of widgets produced per day. Then

$$\text{Maximize profit } P = 360x + 200y$$
$$\text{Subject to:}$$
$$2x + 2y \leq 80 \text{ (kg. steel)}$$
$$5x + 3y \leq 150 \text{ (kg. plastic)}$$
$$y \geq 10 \text{ (contractual obligation)}$$
$$x \geq 0, \quad y \geq 0$$

The *Mathematica* command is:

DualLinearProgramming$[\{-\mathbf{360}, -\mathbf{200}\}, \{\{\mathbf{2}, \mathbf{2}\}, \{\mathbf{5}, \mathbf{3}\}, \{\mathbf{0}, \mathbf{1}\}\}, \{\{\mathbf{80}, -\mathbf{1}\}, \{\mathbf{150}, -\mathbf{1}\}, \{\mathbf{10}, \mathbf{1}\}\}]$

Mathematica returns $\{\{24,10\}, \{0, -72, 16\}, \{0,0\}, \{0,0\}\}$. Thus

$$x = 24 \text{ cases gadgets}$$
$$y = 10 \text{ cases widgets}$$
$$P = 360(24) + 200(10) = \$10{,}640 \text{ (Maximized)}$$
$$M_1 = 0$$
$$M_2 = \$72/\text{kg plastic}$$
$$M_3 = \$16/\text{case widgets}$$

By complementary slackness, the second and third constraints are binding. To find any leftover steel, observe we have used $2(24) + 2(10) = 68$, so:

$$S_1 = 80 - 68 = 12 \text{ kg leftover steel}$$
$$S_2 = 0 \text{ leftover plastic}$$
$$S_3 = 0 \text{ (cases widgets produced beyond the contractually required 10)}$$

Excel produces the same answer.

7. Same directions as problem 1 (both parts a and b) for the following problem (Problem 14 of Section 3.4.) Mr. Cooder, a farmer in the Purple Valley, has at most 400 acres to devote to two crops: rye and barley. Each acre of rye yields \$100 profit per week, while each acre of barley yields \$80 profit per week. Due to local demand, Mr. Cooder must plant at least 100 acres of barley. The federal government provides a subsidy to grow these crops in the form of tax credits. They credit Mr. Cooder 4 units for each acre of rye and 2 units for each acre of barley. The exact value of a 'unit' of tax credit is not important. Mr. Cooder has decided that he needs at least 600 units of tax credits in order to be able to afford his loan payments on a new harvester. How many acres of each crop should he plant in order to maximize his profit?

Solution: Recall the set-up: Let x be the number of acres of rye and y the number of acres of barley. Then

$$\text{Maximize weekly profit } P = 100x + 80y$$
$$\text{Subject to:}$$
$$x + y \leq 400 \text{ (acres land)}$$
$$y \geq 100 \text{ (acres barley demand)}$$
$$4x + 2y \geq 600 \text{ (units tax credit)}$$
$$x \geq 0, \quad y \geq 0$$

The command is

$$\textbf{DualLinearProgramming}[\{-100,-80\},\{\{1,1\},\{0,1\},\{4,2\}\},\{\{400,-1\},\{100,1\},\{600,1\}\}]$$

Mathematica returns $\{\{300,100\},\{-100,20,0\},\{0,0\},\{0,0\}\}$ Thus,

$$x = 300 \text{ acres rye}$$
$$y = 100 \text{ acres barley}$$
$$P = 100(300) + 80(100) = \$38{,}000 \text{ (Maximized)}$$
$$M_1 = \$100/\text{acre}$$
$$M_2 = \$20/\text{acre barley}$$
$$M_3 = 0$$

Observe that $4(300) + 2(100) = 1400$, so there is a surplus of tax credits. By complementary slackness, the other constraints are binding. So

$$S_1 = 0$$
$$S_2 = 0$$
$$S_3 = 1400 - 600 = 800 \text{ surplus units tax credit}$$

Excel gives the same answer.

8. Same directions as problem 1 (both parts a and b) for the following problem (Problem 7 of Section 3.5.) The Spooky Boogie Costume Salon makes and sells four different Halloween costumes: the witch, the ghost, the goblin, and the werewolf. Each witch costume uses 3 yards material and takes 2 hours to sew. Each ghost costume uses 2 yards of material and takes 1 hour to sew. Each goblin costume uses 2 yards of material and takes 3 hours to sew. Each werewolf costume uses 2 yards of material and takes 4 hours to sew. The profits for each costume are as follows: \$10 for the witch, \$8 for the ghost, \$12 for the goblin, and \$16 for the werewolf. If they have 600 yards of material and 510 sewing hours available before the holiday, how many of each costume should they make in order to maximize profit, assuming they can sell everything they make?

Solution: Recall the set-up: Let w be the number of witch costumes, x the number of ghost costumes, y the number of goblin costumes, and z the number of werewolf costumes. Then

$$\text{Maximize profit } P = 10w + 8x + 12y + 16z$$
$$\text{Subject to:}$$
$$3w + 2x + 2y + 2z \leq 600 \text{ (yards material)}$$
$$2w + x + 3y + 4z \leq 510 \text{ (hours for sewing)}$$
$$w \geq 0, \quad x \geq 0, \quad y \geq 0, \quad z \geq 0$$

The *Mathematica* command is

$$\textbf{DualLinearProgramming}[\{-10,-8,-12,-16\},\{\{3,2,2,2\},\{2,1,3,4\}\},\{\{600,-1\},\{510,-1\}\}]$$

Mathematica returns $\left\{\{0,230,0,70\},\left\{-\frac{8}{3},-\frac{8}{3}\right\},\left\{\frac{10}{3},0,\frac{4}{3},0\right\},\{0,0,0,0\}\right\}$ Thus,

$$w = 0 \text{ witch costumes}$$
$$x = 230 \text{ ghost costumes}$$
$$y = 0 \text{ goblin costumes}$$
$$z = 70 \text{ werewolf costumes}$$
$$P = 10(0) + 8(230) + 12(0) + 16(70) = \$2{,}960 \text{ (Maximized)}$$
$$M_1 = \frac{8}{3}$$
$$M_2 = \frac{8}{3}$$

By complementary slackness, both constraints are binding, so

$$S_1 = 0 \text{ (leftover yards material)}$$
$$S_2 = 0 \text{ (leftover hours time)}$$

Excel gives the same answer.

9. Solve the problem of the *Gigantic* ocean liner (Problem 14 of Section 3.1). You may use either Mathematica or Excel. In your solution, find values of all decision and slack variables, as well as the objective function, but don't bother with the Marginal values.

Solution: Recall the set-up: Let x be the number of wooden lifeboats, y the number of large rubber rafts, and z the number of small rubber rafts. Then

$$\text{Minimize cost } C = 6000x + 4000y + 1500z$$
Subject to:

$$1000x + 400y + 250z \leq 38000 \text{ (lbs.)}$$
$$x + \frac{1}{2}y + \frac{1}{4}z = 40 \text{ (storage berths)}$$
$$40x + 24y + 10z \geq 1600 \text{ (people)}$$
$$40x + 30y + 8z \geq 1500 \text{ (life jackets)}$$
$$6x + 4y + z \geq 200 \text{ (first aid kits)}$$
$$x \geq 12 \text{ (wooden lifeboats)}$$
$$y + z \geq 80 \text{ (rubber rafts)}$$
$$x \geq 0, \quad y \geq 0, \quad z \geq 0$$

Since we are not asked to find the marginal values, if using *Excel* to solve, one only need the answer report, not the sensitivity report. If using *Mathematica*, one only needs the LiearProgramming command, rather than the DualLinearProgramming command.

The *Mathematica* command is

LinearProgramming

$$\left[\{6000,4000,1500\},\ \left\{\{1000,400,250\},\left\{1,\frac{1}{2},\frac{1}{4}\right\},\{40,24,10\},\{40,30,8\},\{6,4,1\},\{1,0,0\},\{0,1,1\}\right\},\ \{\{38000,-1\},\{40,0\},\{1600,1\},\{1500,1\},\{200,1\},\{12,1\},\{80,1\}\}\right]$$

Mathematica returns {15,20,60}. The full solution (minus the marginal values) requires a bit of calculation (but is readily available if you use *Excel*.) It is:

$$x = 15 \text{ wooden life boats}$$
$$y = 20 \text{ large rubber rafts}$$
$$z = 60 \text{ small rubber rafts}$$
$$C = 6000(15) + 4000(20) + 1500(60) = \$260{,}000 \text{ (Minimized)}$$
$$1000(15) + 400(20) + 250(60) = 38{,}000 \text{ lbs., so } S_1 = 0$$
$$1(15) + \frac{1}{2}(20) + \frac{1}{4}(60) = 40 \text{ berths, so } S_2 = 0$$
$$40(15) + 24(20) + 10(60) = 1680 \text{ people, so } S_3 = 80$$
$$40(15) + 30(2) + 8(60) = 1680 \text{ life jackets, so } S_4 = 180$$
$$6(15) + 4(20) + 1(60) = 230 \text{ first aid kits, so } S_5 = 30$$
$$x = 15 \text{ wooden life boats, so } S_6 = 3$$
$$y + z = 80 \text{ rubber rafts, so } S_7 = 0$$

10. Same directions as problem 9 for *Derek's diet problem* (Problem 16 of Section 3.1.)

Solution: Recall the set-up: Let s be the number of ounces of garden salad, v the number of ounces of grilled vegetables, p the number of ounces of pasta, m the number of ounces of meatballs, and x the number of ounces of curried chicken salad. Then Derek must

$$\text{Minimize Cost } C = 25s + 40v + 30p + 50m + 60x$$
Subject to:
$$s + v + 3p + 6m + 4x \leq 40 \text{ (grams fat)}$$
$$v + 2p + 10m + 12x \geq 80 \text{ (grams protein)}$$
$$3s + 4v + 12p + 6m + 3x \geq 60 \text{ (grams carbohydrates)}$$
$$10s + 12v + 15p + 25m + 20x \leq 200 \text{ (mg sodium)}$$
$$15s + 20v + 40p + 60m + 50x \leq 700 \text{ (calories)}$$
$$s \geq 0,\ \ v \geq 0,\ \ p \geq 0,\ \ m \geq 0,\ \ x \geq 0$$

The Mathematic command is

LinearProgramming

$$[\{25,40,30,50,60\},\{\{1,1,3,6,4\},\{0,1,2,10,12\},\{3,4,12,6,3\},\{10,12,15,25,20\},\{15,20,40,60,50\}\},\ \{\{40,-1\},\{80,1\},\{60,1\},\{200.-1\},\{700,-1\}\}]$$

Mathematica returns $\left\{0,0,\frac{8}{3},\frac{8}{3},4\right\}$. Thus, the solution is

$$s = 0 \text{ oz. salad}$$
$$v = 0 \text{ oz. grilled vegetables}$$
$$p = \frac{8}{3} \text{ oz. pasta}$$
$$m = \frac{8}{3} \text{ oz. meatballs}$$
$$x = 4 \text{ oz. curried chicken}$$
$$C = \$4.53 \text{ (minimized)}$$
$$S_1 = 0 \text{ g fat}$$
$$S_2 = 0 \text{ g protein}$$
$$S_3 = 0 \text{ g carbohydrates}$$
$$S_4 = \frac{40}{3} \approx 13.33 \text{ mg sodium}$$
$$S_5 = \frac{700}{3} \approx 233.33 \text{ calories}$$

11. Same directions as problem 9 for *Ray's diet problem* (Problem 17 of Section 3.1.)

Solution: Recall the set-up: Let s be the number of ounces of garden salad, v the number of ounces of grilled vegetables, p the number of ounces of pasta, m the number of ounces of meatballs, and x the number of ounces of curried chicken salad. Then Ray must

$$\text{Maximize Protein } P = v + 2p + 10m + 12x$$
$$\text{Subject to:}$$
$$s + v + 3p + 6m + 4x \leq 60 \text{ (grams fat)}$$
$$3s + 4v + 12p + 6m + 3x \geq 120 \text{ (grams carbohydrates)}$$
$$12s + 5v + 6p + 4m + x \geq 90 \text{ (mg sugar)}$$
$$12s + 5v + 6p + 4m + x \leq 200 \text{ (mg sugar)}$$
$$25s + 40v + 30p + 50m + 60x \leq 750 \text{ (cost)}$$
$$s \geq 0,\ \ v \geq 0,\ \ p \geq 0,\ \ m \geq 0,\ \ x \geq 0$$

The Mathematica command is

LinearProgramming

$$\left[\{\mathbf{0}, -\mathbf{1}, -\mathbf{2}, -\mathbf{10}, -\mathbf{12}\}, \{\{\mathbf{1}, \mathbf{1}, \mathbf{3}, \mathbf{6}, \mathbf{4}\}, \{\mathbf{3}, \mathbf{4}, \mathbf{12}, \mathbf{6}, \mathbf{3}\}, \{\mathbf{12}, \mathbf{5}, \mathbf{6}, \mathbf{4}, \mathbf{1}\}, \{\mathbf{12}, \mathbf{5}, \mathbf{6}, \mathbf{4}, \mathbf{1}\}, \{\mathbf{25}, \mathbf{50}, \mathbf{30}, \mathbf{50}, \mathbf{60}\}\}, \{\{\mathbf{60}, -\mathbf{1}\}, \{\mathbf{120}, \mathbf{1}\}, \{\mathbf{90}, \mathbf{1}\}, \{\mathbf{200}, -\mathbf{1}\}, \{\mathbf{750}, -\mathbf{1}\}\}\right]$$

Mathematica returns $\left\{\frac{290}{71}, 0, \frac{610}{91}, \frac{180}{91}, \frac{570}{91}\right\}$. The optimal diet for Ray is

$$s = \frac{290}{71} \approx 2.97 \text{ oz. garden salad}$$
$$v = 0 \text{ oz. grilled vegetables}$$
$$p = \frac{610}{91} \approx 6.7 \text{ oz pasta}$$
$$m = \frac{180}{91} \approx 1.98 \text{ oz meatballs}$$
$$x = \frac{570}{91} \approx 6.26 \text{ oz curried chicken salad}$$
$$\text{Protein } P = \frac{9860}{91} \approx 108.35 \text{ g (Maximized)}$$
$$S_1 = \frac{270}{91} \approx 2.97 \text{ (g fat)}$$

$$S_2 = 0 \text{ (g carbohydrates)}$$
$$S_3 = 0 \text{ (g sugar)}$$
$$S_4 = 110 \text{ (g sugar)}$$
$$S_5 \approx \$0.96$$

12. Same directions as problem 9 for the *Creative Thought Matters Biology Research problem* (Problem 20 of Section 3.1.)

Solution: Recall the Set-up: Let u be the number of kilograms shipped from Atlanta to Chicago, v the number of kilograms shipped from Atlanta to Denver, w the number of kilograms shipped from Atlanta to San Francisco, x the number of kilograms shipped from Boston to Chicago, y the number of kilograms shipped from Boston to Denver, and z the number of kilograms shipped from Boston to San Francisco. Then:

$$\text{Minimize Cost } C = 30u + 45v + 80w + 40x + 40y + 70z$$

Subject to

$$u + v + w \leq 330 \text{ (kg. stored in Atlanta)}$$
$$x + y + z \leq 450 \text{ (kg. stored in Boston)}$$
$$u + x \geq 300 \text{ (kg. requested by Chicago)}$$
$$v + y \geq 120 \text{ (kg. requested by Denver)}$$
$$w + z \geq 360 \text{ (kg. requested by San Francisco)}$$
$$u \geq 0, \quad v \geq 0, \quad w \geq 0, \quad x \geq 0, \quad y \geq 0, \quad z \geq 0$$

The Mathematica command is

LinearProgramming
$$\left[\{\mathbf{30}, \mathbf{45}, \mathbf{80}, \mathbf{40}, \mathbf{40}, \mathbf{70}\}, \{\{\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{0}\}, \{\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{1}, \mathbf{1}\}, \{\mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{0}\}, \{\mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{0}\}, \{\mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{1}\}\}, \{\{\mathbf{330}, -\mathbf{1}\}, \{\mathbf{450}, -\mathbf{1}\}, \{\mathbf{300}, \mathbf{1}\}, \{\mathbf{120}, \mathbf{1}\}, \{\mathbf{360}, \mathbf{1}\}\}\right]$$

Mathematica returns {300,30,0,0,90,360}. The optimal solution is then:

$$u = 300 \text{ (kg shipped from Atlanta to Chicago)}$$
$$v = 30 \text{ (kg shipped from Atlanta to Denver)}$$
$$w = 0 \text{ (kg shipped from Atlanta to San Francisco)}$$
$$x = 0 \text{ (kg shipped from Boston to Chicago}$$
$$y = 90 \text{ (kg shipped from Boston to Denver)}$$
$$z = 360 \text{ (kg shipped from Boston to San Francisco)}$$
$$C = \$39{,}150 \text{ (Minimized)}$$
$$S_1 = 0 \text{ leftover kg in Atlanta}$$
$$S_2 = 0 \text{ leftover kg in Boston}$$
$$S_3 = 0 \text{ surplus kg shipped to Chicago}$$
$$S_4 = 0 \text{ surplus kg shipped to Denver}$$
$$S_5 = 0 \text{ surplus kg shipped to San Francisco}$$

13. Consider *Green's Heavy Metal Foundry problem* (Problem 19 of Section 3.1.)
a) Use Excel to solve this problem

Solution: Recall the set-up: Let x be the number of (100 lb.) units of Alloy I, y the number of units of Alloy II, and z the number of units of Alloy III. Then Mr. Giante must solve the following linear programming problem:

$$\text{Maximize profit } P = 100x + 80y + 40z$$
$$\text{Subject to:}$$
$$50x + 30y + 50z \leq 12000\ (lbs. copper)$$
$$50x + 30y + 20z \leq 10000\ (lbs. zinc)$$
$$40y + 30z \leq 12000\ (lbs. iron)$$
$$x \geq 0, \quad y \geq 0, \quad z \geq 0$$

We leave it to the reader to solve this via Excel. The optimal solution should be $(x, y, z) = (20,300,0)$ and $P = \$26{,}000$, with 2,000 lbs. leftover copper.

b) From the sensitivity report, find the stable range for each constraint; that is, for each resource.

Solution: First consider copper. The sensitivity report gives an allowable decrease of 2000 lbs., and an infinite allowable increase. Thus, the stable range for copper is $[10{,}000\ lbs., \infty)$.
Next, consider zinc. The sensitivity report givens an allowable decrease of 1000 lbs, and an allowable increase of 2,000 lbs. Thus, the stable range for zinc is $[9{,}000\ lbs., 12{,}000\ lbs.]$
Finally, consider iron. The sensitivity report gives an allowable decrease of $\frac{4000}{3} \approx 1333.3$ lbs, and an allowable increase of 12,000 lbs. Thus, the stable range for iron is $\left[\frac{32000}{3}\ lbs.,\ 24{,}000\ lbs.\right]$

14. Consider the *Court and Sparkle Jewelry Emporium* (see problem 2 above.) From the Excel solution, determine the stable range for each of the resources and the stable range for the two objective coefficients.

Solution: Recall that we have leftover rubies at the optimal solution. The allowable decrease is 9 rubies (matching the number of leftover rubies in the optimal solution), and an infinite allowable increase (since rubies are already in surplus.) (Note: the situation is analogous to that of copper in the previous problem.) Thus, the stable range for rubies is $[45, \infty)$.
For pearls, the allowable decrease is 36 and the allowable increase is 60, so the stable range for pearls is $[84{,}180]$.
Finally, for opals, the allowable decrease is 100 and the allowable increase is 36, so the stable range for opals is $[200{,}336]$.
We leave it to the reader to also find the allowable decreases and increases for the objective coefficients (which are the profits for each type of bracelet) on the sensitivity report, and thereby determine the stable range for these coefficients.

15. In the Court and Sparkle Jewelry Emporium (see problem 2 above), determine the new optimal data:

a) If the supply of pearls is increased to 150 and no other changes are made.

Solution: Since 150 is within the stable range for pearls, we know the revised solution will have the same basic variables as before, so the optimal point is the intersection of the pearl and opal lines. Furthermore, the marginal value of a pearl is \$150, and since we have increased the pearl supply by 30 pearls, this means the profit should increase by $30(150) = \$4{,}500$ to a revised profit of \$49,500. Also, the marginal values are unchanged at this point. This checks:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{60}\begin{bmatrix} 15 & -3 \\ -10 & 6 \end{bmatrix}\begin{bmatrix} 150 \\ 300 \end{bmatrix} = \begin{bmatrix} \frac{45}{2} \\ 5 \end{bmatrix}$$

$$P = 1800\left(\frac{45}{2}\right) + 1800(5) = 49500$$

The solution is:

$$x = \frac{45}{2} = 22.5 \text{ Dawntreader bracelets}$$
$$y = 5 \text{ Hejira bracelets}$$
$$P = \$49{,}500 \text{ (Maximized)}$$
$$S_1 = 16.5 \text{ leftover rubies}, \quad M_1 = 0$$
$$S_2 = 0 \text{ leftover pearls}, \quad M_2 = \$150/\text{pearl}$$
$$S_2 = 0 \text{ leftover opals}, \quad M_3 = \$90/\text{opal}$$

b) If the profit for a Hejira model bracelet increases to \$2100 and no other changes are made.

Solution: As can be seen on the sensitivity report in the Excel solution, the stable range for the coefficient of profit for a Hejira bracelet is $[900, 2700]$. Since 2100 is within that range, the optimal point remains unchanged at (15,10), and 9 leftover rubies. However, the marginal values do change. Since we have leftover rubies, it is still the case that $M_1 = 0$ (complementary slackness). To find the other marginal values, we use the computation FA^{-1} from Chapter 4:

$$[M_2, \quad M_3] = [1800 \quad 2100]\left(\frac{1}{60}\begin{bmatrix} 15 & -3 \\ -10 & 6 \end{bmatrix}\right) = [100 \quad 120]$$

Thus, the complete solution is:

$$x = 15 \text{ Dawntreader bracelets}$$
$$y = 10 \text{ Hejira bracelets}$$
$$P = \$48{,}000 \text{ (Maximized)}$$
$$S_1 = 9 \text{ leftover rubies}, \quad M_1 = 0$$
$$S_2 = 0 \text{ leftover pearls}, \quad M_2 = \$100/\text{pearl}$$
$$S_2 = 0 \text{ leftover opals}, \quad M_3 = \$120/\text{opal}$$

16. In the Jefferson Plastic Fantastic Assembly Corporation (see problem 6 above), determine the new optimal data if the supply of plastic is increased to 175 kg., and no other changes are made.

Solution: From the sensitivity report, the stable range for plastic is $[30,180]$. Since 175 is within that range, the new optimal solution has the same basic variables as the original problem and the marginal values are unchanged. Thus, the optimal point is the intersection of the plastic line with the contractual constraint line. Since the marginal value of plastic is $\$72/\text{kg}$, and we have increased the supply from 150 to 175, the additional profit should be $25(72) = \$1800$. This checks:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 & -3 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 175 \\ 10 \end{bmatrix} = \begin{bmatrix} 29 \\ 10 \end{bmatrix}$$

The new profit is $360(29) + 200(10) = \$12{,}440$, which is indeed an increase of \$1800 over the previous optimal amount of \$10,640. The complete solution is:

$$x = 29 \text{ cases gadgets}$$
$$y = 10 \text{ cases widgets}$$
$$P = \$12{,}440 \text{ (Maximized)}$$
$$S_1 = 2 \text{ kg. steel}, \quad M_1 = 0$$
$$S_2 = 0 \text{ kg. plastic}, \quad M_2 = \$72/\text{kg}$$
$$S_3 = 0 \text{ surplus cases widgets}, \quad M_3 = \$10/\text{case}$$

Chapter 6. Game Theory

Section 6.1 Introduction

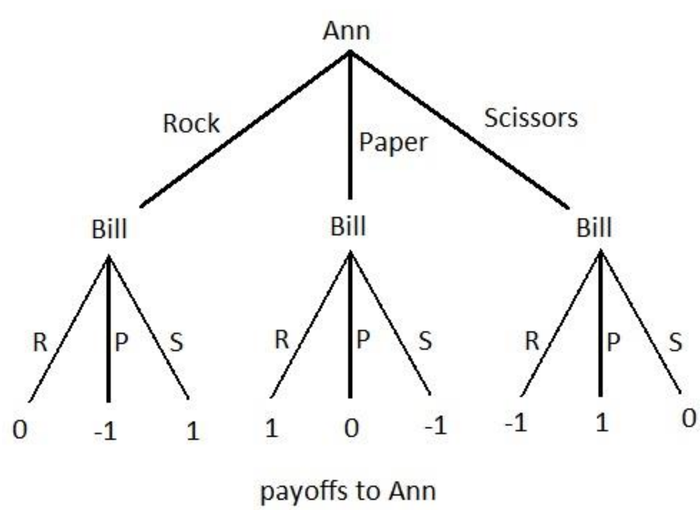
Music References in the Text:

In the marketing game of Example 6.5, the companies Antares and Bellatrix are, of course, names of stars. The computer chip the companies are producing is named after the folk/jazz band Pentangle. In 1968 they recorded the traditional song 'Watch the Stars' for their album Sweet Child. (Website: https://en.wikipedia.org/wiki/Pentangle)

Solutions to Exercises:

1. Express the rock-paper-scissors game from example 6.2 in both normal form and extensive form. That is, draw the game tree and find the payoff matrix as well.

Solution: Extensive form:



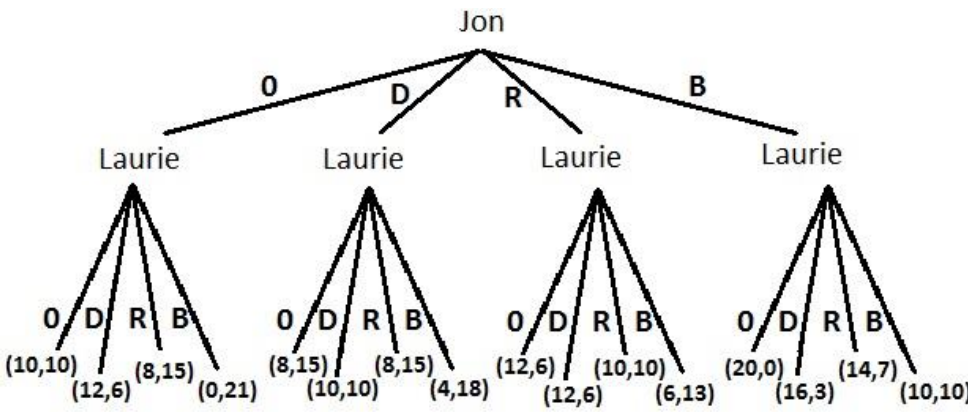
Normal form:

		Bill		
		R	P	S
Ann	R	$(0,0)$	$(-1,1)$	$(1,-1)$
	P	$(1,-1)$	$(0,0)$	$(-1,1)$
	S	$(-1,1)$	$(1,-1)$	$(0,0)$

Note that, because the game is zero-sum, in either the game tree or the payoff matrix, we can just display the playoffs to the row player Ann instead of the ordered pair payoffs.

2. Draw the game tree for the game in Example 6.4, Saturday chores at the Anderson Household, version II.

Solution:



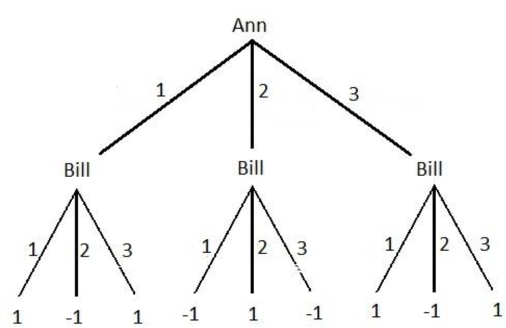
payoffs to (Jon, Laurie)

3. The game of two finger Morra (Example 6.1) can be extended to three finger Morra. The same rules apply – if the total number of fingers revealed is even, the column player pays the row player \$1, while if the total is odd, the row player pays the column player \$1. Draw the game tree and the payoff matrix for three finger Morra.

Solution: We write the payoffs to the row player only. The payoff matrix is:

		Column Player		
		1	2	3
Row Player	1	1	−1	1
	2	−1	1	−1
	3	1	−1	1

The game tree (with Ann as the row player and Bill as the column player) is:



payoffs to Ann

4. The game of two finger Morra (Example 6.1) can be extended to four-finger Morra. The same rules apply – if the total number of fingers revealed is even, the column player pays the row player $1, while if the total is odd, the row player pays the column player $1. Find the payoff matrix for four finger Morra.

Solution: The payoff matrix is:

		Column player			
Row player		1	2	3	4
	1	1	−1	1	−1
	2	−1	1	−1	1
	3	1	−1	1	−1
	4	−1	1	−1	1

5. In the game of Example 6.3, Saturday chores at the Anderson Household, version I, it was mentioned that each [layer has eight strategies open to them. List the strategies.

Solution: The strategies are: Choose all three jobs, choose washing dishes and raking, choose washing dishes and cleaning basement, choose raking and cleaning basement, choose only washing dishes, choose only raking, choose only cleaning basement, and choose no job. If we abbreviate the jobs as D (dishes), R (raking), C (cleaning), we see the strategies are precisely the set of all subset of $\{D, R, C\}$ and can be listed as $DRC, DR, DC, RC, D, R, C, \emptyset$.

6. Consider the marketing game Antares vs. Bellatrix of Example 6.5. Construct a finite version of the game as follows: Instead of allowing the set prices to be any number in the interval $[80,120]$, assume the companies are debating between just a few specific choices (all of which lie within this interval.) For example, suppose Antares is choosing between the two prices $80 and $110, while Bellatrix is choosing between the two prices $90 and $110. Under these assumptions the game is reduced to a 2×2 game. Let Antares be the row player and Bellatrix the column player. Find the payoff matrix, assuming each company still wants to obtain the largest possible market share. [**Hint**: Use the formula (6.1) for $E(x, y)$ from the example to determine the payoffs.

Solution:

		Bellatrix	
Antares		$90	$110
	$80	63	57
	$110	54	66

7. Repeat Exercise 6 if Antares is deciding between the two prices \$80 and \$110, while Bellatrix is deciding between the three prices $\$80, \$100, \$110$. (This exercise illustrates that payoff matrices need not be square matrices.)

Solution:

		Bellatrix		
Antares		\$80	\$100	\$110
	\$80	66	60	57
	\$110	48	60	66

8. Repeat Exercise 6 if Antares and Bellatrix are both deciding between the five prices $\$80, \$90, \$100, \$110, \$120$.

Solution:

		Bellatrix				
		\$80	\$90	\$100	\$110	\$120
	\$80	66	63	60	57	54
Antares	\$90	60	60	60	60	60
	\$100	54	57	60	63	66
	\$110	48	54	60	66	72
	\$120	42	51	60	69	78

9. This is a variation on four-finger Morra. Each player simultaneously puts out either 1,2,3 or 4 fingers. Again, if the total number of fingers revealed is even, the column player pays the row player, while if the total number is odd, the row player pays the column player. However, instead of the payment being \$1, the number of dollars will be the (non-negative) difference between the number of fingers they show. Find the payoff matrix.

Solution:

		Column player			
		1	2	3	4
	1	0	-1	2	-3
Row player	2	-1	0	-1	2
	3	2	-1	0	-1
	4	-3	2	-1	0

10. A guessing game. Bill secretly puts $\$x$ in his hand, where $x = 0,1,3$, or 5. Ann must guess how many dollars Bill is holding. If she guesses correctly, Bill pays her twice the number of dollars he is holding, plus one dollar more; that is, he pays her $\$(2x + 1)$. If she guesses

incorrectly, she must pay Bill the (positive) difference between her guess and the actual number, plus a $2 penalty. Assume Ann is the row player and find the payoff matrix.

Solution:

	Bill				
		0	1	3	5
Ann	0	1	−3	−5	−7
	1	−3	3	−4	−6
	3	−5	−4	7	−4
	5	−7	−6	−4	11

Section 6.2 Dominant Strategies and Nash Equilibrium Points

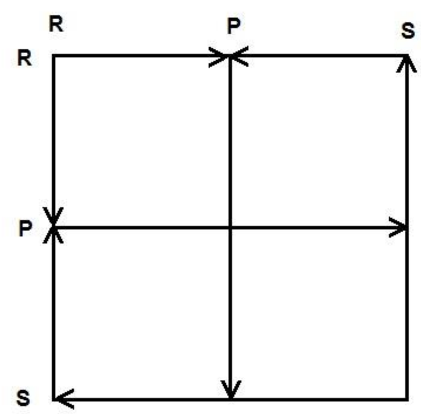
Solutions to exercises:

1. Draw the movement diagram for the game rock-paper-scissors. Conclude that, in this game, there are no dominant strategies and none of the outcomes is a Nash equilibrium,

Solution: We display the payoff matrix first, using just payoffs to the row player (compare with Exercise 1 of Section 6.1):

		Column player		
		R	P	S
Row player	R	0	-1	1
	P	1	0	-1
	S	-1	1	0

The movement diagram is then:



Since there is no outcome which has both horizontal and vertical arrows pointing to it, there is no Nash equilibrium. Also, every row and every column has arrows pointing to that row or column and away from that row or column, so no strategy dominates any other strategy.

The matrices in Exercises 2-10 are assumed to be payoff matrices for zero-sum games. For each one, do the following: i) determine if there are any dominant and dominated strategies for either player and list them, ii) locate all saddle points, if any. You may use your favorite method to locate them (i.e., movement diagrams, comparing row minima to column maxima, or the overline/underline method, iii) if the game is strictly determined, give the complete solution (optimal strategies for each player and the value of the game.)

2.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Solution: The second row dominates the first, and the first column dominates the second. There is a saddle point at the intersection of the dominant row and the dominant column, so the game is strictly determined. The row player should play the second row, and column player should play the first column, and the value of the game is $v = 3$ (the value of the saddle point.)

3.

$$\begin{bmatrix} 0 & 1 & -3 \\ 3 & 2 & 4 \end{bmatrix}$$

Solution: The second row dominates the first. No column dominates any other, but there is higher order dominance. If you delete the dominated first row, only the second row remains, in which the 2 in the second column dominates both of the other columns. Thus, $a_{22} = 2$ is a saddle point of this matrix and the game is strictly determined. The row player should play the second row and the column player should play the second column, and $v = 2$.

4.

$$\begin{bmatrix} 0 & 1 & -3 \\ -2 & 3 & 4 \end{bmatrix}$$

Solution: Neither row dominates the other. The first column dominates the second, so deleting the second column recues the matrix to:

$$\begin{bmatrix} 0 & -3 \\ -2 & 4 \end{bmatrix}$$

This reduced matrix has no dominance, so there is no higher order dominance in the original matrix. Also, there is no saddle point so the game is not strictly determined.

5.

$$\begin{bmatrix} -4 & -1 & 7 \\ 2 & -1 & 3 \\ 8 & -3 & 11 \end{bmatrix}$$

Solution: No row dominates another row. The second column dominates the third column, so deleting the third column reduces the matrix to:

$$\begin{bmatrix} -4 & -1 \\ 2 & -1 \\ 8 & -3 \end{bmatrix}$$

Now the second row dominates the first, so deleting the dominated first row leads to:

$$\begin{bmatrix} 2 & -1 \\ 8 & -3 \end{bmatrix}$$

Now the second column dominates the first, so delete the first column to obtain:

$$\begin{bmatrix} -1 \\ -3 \end{bmatrix}$$

Finally, the -1 dominates the -3 for the row player. So, by successively deleting higher order strategies, we arrive at the saddle point $a_{22} = -1$. The game is strictly determined – the row player should play the second row, the column player should play the second column, and $v = -1$.

6.

$$\begin{bmatrix} 1 & 0 & -4 \\ -3 & 1 & -6 \\ 0 & 6 & -3 \end{bmatrix}$$

Solution: The third row dominates the second. Also, the third column dominates both of the other tow columns. Deleting all the dominated outcomes leads to a saddle point at $a_{33} = -3$. The game is strictly determined – the row player should play the third row and the column player should play the third column. The value of the game is $v = -3$.

7.

$$\begin{bmatrix} 1 & 0 & 1 & -3 \\ -2 & 0 & 1 & 1 \\ 3 & 2 & -5 & 0 \end{bmatrix}$$

Solution: No row dominates any other, and no column dominates any other. Also, there is no saddle point, so the game is not strictly determined.

8.

$$\begin{bmatrix} -1 & -3 & 1 & -9 \\ 5 & -3 & 7 & -4 \\ 4 & -2 & 0 & -2 \end{bmatrix}$$

Solution: The second row dominates the first. The fourth column dominates every other column. There is a saddle point at a_{32} and at a_{34}, so the game is strictly determined. The row player should play the third row, while the column player could play either the second or the fourth column. The value is $v = -2$.

9.

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 3 \\ -3 & -1 & 1 & -3 & 0 \\ 3 & 0 & 4 & 0 & 5 \\ 0 & -1 & 0 & -4 & 1 \end{bmatrix}$$

Solution: The first row dominates the second row and also dominates the fourth row. But the third row dominates all the other rows. For columns, the fourth dominates the second and the fifth, while the second dominates the fifth as well. Also, once dominated rows are deleted, we are left only with the third row, and then the second and fourth columns dominate the other three (higher order dominance.) This leads to saddle points at a_{32} and a_{34}. The row player should play the third row, and column player should play either the second or fourth column. The value of the game is $v = 0$ (a fair game.)

10.

$$\begin{bmatrix} -4 & 1 & 0 & 0 & -2 \\ -3 & 3 & 3 & 0 & -1 \\ 1 & 1 & 1 & 1 & 0 \\ 5 & -3 & -2 & -4 & 5 \end{bmatrix}$$

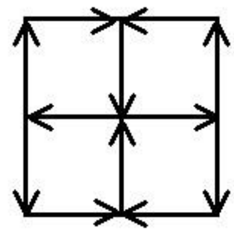
Solution: The second and third row each dominate the first row. The fourth column dominates both the second and third columns. However, there is no saddle point, so the game is not strictly determined.

11. In Exercise 3 of Section 6.1, we extended two-finger Morra to a version for three fingers. Draw the movement diagram and determine whether or not this is a strictly determined game.

Solution: Recall the payoff matrix

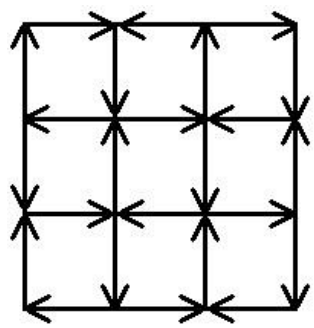
		Column Player		
		1	2	3
Row Player	1	1	-1	1
	2	-1	1	-1
	3	1	-1	1

In this matrix, every row minimum is negative (with value -1), while every column maximum is positive (with value 1). So a row minimum could never equal a column maximum. Therefore, there is no saddle point and so the game is not strictly determined. Here's the movement diagram:



12. In Exercise 4 of Section 6.1, we extended 2-finger Morra to a version for four fingers. Draw the movement diagram and determine whether or not this is a strictly determined game.

Solution: The exact same argument as in Exercise 11 above shows that the game is not strictly determined. Here is the movement diagram confirming the absence of Nash equilibria:



13. In Exercise 7 of Section 6.1, we considered a finite version of the Antares vs. Bellatrix marketing game. Determined whether or not this is a strictly determined game, and if so, give the solution.

Solution: Recall the payoff matrix from Exercise 7 of Section 6.1:

Antares	Bellatrix			
		\$80	\$100	\$110
	\$80	66	60	57
	\$110	48	60	66

In this matrix there is no saddle point (and not any dominance either), so the game is not strictly determined.

14. In Exercise 8 of Section 6.1, we considered a finite version of the Antares vs. Bellatrix marketing game. Determine whether or not this is a strictly determined game, and if so, give the solution:

Solution: Recall the payoff matrix from Exercise 8 of Section 6.1:

		Bellatrix				
		$80	$90	$100	$110	$120
Antares	$80	66	63	60	57	54
	$90	60	60	60	60	60
	$100	54	57	60	63	66
	$110	48	54	60	66	72
	$120	42	51	60	69	78

In this matrix, note that there is a row and a column which are both constant – every entry has the same value - 60. Where this row and column cross must be a saddle point, by definition! So, this version of the game is strictly determined. The solution is Antares sets their price at $90, and Bellatrix sets theirs at $100. At those optimal prices the value of the game is $v = 60$, so Antares gets 60% of the market. Notice that if either player unilaterally deviates from their optimal strategy, it has no effect whatsoever on the value of the outcome of the game. The payoff is still $v = 60$, so neither player has any incentive to unilaterally deviate from their optimal strategy. (This shows directly that the outcome is a Nash equilibrium.)

15. In Exercise 9 of section 6.1, we considered a version of four-finger Morra. Determine whether or not this version is a strictly determined game, and if so, give the solution.

Solution: Recall the payoff matrix from Exercise 9 of Section 6.1:

		Column player			
		1	2	3	4
Row player	1	0	-1	2	-3
	2	-1	0	-1	2
	3	2	-1	0	-1
	4	-3	2	-1	0

Observe that even with the modified payoffs, it is still the case that each row minimum is negative and each column maximum is positive. Therefore, they can never be equal to one another so there is no saddle point. Thus, the game is not strictly determined.

16. Consider the guessing game of Exercise 10 from Section 6.1. Determine whether or not this is a strictly determined game, and if so, give the solution.

Solution: Recall the payoff matrix from Exercise 10 of Section 6.1:

		Bill			
		0	1	3	5
Ann	0	1	−3	−5	−7
	1	−3	3	−4	−6
	3	−5	−4	7	−4
	5	−7	−6	−4	11

We leave the movement diagram to the interested reader. In either case, it is still the case that the row minima are negative while the column maxima are positive, hence they can never be equal and so there are no saddle points. This game is not strictly determined.

17. Consider the following variation of four-finger Morra. Ann and Bill simultaneously put out either 1,2,3, or 4 fingers. If the total sum of the fingers shown is odd, Ann pays Bill the number of dollars equal to the smaller of the two numbers shown. If the total number of fingers revealed is even, then Bill pays Ann a number of dollars equal to the smaller of the two numbers shown. Write down the payoff matrix, draw the movement diagram, list any dominant strategies and saddle points, if any. If the game is strictly determined, give the solution.

Solution: Here is the payoff matrix:

		Bill			
		1	2	3	4
Ann	1	1	−1	1	-1
	2	−1	2	−2	2
	3	1	−2	3	−3
	4	−1	2	−3	4

We leave the movement diagram to the reader. It is still the case that that the row minima and column maxima have opposite signs, so cannot be equal. Thus, there are no saddle points. Also, by looking at either the matrix or the movement diagram you can see there is no dominance. The game is not strictly determined.

18. a) Consider the following variation of four-finger Morra. Ann and Bill simultaneously put out either 1,2,3, or 4 fingers. If the total number of fingers shown is a number that has a remainder of 1 when it is divided by 3, then Ann pays Bill \$1. If the total number of fingers shown is a number which gives a remainder of 2 when divided by 3, then Bill pays Ann \$1. However, if the total number of fingers shown is exactly divisible by 3 with no remainder, then no payment is made. Write down the payoff matrix, draw the movement diagram, list any

dominant strategies and saddle points, if any. If the game is strictly determined, give the solution.

Solution: Here is the payoff matrix:

		Bill			
		1	2	3	4
Ann	1	1	0	−1	1
	2	0	−1	1	0
	3	−1	1	0	−1
	4	1	0	−1	1

We leave the movement diagram to the reader. Again, none of the row minima can equal any of the column maxima because they have opposite signs, so there are no saddle points, so the game is not strictly determined. Also, there are no instances of dominance.

b) Repeat this exercise with the same rules except using up to three fingers only.

Solution: In this case, the payoff matrix is:

		Bill		
		1	2	3
Ann	1	1	0	−1
	2	0	−1	1
	3	−1	1	0

However, the conclusions are the same as part (a) – there is no dominance, nor are thre any saddle points, so the game is not strictly determined.

19. This exercise is about *translations* of games. Let A be a matrix consisting of the payoffs for a constant-sum game, which is strictly determined, and suppose the payoffs to the row player and to the column player sum to the constant c. To clarify things, think of the entries of A as ordered pairs to both players, rather than just the usual payoff to the row player. Suppose we modify the game by adding a real constant b to each entry in A to obtain a new matrix B. That is, we add b to both coordinates, so if the ij entry of A is $\left(a_{ij}, a'_{ij}\right)$, then the ij entry of B is $\left(a_{ij}, a'_{ij}\right) + (b, b) = \left(a_{ij} + b, a'_{ij} + b\right)$ for all i and j. The game with payoff matrix B is said to be a *translation* of the game with payoff matrix A. Since we are assuming A is constant sum, we typically write just the payoffs to the row player rather than the complete ordered pair. In that case, we can think of B being obtained from A by $b_{ij} = a_{ij} + b$ for all i and j, provided we remember the convention that b is also added to the column player's payoffs.

a) Explain why the resulting matrix B still represents the matrix of payoffs for the row player in a constant-sum game.

Solution. For each outcome, the sum of the payoffs is $(a_{ij} + b) + (a'_{ij} + b) = a_{ij} + a'_{ij} + 2b = c + 2b$, which is a constant.

b) What constant do the payoffs sum to in this modified game?

Solution: $c + 2b$ (see part a.)

c) Explain why this game is still strictly determined, and that the saddle points in B occur in the exactly the same locations as in A.

Solution: Ignoring the second coordinates, one can see that we are simply adding b to every entry for the row player's payoffs, the location of row minima and the column maxima haven't changed (although their values have all increased by b.) So any saddle point of A would yield another one at the same location in B. Alternately, observe that that the movement diagram does not change at all when a constant b is added to every payoff.

d) If the value of the original game is $v = v_A$, what is the value v_B of the modified game?

Solution: $v_B = v_A + b$, because the value of each saddle point has increased by b, and the value of a strictly determined game is by definition the value of a saddle point.

(Observe that the procedure for translating a game works for all games, not just strictly determined ones.)

20. This exercise is about *scaling*s of games. Let A be a matrix consisting of the payoffs for a constant-sum game, which is strictly determined, and suppose the payoffs to the row player and to the column player sum to the constant c. To clarify things, think of the entries of A as ordered pairs to both players, rather than just the usual payoff to the row player. Suppose we modify the matrix A by multiplying each entry by a positive real constant $m > 0$ to obtain a new matrix B. That is, we multiply both coordinates by m, so if the ij entry of A is (a_{ij}, a'_{ij}), then the ij entry of B is $m(a_{ij}, a'_{ij}) = (ma_{ij}, ma'_{ij})$ for all i and j. The game with payoff matrix B is said to be a *scaling* of the game with payoff matrix A. Since we are assuming A is constant sum, we typically write just the payoffs to the row player rather than the complete ordered pair. In that case, we can think of B being obtained from A by $b_{ij} = ma_{ij}$ for all i and j, provided we remember the convention that the columns player's payoffs are scaled by the same factor m.

a) Explain why the resulting matrix B still represents the payoff matrix for a constant sum game, and what to the entries sum to in B?

Solution: For any entry in B, the sum of the payoff is ma_{ij} +m$a'_{ij} = m(a_{ij} + a'_{ij}) = mc$, which is a constant. Thus, the scaled game is also constant sum.

b) Explain why this new game is still strictly determined, and why the saddle points in B occur in exactly the same locations as in A.

Solution: Ignoring the second coordinates, one can see that since we simply scaled every entry of the row player's payoffs by the same factor $m > 0$, the location of the row minima and column maxima haven't changed (although their values have changed by a factor of m.) So any saddle point of A would yield one in B at the same location. Alternately, observe that the movement diagram does not change at all when every payoff is scaled by $m > 0$ (Note that if $m < 0$, the directions of the arrows reverse, since the signs of every entry changed.)

c) If the value of the original game is v_A, what is the value v_B of the modified game?

Solution: Since the value of a strictly determined game is just the value of the saddle point, which has been scaled by a factor of m, it follows that $v_B = mv_A$.

(Observe that, as with translating, scaling can be done to any game, not just strictly determined ones.)

Section 6.3 Mixed-Strategy Constant-Sum Games.

6.3.1 Antares vs. Bellatrix

No new music references or exercises in this section.

6.3.2 Mixed Strategies, Probabilities, and Expected Payoffs

Music reference in the text: Danny Steele, the reporter who covered the jewel robbery in the example (bottom of page 274) used to illustrate expected values is a pun on the band Steely Dan, and the example contains a number of references to their songs, including Your Gold Teeth, Your Gold Teeth II, and the Boston Rag. Also, their hit song Reelin' in the Years contains the lyric 'you wouldn't know a diamond if you held it in your hand'. (Website: https://www.steelydan.com)

Solutions to exercises and more music references:

1. In the marketing game Antares vs. Bellatrix, suppose that, instead of Eq. (6.1), the market share for Antares is given by this equation:

$$E(x, y) = .04xy - 3.6x - 4.4y + 441$$

Solve the game under this hypothesis. Give the optimal prices for both companies, the value of the game in terms of market shares, and also the revenue that each company will enjoy, assuming a total market of 400,000 chips.

Solution: Again, $E(x, y)$ is a saddle surface, so we must put it into standard form. Observe that

$$E(x, y) = .04(xy - 90x - 110y) + 441$$

$$= .04(x(y - 90) - 110(y - 90) - 9900)$$

$$= .04(x - 110)(y - 90) - 396 + 441$$

$$= .04(x - 110)(y - 90) + 45$$

Thus, the solution is Antares sets their price at \$110, and Bellatrix sets theirs at \$90. At these prices, Antares gets 45% of the market. The revenues are $(.45)(400{,}000)(110) =$ \$19,800,000 for Antares and $(.55)(400{,}000)(90) =$ \$19,800,000 for Bellatrix.

2. Consider a probability experiment which consists of rolling an ordinary, six-sided die which is labeled with the integers 1,2,3,4,5,6 on the sides.

a) Lest the outcomes in the sample space S. If the die is fair, what are the probabilities associated with each outcome?

Solution: The sample space is $S = \{1,2,3,4,5,6\}$. If the die is fair, each outcome is equally likely, so the space is uniform. The probability of any one outcome is $\frac{1}{6}$.

b) Let A be the event of rolling an even number, and let B be the event of rolling a number (strictly) greater than 2. Determine the probabilities of A and B.

Solution: $A = \{2,4,6\}$ and $B = \{3,4,5,6\}$. Thus $P(A) = P(2) + P(4) + P(6) = \frac{3}{6}$. Similarly, $P(B) = \frac{4}{6}$. (These results also follow from Theorem A34 in the Appendix)

c) Find $P(A \cap B)$ and determine whether or not A and B are independent events.

Solution: $A \cap B = \{4,6\}, so\ P(A \cap B) = \frac{2}{6} = \frac{1}{3}$. On the other hand, $P(A) \cdot P(B) = \frac{3}{6} \cdot \frac{4}{6} = \frac{12}{36} = \frac{1}{3}$. Since $P(A \cap B) = P(A) \cdot P(B)$, the events are independent.

3. Suppose that the die in Exercise 2 is not fair, but weighted so that the probabilities of each outcome is as follows, for some unknown x:

outcome	p_i
1	$.1$
2	$.1$
3	$.1$
4	$.2$
5	$.2 + x$
6	x

a) Find x.

Solution: The sum of the probabilities is 1:

$$.1 + .1 + .1 + .2 + (.2 + x) + x = 1$$
$$.7 + 2x = 1$$
$$2x = .3$$
$$x = .15$$

b) Let A be the event of rolling an even number, and let B be the event of rolling a number (strictly) greater than 2. Determine the probabilities of A and B.

Solution: $P(A) = P(2) + P(4) + P(6) = .1 + .2 + .15 = 0.45$. $P(B) = P(3) + P(4) + P(5) + P(6) = .1 + .2 + .35 + .15 = 0.8$.

c) Find $P(A \cap B)$ and determine whether or not A and B are independent events.

Solution: $A \cap B = \{4,6\}$, so $P(A \cap B) = P(4) + P(6) = .2 + .15 = 0.35$. On the other hand, $P(A) \cdot P(B) = (.45)(.8) = 0.36$. Since $P(A \cap B) \neq P(A) \cdot P(B)$, the events are NOT independent.

4. Consider a probability experiment which consists of rolling two fair dice, and recording the outcomes showing on the top face of each as an ordered pair. (Assume the two dice are distinguishable – for example they could be different colors, so the first entry in the ordered pair always corresponds to the dame die, for example the red die if you are rolling a red die and a white one.)

a) List the entire sample space S of ordered pairs.

Solution:

$$S = \begin{Bmatrix} (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), \\ (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), \\ (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), \\ (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), \\ (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), \\ (6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \end{Bmatrix}$$

b) Since the dice are fair, each outcome is equally likely. What is the probability of any one outcome?

Solution: $\frac{1}{36}$.

c) Let A be the event of rolling a sum of 10 or greater. Let B be the event of rolling doubles (i.e., the same number on each die.) Let C be the event of rolling a 5 or a 6 on the red die. Find $P(A), P(B)$, and $P(C)$.

Solution: $A = \{(4,6), (5,5), (6,4), (5,6), (6,5), (6,6)\}$, so $P(A) = \frac{6}{36} = \frac{1}{6}$.

$B = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$, so $P(B) = \frac{6}{36} = \frac{1}{6}$.

$C = \begin{Bmatrix} (5,1), (5,2), (5,3), (5,4), (5,5), (5,6) \\ (6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \end{Bmatrix}$, sp$P(C) = \frac{12}{36} = \frac{1}{3}$.

d) Find $P(A \cap B), P(A \cap C)$, and $P(B \cap C)$. Are any two of A, B, or C independent events?

Solution: $A \cap B = \{(5,5), (6,6)\}$, so $P(A \cap B) = \frac{2}{36} = \frac{1}{18}$.

$A \cap C = \{(5,5), (6,4), (5,6), (6,5), (6,6)\}$, so $P(A \cap C) = \frac{5}{36}$.

$B \cap C = \{(5,5), (6,6)\} = A \cap B$, so $P(B \cap C) = \frac{1}{18}$.

Observe: $P(A) \cdot P(B) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36} \neq \frac{1}{18} = P(A \cap B)$, so A and B are NOT independent.

$P(A) \cdot P(C) = \frac{1}{6} \cdot \frac{1}{3} = \frac{1}{18} \neq \frac{5}{36} = P(A \cap C)$, so A and C are NOT independent.

$P(B) \cdot P(C) = \frac{1}{6} \cdot \frac{1}{3} = \frac{1}{18} = P(B \cap C)$, so B and C are independent events.

5. The Saratoga Spitfires Soccer Club is selling raffle tickets as a fund-raiser for their upcoming trip to a tournament. They print a total of 300 tickets to sell. There is one grand prize ticket worth \$250, four second prize tickets worth \$50 each, and ten third prize tickets worth \$25 each. The remaining 285 tickets have no value. Assume that all 300 tickets will be sold.

a) Considering the purchase of a ticket as a probability experiment, there are four possible outcomes (grand prize ticket, second prize ticket, third prize ticket, worthless ticket.) Compute the expected value of a ticket (construct a table as we did in the text for the household polling example.) What is the interpretation of this result?

Solution:

outcome	p_i	v_i	$p_i v_i$
Grand prize ticket	$\frac{1}{300}$	250	$\frac{5}{6}$
Second prize ticket	$\frac{4}{300}$	50	$\frac{2}{3}$
Third prize ticket	$\frac{10}{300}$	25	$\frac{5}{6}$
Worthless ticket	$\frac{285}{300}$	0	0

Thus, the expected value is

$$\frac{5}{6} + \frac{2}{3} + \frac{5}{6} + 0 = \frac{7}{3}$$

It means that an "average" ticket is worth \$2.33 in prize money.

b) If the price of a raffle ticket is \$10, how much money will the team make towards their trip?

Solution: The average value is $\frac{7}{3}$, and there are 300 tickets, so the total prize money is $\frac{7}{3} \cdot 300 = \$700$. If 300 tickets are sold at \$10 each, that is \$3,000 revenue, minus the \$700 prize money, leaving \$2,300 for the team toward their trip.

6. Past records at the local $Bob\ \&\ Jerry's\ Grateful\ Scoop\ Ice\ Cream\ Shop$ indicate that they get 180 customers on a hot and sunny day, 120 customers on a hot and cloudy day, 100 customers on a mild and sunny day, 80 customers on a mild and cloudy day, and 30 customers on a cool day. Suppose that for the next month, the probability that it is hot and sunny is $.2$, the probability that is it hot and cloudy is $.4$, the probability that it is mild and sunny is $.1$, the probability that it is mild and cloudy is $.2$, and the probability that it is cool is $.1$. Find the expected number of customers per day to visit the store.

Music allusion to: The Grateful Dead. Bob Weir and Jerry Garcia were two of the founding members of the band. As an extra layer of cultural reference, this example also alludes to the well-known real-life ice cream chain which already has a flavor called Cherry Garcia in honor of Jerry Garcia! (Website: https://www.dead.net/)

Solution: Let the value v_i be the number of customers on that type of day.

Outcome	p_i	v_i	$p_i v_i$
Hot and sunny	$.2$	180	36
Hot and cloudy	$.4$	120	48
Mild and sunny	$.1$	100	10
Mild and cloudy	$.2$	80	16
Cool	$.1$	30	3

Thus, the expected number of customers per day is

$$36 + 48 + 10 + 16 + 3 = 113.$$

7. Mick and Keith are playing rock-paper-scissors, and Mick (the row player) says he will not be satisfied unless he wins an average of $\$.50$ per game. Meanwhile, Keith (the column player) wants to win at least an average of $\$.40$ per game. Mick decides never to play paper, but to play rock half the time and scissors half the time. Keith decides to play rock a quarter of the time, paper half the time, and scissors a quarter of the time. If both players tick to these mixed strategies in repeated play, what is the expected payoff for both players? Use (6.5). Does Mick have Satisfaction? Does Keith get what he wants?

Musical homage to: The Rolling Stones. Mick Jagger and Keith Richards are the vocalist and main guitarist for the band. Two of their songs were '(I Can't Get No) Satisfaction' (1965) and 'You can't Always Get What You Want' (1969). (Website: https://rollingstones.com/)

Solution: By (6.5) we have

$$E = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{4} \end{bmatrix} = [0]$$

Thus, in the long run, both players break even. Neither one of them gets what they want.

In Exercises 8-15, you are given the payoff matrix A of a zero-sum game. Compute the expected payoff to the row player using the indicated mixed strategies. Use (6.5).

8. $A = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}, \quad p = [.5 \quad .5], \quad q = \begin{bmatrix} .6 \\ .4 \end{bmatrix}$

Solution: $E = pAq = [.5 \quad .5] \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} .6 \\ .4 \end{bmatrix} = [-.2]$

9. $A = \begin{bmatrix} -2 & 5 \\ 4 & -1 \end{bmatrix}, \quad p = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \end{bmatrix}, \quad q = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$

Solution: $E = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} -2 & 5 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \end{bmatrix}$

10. $A = \begin{bmatrix} 1 & 0 \\ 0 & -3 \\ -2 & 1 \end{bmatrix}, \quad p = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}, \quad q = \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix}$

Solution: $E = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix} = [-1]$

11. $A = \begin{bmatrix} 2 & -1 & 3 & -2 \\ -3 & 0 & 1 & 2 \end{bmatrix}, \quad p = [.7 \quad .3], \quad q = \begin{bmatrix} .3 \\ .4 \\ .1 \\ .2 \end{bmatrix}$

Solution: $E = [.7 \quad .3]\begin{bmatrix} 2 & -1 & 3 & -2 \\ -3 & 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} .3 \\ .4 \\ .1 \\ .2 \end{bmatrix} = [-.05]$

12. $A = \begin{bmatrix} 1 & -3 & 5 \\ -2 & 4 & -6 \\ 0 & -1 & 2 \end{bmatrix}, \quad p = [.4 \quad .3 \quad .3], \quad q = \begin{bmatrix} .2 \\ .6 \\ .2 \end{bmatrix}$

Solution: $E = [.4 \quad .3 \quad .3]\begin{bmatrix} 1 & -3 & 5 \\ -2 & 4 & -6 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} .2 \\ .6 \\ .2 \end{bmatrix} = [-.06]$

13. Same matrix as Exercise 12, but with $p = [.5 \quad .2 \quad .3], \quad q = \begin{bmatrix} .5 \\ .5 \\ 0 \end{bmatrix}$

Solution: $E = [.5 \quad .2 \quad .3]\begin{bmatrix} 1 & -3 & 5 \\ -2 & 4 & -6 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} .5 \\ .5 \\ 0 \end{bmatrix} = [-.45]$

14. $A = \begin{bmatrix} 1 & -4 & 2 & 0 \\ 1 & 1 & 1 & -5 \\ -3 & -1 & 8 & 4 \\ 0 & 2 & -9 & -1 \end{bmatrix}, \quad p = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} & \frac{2}{5} & 0 \end{bmatrix}, \quad q = \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \\ 0 \\ \frac{3}{5} \end{bmatrix}$

Solution: $E = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} & \frac{2}{5} & 0 \end{bmatrix}\begin{bmatrix} 1 & -4 & 2 & 0 \\ 1 & 1 & 1 & -5 \\ -3 & -1 & 8 & 4 \\ 0 & 2 & -9 & -1 \end{bmatrix}\begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \\ 0 \\ \frac{3}{5} \end{bmatrix} = \begin{bmatrix} -\frac{13}{25} \end{bmatrix}$

15. $A = \begin{bmatrix} 1.4 & -2.1 & 3.9 & -1.3 & .1 \\ -2.2 & 1.7 & 0 & 3.3 & -.8 \\ -.9 & 1.2 & -4.1 & -2.6 & 4.9 \\ 2.0 & -4.5 & 2.8 & 1.9 & -5.6 \end{bmatrix}$, $p = [.1 \quad .2 \quad .3 \quad .4]$, $q = \begin{bmatrix} .1 \\ 0 \\ .2 \\ .5 \\ .2 \end{bmatrix}$

Solution: $E = [.1 \quad .2 \quad .3 \quad .4] \begin{bmatrix} 1.4 & -2.1 & 3.9 & -1.3 & .1 \\ -2.2 & 1.7 & 0 & 3.3 & -.8 \\ -.9 & 1.2 & -4.1 & -2.6 & 4.9 \\ 2.0 & -4.5 & 2.8 & 1.9 & -5.6 \end{bmatrix} \begin{bmatrix} .1 \\ 0 \\ .2 \\ .5 \\ .2 \end{bmatrix} = [.15]$

16. a) In Exercise 9 above, what is the expected payoff for the column player?

Solution: Since the expected payoff to the row player is $\frac{5}{3}$, and it is a zero-sum game, the expected payoff to the column player must be $-\frac{5}{3}$.

b) Suppose the given matrix A in Exercise 9 is the matrix of payoffs for the row player in a constant-sum game, rather than a zero-sum game, where the payoffs sum to the constant 2. What are the expected payoffs to the row player and columns player now?

Solution: For the row player it remains $\frac{5}{3}$, and for the column player it is $2 - \frac{5}{3} = \frac{1}{3}$.

17. Jimmy and Robert are sitting at the *Four Sticks Tavern*, playing the variation on four-finger Morra from Exercise 9 in Section 6.1. Recall that each player simultaneously put out 1,2,3, or 4 fingers. If the total number of revealed fingers is even, the column player pays the row player, while if it is odd, the row player pays the column player. The difference is, instead of the payoff being $1, the number of dollars is equal to the (non-negative) difference between the number of fingers shown. Suppose both players use the same mixed strategy (so $p = q^T$) of playing 1 finger 10% of the time, 2 fingers 20% of the time, 3 fingers 30% of the time, and 4 fingers 40% of the time. What is the expected payoff to Robert if he is the row player?

Music reference to: the band Led Zeppelin. Jimmy Page and Robert Plant are the guitarist and vocalist for the band. And their fourth album (1971) contained the song 'Four Sticks'. (Website: https://lz50.ledzeppelin.com/)

Solution: If Robert is the row player, the expected payoff to Robert is $E = pAq$:

$$E = [.1 \quad .2 \quad .3 \quad .4] \begin{bmatrix} 0 & -1 & 2 & -3 \\ -1 & 0 & -1 & 2 \\ 2 & -1 & 0 & -1 \\ -3 & 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} .1 \\ .2 \\ .3 \\ .4 \end{bmatrix} = [-.20]$$

18. Recall the guessing game from Exercise 10 in Section 6.1. Bill secretly puts $\$x$ in his hand, where $x = 0{,}1{,}3$, or 5. Ann must guess how many dollars Bill is holding. If she guesses correctly, Bill pays her twice the number of dollars he is holding, plus one dollar more; that is, he pays her $\$(2x + 1)$. If she guesses incorrectly, she must pay Bill the (positive) difference between her guess and the actual number, plus an extra \$2. Assume Ann is the row player. Ann decides to guess according to the mixed strategy of guessing $\$0$ with probability $\frac{1}{2}$, guessing $\$1$ with probability $\frac{1}{4}$, guessing $\$3$ with probability $\frac{1}{8}$, and guessing $\$5$ with probability $\frac{1}{8}$. Bill, on the other hand, decides never to put \$5 in his hand but to use the other three strategies with equal probability, so with probability $\frac{1}{3}$ each. Under repeated play with these assumptions, what is the expected payoff to Ann?

Solution: Since Ann is the row player, her expected payoff is $E = pAq$:

$$E = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} \end{bmatrix} \begin{bmatrix} 1 & -3 & -5 & -7 \\ -3 & 3 & -4 & -6 \\ -5 & -4 & 7 & -4 \\ -7 & -6 & -4 & 11 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{55}{24} \end{bmatrix}$$

Thus, Ann loses about \$2.29 per game on average.

Section 6.4 Solving Mixed-Strategy Games: The Minimax Theorem in the 2×2 Case.

6.4.1 The Derived Game

No musical references or exercises in this section.

6.4.2 Equalizing Expectation

Solutions to exercises (and more musical references):

1. One hazy day in winter, Paul and Art are playing the game of two-finger Morra, with zero-sum payoff matrix given in Section 6.2:

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Neither of them has ever studied game theory, but because of the symmetry of the payoff matrix, they both intuitively believe that the optimal solution is for them to play each strategy half the time, and doing so, in the long run, they will break even. (Such xero-sum games, where $v = 0$ are called *fair games*.) Find the optimal solution, using the derived game approach, and see if their intuition is correct.

Musical homage to: The folk/rock duo (Paul) Simon and (Art) Garfunkel. They had a 1966 song entitled 'A Hazy Shade of Winter' (a song that has been covered by dozens of other artists, including the Bangles.) (Website: https://www.simonandgarfunkel.com/)

Solution: To solve this game via the derived game, we must compute $E(x, y)$:

$$E(x, y) = [x \quad 1 - x] \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} y \\ 1 - y \end{bmatrix} = [4xy - 2y - 2x + 1]$$

We then put this surface into standard form:

$$E(x, y) = 4\left(xy - \frac{1}{2}y - \frac{1}{2}x\right) + 1$$

$$= 4\left(y\left(x - \frac{1}{2}\right) - \frac{1}{2}x\right) + 1$$

$$= 4\left(y\left(x - \frac{1}{2}\right) - \frac{1}{2}\left(x - \frac{1}{2}\right) - \frac{1}{4}\right) + 1$$

$$= 4\left(\left(x - \frac{1}{2}\right)\left(y - \frac{1}{2}\right)\right) - 1 + 1$$

$$= 4\left(\left(x - \frac{1}{2}\right)\left(y - \frac{1}{2}\right)\right) + 0$$

Hence, the optimal solution is:

$$\hat{p} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\hat{q} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$v = 0$$

Thus, their intuition was correct.

2. Recall the zero-sum game from Exercise 8 in Section 6.3, with

$$A = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$$

Find the optimal solution using the derived game approach.

Solution:

$$E(x, y) = \begin{bmatrix} x & 1-x \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} y \\ 1-y \end{bmatrix} = [6xy - 5y - 2x + 2]$$

$$= 6\left(xy - \frac{5}{6}y - \frac{2}{6}x\right) + 2$$

$$= 6\left(y\left(x - \frac{5}{6}\right) - \frac{2}{6}x\right) + 2$$

$$= 6\left(y\left(x - \frac{5}{6}\right) - \frac{2}{6}\left(x - \frac{5}{6}\right) - \frac{10}{36}\right) + 2$$

$$= 6\left(x - \frac{5}{6}\right)\left(y - \frac{2}{6}\right) + 2 - \frac{10}{6}$$

$$= 6\left(x - \frac{5}{6}\right)\left(y - \frac{2}{6}\right) + \frac{2}{6}$$

So, the optimal solution is:

$$\hat{p} = \begin{pmatrix} \frac{5}{6} & \frac{1}{6} \end{pmatrix}$$

$$\hat{q} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$$

$$v = \frac{1}{3}$$

3. Recall the zero-sum game from Exercise 9 in Section 6.3, with

$$A = \begin{bmatrix} -2 & 5 \\ 4 & -1 \end{bmatrix}$$

Find the optimal solution using the derived game approach.

Solution:

$$E(x,y) = \begin{bmatrix} x & 1-x \end{bmatrix} \begin{bmatrix} -2 & 5 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} y \\ 1-y \end{bmatrix} = [-12xy + 6x + 5y - 1]$$

$$= -12\left(xy - \frac{1}{2}x - \frac{5}{12}y\right) - 1$$

$$= -12\left(x\left(y - \frac{1}{2}\right) - \frac{5}{12}\left(y - \frac{1}{2}\right) - \frac{5}{24}\right) - 1$$

$$= -12\left(x - \frac{5}{12}\right)\left(y - \frac{1}{2}\right) + \frac{3}{2}$$

Thus,

$$\hat{p} = \begin{pmatrix} \frac{5}{12} & \frac{7}{12} \end{pmatrix}$$

$$\hat{q} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$v = \frac{3}{2}$$

4. When Ann and Nancy tire of playing the card game Hearts, they dream up other games. One game they call Little Queen is a zero-sum game with the following payoff matrix, where Ann is the row player:

$$\begin{bmatrix} 6 & -1 \\ -3 & 2 \end{bmatrix}$$

Find the optimal solution using the derived game approach.

Musical reference to: the rock band Heart, led by sisters Ann and Nancy Wilson. Their songs included ‘These Dreams’ (1986) and ‘Little Queen’ (1977). (Website: https://www.heart-music.com/)

Solution:

$$E(x,y) = [x \quad 1-x]\begin{bmatrix} 6 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} y \\ 1-y \end{bmatrix} = [12xy - 5y - 3x + 2]$$

$$= 12\left(xy - \frac{5}{12}y - \frac{3}{12}x\right) + 2$$

$$= 12\left(y\left(x - \frac{5}{12}\right) - \frac{3}{12}\left(x - \frac{5}{12}\right) - \frac{15}{144}\right) + 2$$

$$= 12\left(x - \frac{5}{12}\right)\left(y - \frac{3}{12}\right) - \frac{15}{12} + 2$$

$$= 12\left(x - \frac{5}{12}\right)\left(y - \frac{3}{12}\right) + \frac{3}{4}$$

The optimal solution:

$$\hat{p} = \begin{pmatrix} \frac{5}{12} & \frac{7}{12} \end{pmatrix} \text{ (Ann)}$$

$$\hat{q} = \begin{pmatrix} \frac{1}{4} \\ \frac{3}{4} \end{pmatrix} \text{ (Nancy)}$$

$$v = \frac{3}{4}$$

5. Notice that in the payoff matrix A from Exercise 4, the largest entry is a_{11} and the second largest is a_{22}. Thus, the corollary to the minimax theorem applies. Solve the game using the corollary, and verify that the answer agrees with that in Exercise 4.

Solution: We have $D = 12$, so

$$\hat{p} = \begin{pmatrix} \frac{d-c}{D} & \frac{a-b}{D} \end{pmatrix} = \begin{pmatrix} \frac{2-(-3)}{12} & \frac{6-(-1)}{12} \end{pmatrix} = \begin{pmatrix} \frac{5}{12} & \frac{7}{12} \end{pmatrix}$$

$$\hat{q} = \begin{pmatrix} \frac{d-b}{D} \\ \frac{a-c}{D} \end{pmatrix} = \begin{pmatrix} \frac{2-(-1)}{12} \\ \frac{6-(-3)}{12} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{3}{4} \end{pmatrix}$$

$$v = \frac{ad-bc}{D} = \frac{(6)(2)-(-1)(-3)}{12} = \frac{3}{4},$$

which agrees with our previous solution.

6. Consider the payoff matrix $A = \begin{bmatrix} -2 & 5 \\ 4 & -1 \end{bmatrix}$ from Exercise 3. The largest two entries are not on the main diagonal, so we cannot use the corollary to the minimax theorem directly. However, if we list the column player's strategies in the reverse order, the effect is to switch the two columns of A so we arrive a t a new matrix $B = \begin{bmatrix} 5 & -2 \\ -1 & 4 \end{bmatrix}$, which represents essentially the same game as A.

a) the Corollary does apply to B, so use it to find the optimal solution of B.

Solution:

$$\hat{p} = \begin{pmatrix} \frac{d-c}{D} & \frac{a-b}{D} \end{pmatrix} = \begin{pmatrix} \frac{4-(-1)}{12} & \frac{5-(-2)}{12} \end{pmatrix} = \begin{pmatrix} \frac{5}{12} & \frac{7}{12} \end{pmatrix}$$

$$\hat{q} = \begin{pmatrix} \frac{d-b}{D} \\ \frac{a-c}{D} \end{pmatrix} = \begin{pmatrix} \frac{4-(-2)}{12} \\ \frac{5-(-1)}{12} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$v = \frac{ad-bc}{D} = \frac{(5)(4)-(-2)(-1)}{12} = \frac{3}{2}$$

b) Since the difference between A and B only involved transposing the columns, it follows that the optimal mix $\hat{p}$ for the row player should be the same in matrix A as in B. That is, $\hat{p}_A = \hat{p}_B$. Onn the other hand, because the columns were interchanged, if follows that to obtain q_A from q_B, we merely switch the coordinates. Explain why these statements are true and find the optimal mixes for A using your solution to B from part A. [**Hint**: to explain why the statements are true, think of the method of equalizing expectations. How do the calculations change when the columns are interchanged?]

Solution: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and let $B = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$, obtained from A by switching the columns. Observe that:

$$[x \quad 1-x] \begin{bmatrix} a & b \\ c & d \end{bmatrix} = [ax - c(x-1) \quad bx - d(x-1)]$$

We solve for x to find $\hat{p}_A$ by setting the two coordinates equal to each other. To find $\hat{p}_B$ we do the same with B in place of A:

$$[x \quad 1-x] \begin{bmatrix} b & a \\ d & c \end{bmatrix} = [bx - d(x-1) \quad ax - c(x-1)]$$

But these are the same two equations so when we solve for x by setting them equal, we obtain the same value of x in both answers, so $\hat{p}_A = \hat{p}_B$.

As for the column player's solution:

$$A\begin{bmatrix} y \\ 1-y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} y \\ 1-y \end{bmatrix} = \begin{bmatrix} ay - b(y-1) \\ cy - d(y-1) \end{bmatrix}$$

We solve for y to obtain $\hat{q}_A$ by setting the two coordinates equal.

$$ay - b(y-1) = cy - d(y-1)$$

Now define $z = 1 - y$, so that $y = 1 - z$, and substitute in the above:

$$a(1-z) + bz = c(1-z) + dz$$

However, these are exactly the calculations needed to find $\hat{q}_B$:

$$B\begin{bmatrix} z \\ 1-z \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}\begin{bmatrix} z \\ 1-z \end{bmatrix} = \begin{pmatrix} bz + a(1-z) \\ dz + c(1-z) \end{pmatrix}$$

This shows that $\hat{q}_B = \begin{pmatrix} z \\ 1-z \end{pmatrix} = \begin{pmatrix} 1-y \\ y \end{pmatrix}$, which is just $\hat{q}_A$ with the coordinates switched, proving the claim.

c) Explain why $v_A = v_B$, and rewrite v_A in terms of the determinant of A. Find v_A and check that your answer agrees with your solution from Exercise 3.

Solution: Recall that to find the value, one substitutes the value of x into either one of the equations for E, such as $ax - c(x-1)$. Since we got the same value of x, we get the same value of E as well. In terms of determinants, observe that $-detA = detB$. It follows that

$$v_A = v_B = \frac{-detA}{D}.$$

Applying this to the example, $v = \frac{-detA}{12} = \frac{-(-18)}{12} = \frac{3}{2}$, as expected.

7. Consider the matrix $A = \begin{bmatrix} -2 & 1 \\ 6 & -3 \end{bmatrix}$. The corollary to the minimax theorem does not apply directly to A. However, if you list the row player's strategies in the reverse order, the effect is to interchange the rows of A to obtain the matrix $C = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$, and the corollary does apply to C. Repeat the steps in Exercise 6 to solve the game with payoff matrix A by comparing it to the solution of C.

Solution: This is just like exercise 6, so is left for the reader. (In fact, you could even deduce it from Exercise 6 by taking transposes of everything!)

8. Consider the zero-sum payoff matrix $A = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$ from Exercise 2. We continue to study the effect of interchanging rows and columns on the game. Exercises 6 and 7 explored what happened when interchanging just the rows, or just the columns. Interchanging just the rows or just the columns to this matrix will not yield a matrix to which the corollary applies. However, if you switch both the rows and the columns (that is, list the strategies for both players in reverse order), you obtain the matrix $B = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}$ to which the corollary does apply. By combining the results of Exercises 6 and 7, show how to obtain the solution of A from the solution of B. In particular, pay close attention to the formula for v, and observe that $detA = detB$. Similarly, the denominators D are the same for both matrices (why?) This explains why the formulas in Corollary 6.25 still hold when d is the largest entry instead of a.

Solution: Think of obtaining B from A by a two-step process. First switch just the columns of A to get a matrix C, then switch just the rows of C to get B. By exercises 6 and 7, the effect on the optimal solutions in $A \to C \to B$ is to switch both coordinates (since in one step we switch the coordinates in the row player's solution , and in the other we switch the coordinates in the column player's solution.) When we switch just the rows, or just the columns, the effect on the determinant is to multiply by -1. Since we are doing both, we multiply by $(-1)^2 = 1$. That is, $detA = detB$. Similarly, D is unchanged. (Indeed, $D = (d - c) + (a - b)$ from matrix A, but after switching rows and columns, we get $D = (a - b) + (d - c)$ from the matrix B, which is the same value as from A.) Thus, it still holds that

$$v_A = \frac{detA}{D},$$

as claimed.

The upshot of these exercises is that the formulas form Corollary 6.25 hold whenever the two largest elements of A lie on the main diagonal, and when they are on the "off" diagonal, the formulas hold with the exception of adding a negative sign in the formula for v: $v = \frac{-detA}{D}$ in that case.

9. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be any 2×2 payoff matrix for a zero-sum game without a saddle point. There is a method for finding the optimal solution, called the method of oddments, discovered by J. Williams (1954). The algorithm is as follows:

i) To find $\hat{p}$, the optimal strategy for the row player, take the absolute value of the differences of the entries across the rows. The resulting positive numbers (called *oddments* by Williams) are the odds for playing the rows in $\hat{p}$, except that you must switch the coordinates, so that the oddment in the first row gives the probability of plying the second row, and vice versa. To obtain the probabilities from the odds, simply put them over the sum of the oddments (since "odds" are, by definition, just the ratios of the probabilities.)

ii) To find $\hat{q}$, the optimal strategy for the column player, take the absolute value of the differences of the entries across the columns. The resulting oddments are the odds for playing the rows in $\hat{q}$, except that you must switch the coordinates, so that the oddment in the first column gives the probability of plying the second column, and vice versa. To obtain the probabilities from the odds, simply put them over the sum of the oddments (since "odds" are, by definition, just the ratios of the probabilities.)

iii) To find v, the value of the game, you can just use $v = \hat{p}A\hat{q}$ (or you can derive a formula similar to that in Corollary 6.25, where v is a fraction with the sum of the oddments in the denominator, and $\pm detA$ in the numerator, where the positive sign is taken if the two largest elements of A are on the main diagonal, and the negative sign otherwise.)

Here is an example illustrating the method of oddments. Suppose $A = \begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}$. We find $\hat{p}$ as follows:

	Absolute value of differences	Switch positions	oddments
$2 \quad -1$	$\lvert 2-(-1)\rvert = 3$	↘	7
$-4 \quad 3$	$\lvert (-4)-3\rvert = 7$	↗	3

Thus, the row player should play the rows with odds 7 to 3. This means the probabilities are $\frac{7}{7+3} = \frac{7}{10}$ and $\frac{3}{7+3} = \frac{3}{10}$. Therefore $\hat{p} = (.7 \quad .3)$. Similarly, we find $\hat{q}$

	2 -4	-1 3
Absolute value of differences	$\lvert 2-(-4)\rvert$ $= 6$	$\lvert -1-3\rvert$ $= 4$
Switch positions	↘	↙
oddments	4	6

Thus, the column player should play the columns with odds 4 to 6. Therefore, $\hat{q} = \begin{pmatrix} .4 \\ .6 \end{pmatrix}$. The value of the game is given by

$$v = \hat{p}A\hat{q} = (.7 \quad .3)\begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}\begin{pmatrix} .4 \\ .6 \end{pmatrix} = [.2]$$

This is the optimal solution. Note that the two largest elements of A are on the main diagonal, so an alternate calculation for the value is

$$v = \frac{detA}{D} = \frac{2}{10} = .2$$

a) Solve this game by either the method of derived games, or by Corollary 6.25 and verify that the above solution is correct.

Solution: Since we know Corollary 6.25 holds when the two largest elements of A lie on the main diagonal by Exercise 8, we can solve directly:

$$\hat{p} = \left(\frac{d-c}{D} \quad \frac{a-b}{D}\right) = \left(\frac{3-(-4)}{10} \quad \frac{2-(-1)}{10}\right) = \left(\frac{7}{10} \quad \frac{3}{10}\right)$$

$$\hat{q} = \begin{pmatrix} \frac{d-b}{D} \\ \frac{a-c}{D} \end{pmatrix} = \begin{pmatrix} \frac{3-(-1)}{10} \\ \frac{2-(-4)}{10} \end{pmatrix} = \begin{pmatrix} \frac{4}{10} \\ \frac{6}{10} \end{pmatrix}$$

$$v = \frac{ad-bc}{D} = \frac{(2)(3)-(-1)(-4)}{10} = \frac{2}{10}$$

This agrees with the above solution. The approach via the derived game results in the same outcome and is left for the reader.

b) By combining Exercises 6,7, and 8, Corollary 6.25 extends to any non-strictly determined 2×2 payoff matrix. Show that this extended corollary is the same thing as the method of oddments. [**Hint**: consider cases according to where the two largest entries of A are located.]

Solution: Case 1: The largest entries of A lie on the main diagonal. In this case $d-c$ and $a-b$ are positive, so we can drop the absolute values and note that these numbers ARE the oddments, so the formulas for the optimal solutions in the corollary agree with the method of oddments. Case 2: the largest entries are on the "off" diagonal. In this case, switch the rows to obtain a matrix B where the largest entries are not on the main diagonal of B. Apply case 1 to B, and observe that the calculations are the same as the method of oddments applied to A, where we don't drop the absolute values in the calculation of the oddments. By Exercise 7, the solutions to A and B are the same for $\hat{q}$. Similarly, switch the columns in A to obtain a matrix C, apply case 1 to C, and observe by Exercise 6, the solutions to A and C are the same for $\widehat{p}$. Thus in all cases, the method of oddments reduces to the formulas in Corollary 6.25.

10. Resolve the game with payoff matrix, as in Exercise 2, $A = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$ via the method of oddments. Check that your answer agrees with Exercise 2.

Solution: The row oddments are $|1-0| = 1$ and $|-3-2| = 5$, so $\hat{p} = \begin{pmatrix} \frac{5}{6} & \frac{1}{6} \end{pmatrix}$. The column oddments are $|1-(-3)| = 4$ and $|0-2| = 2$, so $\hat{q} = \begin{pmatrix} \frac{2}{6} \\ \frac{4}{6} \end{pmatrix}$. Also $v = \frac{detA}{6} = \frac{2}{6} = \frac{1}{3}$. This agrees with Exercise 2.

11. Repeat Exercise 10 for the matrix $A = \begin{bmatrix} -2 & 5 \\ 4 & -1 \end{bmatrix}$ of Exercise 3.

Solution: The row oddments are $|-2-5| = 7$ and $|4-(-1)| = 5$, so $\hat{p} = \begin{pmatrix} \frac{5}{12} & \frac{7}{12} \end{pmatrix}$. The column oddments are $|-2-4| = 6$ and $|5-(-1)| = 6$, so $\hat{q} = \begin{pmatrix} \frac{6}{12} \\ \frac{6}{12} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$. Also $v = \frac{-detA}{6} = \frac{18}{12} = \frac{3}{2}$. This agrees with Exercise 3.

12. Repeat Exercise 10 for the matrix $A = \begin{bmatrix} -2 & 1 \\ 6 & -3 \end{bmatrix}$ of Exercise 7.

Solution: The row oddments are $|-2-1| = 3$ and $|6-(-3)| = 9$, so $\hat{p} = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \end{pmatrix}$. The column oddments are $|-2-6| = 8$ and $|1-(-3)| = 4$, so $\hat{q} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$. Also $v = \frac{-detA}{12} = 0$.

In Exercises 13-20, solve the zero-sum game with given payoff matrices using the method of equalizing expectation.

13. $A = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$ as in Exercise 2.

Solution:

$$= (4x - 3 \quad 2 - 2x)$$

Setting the expectations equal:

$$4x - 3 = 2 - 2x$$

$$6x = 5$$

$$x = \frac{5}{6}$$

Thus, $v = E = 2 - 2\left(\frac{5}{6}\right) = \frac{1}{3}$. Also,

$$\begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix} = \begin{pmatrix} y \\ 2-5y \end{pmatrix}$$

Setting the expectations equal:

$$y = 2 - 5y$$

$$6y = 2$$

$$y = \frac{1}{3}$$

Thus, the optimal solution is:

$$\hat{p} = \begin{pmatrix} \frac{5}{6} & \frac{1}{6} \end{pmatrix}$$

$$\hat{q} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$$

$$v = \frac{1}{3}$$

This agrees with our previous solution.

14. $A = \begin{bmatrix} -2 & 5 \\ 4 & -1 \end{bmatrix}$ as in Exercise 3.

Solution:

$$\begin{pmatrix} x & 1-x \end{pmatrix} \begin{bmatrix} -2 & 5 \\ 4 & -1 \end{bmatrix} = \begin{pmatrix} 4-6x & 6x-1 \end{pmatrix}$$

Equalizing:

$$4 - 6x = 6x - 1$$

$$5 = 12x$$

$$x = \frac{5}{12}$$

Therefore, $v = E = 4 - 6\left(\frac{5}{12}\right) = \frac{3}{2}$. Similarly,

$$\begin{bmatrix} -2 & 5 \\ 4 & -1 \end{bmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix} = \begin{pmatrix} 5-7y \\ 5y-1 \end{pmatrix}$$

Equalizing:

$$\begin{gathered} 5 - 7y = 5y - 1 \\ 6 - 12y \\ y = \frac{1}{2} \end{gathered}$$

Thus, the optimal solution is:

$$\hat{p} = \begin{pmatrix} \frac{5}{12} & \frac{7}{12} \end{pmatrix}$$

$$\hat{q} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$v = \frac{3}{2},$$

which agrees with our previous solution.

15. $A = \begin{bmatrix} 6 & -1 \\ -3 & 2 \end{bmatrix}$ as in Exercise 4.

Solution:

$$(x \quad 1-x)\begin{bmatrix} 6 & -1 \\ -3 & 2 \end{bmatrix} = (9x - 3 \quad 2 - 3x)$$

Equalizing:

$$\begin{gathered} 9x - 3 = 2 - 3x \\ 12x = 5 \\ x = \frac{5}{12} \end{gathered}$$

Thus, $v = E = 9\left(\frac{5}{12}\right) - 3 = \frac{3}{4}$. Also

$$\begin{bmatrix} 6 & -1 \\ -3 & 2 \end{bmatrix}\begin{pmatrix} y \\ 1-y \end{pmatrix} = \begin{pmatrix} 7y - 1 \\ 2 - 5y \end{pmatrix}$$

Equalizing:

$$\begin{gathered} 7y - 1 = 2 - 5y \\ 12y = 3 \\ y = \frac{1}{4} \end{gathered}$$

Thus, the optimal solution is:

$$\hat{p} = \begin{pmatrix} \frac{5}{12} & \frac{7}{12} \end{pmatrix}$$

$$\hat{q} = \begin{pmatrix} \frac{1}{4} \\ \frac{3}{4} \end{pmatrix}$$

$$v = \frac{3}{4},$$

which agrees with our previous solution.

16. $A = \begin{bmatrix} -2 & 1 \\ 6 & -3 \end{bmatrix}$ as in Exercise 7.

Solution:

$$(x \quad 1-x)\begin{bmatrix} -2 & 1 \\ 6 & -3 \end{bmatrix} = (6-8x \quad 4x-3)$$

Equalizing:

$$6 - 8x = 4x - 3$$
$$9 = 12x$$
$$x = \frac{3}{4}$$

Thus, $v = E = 6 - 8\left(\frac{3}{4}\right) = 0$. Similarly

$$\begin{bmatrix} -2 & 1 \\ 6 & -3 \end{bmatrix}\begin{pmatrix} y \\ 1-y \end{pmatrix} = \begin{pmatrix} 1-3y \\ 9y-3 \end{pmatrix}$$

Equalizing:

$$1 - 3y = 9y - 3$$
$$4 = 12y$$
$$y = \frac{1}{3}$$

Thus, the optimal solution is:

$$\hat{p} = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

$$\hat{q} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$$

$$v = 0,$$

which agrees with our previous solution.

17. $A = \begin{bmatrix} 12 & -18 \\ -10 & 0 \end{bmatrix}$.

Solution:

$$(x \quad 1-x)\begin{bmatrix} 12 & -18 \\ -10 & 0 \end{bmatrix} = (22x - 10 \quad -18x)$$

Equalizing:

$$22x - 10 = -18x$$
$$40x = 10$$
$$x = \frac{1}{4}$$

Thus, $v = E = 22\left(\frac{1}{4}\right) - 10 = -\frac{9}{2}$. Similarly

$$\begin{bmatrix} 12 & -18 \\ -10 & 0 \end{bmatrix}\begin{pmatrix} y \\ 1-y \end{pmatrix} = \begin{pmatrix} 30y - 18 \\ -10y \end{pmatrix}$$

Equalizing:

$$30y - 18 = -10y$$
$$40y = 18$$
$$y = \frac{9}{20}$$

Thus, the optimal solution is:

$$\hat{p} = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

$$\hat{q} = \begin{pmatrix} \frac{9}{20} \\ \frac{11}{20} \end{pmatrix}$$

$$v = -\frac{9}{2}$$

18. $A = \begin{bmatrix} 1 & 8 \\ 7 & 2 \end{bmatrix}$. Compare your answer to that of exercise 14. Explain.

Solution:

$$(x \quad 1-x)\begin{bmatrix} 1 & 8 \\ 7 & 2 \end{bmatrix} = (7 - 6x \quad 6x + 2)$$

Equalizing:

$$7 - 6x = 6x + 2$$
$$5 = 12x$$
$$x = \frac{5}{12}$$

Thus, $v = E = 7 - 6\left(\frac{5}{12}\right) = \frac{9}{2}$. Similarly,

$$\begin{bmatrix} 1 & 8 \\ 7 & 2 \end{bmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix} = \begin{pmatrix} 8-7y \\ 5y+2 \end{pmatrix}$$

Equalizing:

$$8 - 7y = 5y + 2$$
$$6 = 12y$$
$$y = \frac{1}{2}$$

Thus, the optimal solution is

$$\hat{p} = \begin{pmatrix} \frac{5}{12} & \frac{7}{12} \end{pmatrix}$$

$$\hat{q} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$v = \frac{9}{2}$$

Comparing to the payoff matrix in Exercise 14, we see the optimal strategies are identical; however, the value here is 3 more than the value in Exercise 14. That's because the matrix here is just the matrix in Exercise 14 translated by 3. (Note: it's also clear by using oddments that when you translate a matrix you do not change the optimal strategies, because the differences between the elements of the matrix are unchanged.)

19. $A = \begin{bmatrix} 2 & 9 \\ 8 & 3 \end{bmatrix}$. Compare your answer to that of exercises 14 and 18. Explain.

Solution: This matrix is translated from that in Exercise 14 by 4 (or translated from that in Exercise 18 by 1.) This means the optimal strategies are identical, but the value will be $\frac{9}{2} + 1 = \frac{11}{2}$. The reader should verify this by equalizing expectation in the matrix A.

20. $A = \begin{bmatrix} 12 & -2 \\ -6 & 4 \end{bmatrix}$. Compare your answer to that of Exercise 15. Explain.

Solution: This time, the matrix is obtained from that in Exercise 15 by scaling instead of translating. Note that every payoff is doubled. Again, this leads to the identical optimal strategies, but with the value doubled:

$$\begin{pmatrix} x & 1-x \end{pmatrix} \begin{bmatrix} 12 & -2 \\ -6 & 4 \end{bmatrix} = \begin{pmatrix} 18x-6 & 4-6x \end{pmatrix}$$

Equalizing yields $x = \frac{5}{12}$. Then $v = E = 18\left(\frac{5}{12}\right) - 6 = \frac{3}{2}$, which is double the value of the game from Exercise 15. Finally

$$\begin{bmatrix} 12 & -2 \\ -6 & 4 \end{bmatrix}\begin{pmatrix} y \\ 1-y \end{pmatrix} = \begin{pmatrix} 14y-2 \\ 4-10y \end{pmatrix}$$

Equalizing yields $y = \frac{1}{4}$. Thus, the optimal solution is as claimed:

$$\hat{p} = \begin{pmatrix} \frac{5}{12} & \frac{7}{12} \end{pmatrix}$$

$$\hat{q} = \begin{pmatrix} \frac{1}{4} \\ \frac{3}{4} \end{pmatrix}$$

$$v = \frac{3}{2}$$

21. Based on Exercises 18-20; formulate a conjecture about solutions to translated and scaled games which are not strictly determined. Compare your conjecture to the remark at the end of the exercises in Section 6.2, which explored the same question for strictly determined games. Can you prove your conjecture?

Solution: The conjecture is that is B is obtained from A by translating, so each entry in B is the corresponding entry in A plus a constant b, then the optimal strategies for $A\ and\ B$ are identical, but $v_B = v_A + b$. Also, if B is obtained from A by scaling by a positive constant $m > 0$, then the optimal strategies for $A\ and\ B$ are identical, but $v_B = mv_A$.

As far as a proof goes, this is already known for strictly determined games of any size (and is obvious – just consider that the value of a strictly determined game is the value of the saddle point, and translating and/or positive scaling do not affect the location of the saddle point.)

For games which are not strictly determined, as hinted in the note at the end of Exercise 18, the method of oddments easily proves the theorem for 2×2 payoff matrices (or, just use the formulas of Corollary 6.25). For larger games, a proof will have to wait until we see how to solve larger games.

22. Consider the matrix $A = \begin{bmatrix} -2 & 1 \\ 6 & -3 \end{bmatrix}$ from Exercise 7. Suppose that the column player decides to ignore the optimal mix and instead play the mix $q = \begin{pmatrix} .1 \\ .9 \end{pmatrix}$ regardless of what the row player does. Use the expected value principle to decide how the row player should respond to this.

Solution:

$$Aq = \begin{bmatrix} -2 & 1 \\ 6 & -3 \end{bmatrix} \begin{pmatrix} .1 \\ .9 \end{pmatrix} = \begin{pmatrix} 0.7 \\ -2.1 \end{pmatrix}$$

According to the expected value principle, the row player should always play the first row because it has the better expected payoff.

23. Consider the matrix $A = \begin{bmatrix} -2 & 1 \\ 6 & -3 \end{bmatrix}$ from Exercise 7. Suppose that the row player decides to ignore the optimal mix and instead play the mix $p = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \end{pmatrix}$. Use the expected value principle to decide how the column player should respond to this.

Solution:

$$pA = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \end{pmatrix} \begin{bmatrix} -2 & 1 \\ 6 & -3 \end{bmatrix} = (4 \quad -2)$$

According to the expected value principle, the column player should always play the second column.

24. Consider the matrix $A = \begin{bmatrix} 1 & -4 & 2 & 0 \\ 1 & 1 & 1 & 5 \\ -3 & -1 & 8 & 4 \\ 0 & 2 & -9 & -1 \end{bmatrix}$ from Exercise 14 of the last Section. Suppose that the column player decides to ignore the optimal strategy mix and instead use ply the mix $q = \begin{pmatrix} .2 \\ .6 \\ 0 \\ .2 \end{pmatrix}$, regardless of what the row player does. How should the row player respond?

$$Aq = \begin{bmatrix} 1 & -4 & 2 & 0 \\ 1 & 1 & 1 & 5 \\ -3 & -1 & 8 & 4 \\ 0 & 2 & -9 & -1 \end{bmatrix} \begin{pmatrix} .2 \\ .6 \\ 0 \\ .2 \end{pmatrix} = \begin{pmatrix} -2.2 \\ -0.2 \\ -0.4 \\ 1.0 \end{pmatrix}$$

Thus, the row player should always play the fourth row.

25. Consider the variation of four-finger Morra in Exercise 9 of Section 6.1. If the column plyer decided to always use the mix $q = \begin{pmatrix} .4 \\ 0 \\ .4 \\ .2 \end{pmatrix}$, how should the row player respond?

Solution:

$$Aq = \begin{bmatrix} 0 & -1 & 2 & -3 \\ -1 & 0 & -1 & 2 \\ 2 & -1 & 0 & -1 \\ -3 & 2 & -1 & 0 \end{bmatrix} \begin{pmatrix} .4 \\ 0 \\ .4 \\ .2 \end{pmatrix} = \begin{pmatrix} 0.2 \\ -0.4 \\ 0.6 \\ -1.6 \end{pmatrix}$$

The row player should always play the third row.

26. Consider the guessing game from Exercise 10 of Section 6.1. If Ann decided to use the following mix for her guesses, $p = (.1 \quad .2 \quad .3 \quad .4)$, and Bill knows this, how should Bill respond?

Solution:

$$pA = (.1 \quad .2 \quad .3 \quad .4) \begin{bmatrix} 1 & -3 & -5 & -7 \\ -3 & 3 & -4 & -6 \\ -5 & -4 & 7 & -4 \\ -7 & -6 & -4 & 11 \end{bmatrix} = (-4.8 \quad -3.3 \quad -0.8 \quad 1.3)$$

Bill should respond by always playing the first column – that is, he should put $0 in his hand.

27. Same question as 25, except use the variation of four-finger Morra in Exercise 18 of Section 6.2.

Solution:

$$Aq = \begin{bmatrix} 1 & 0 & -1 & 1 \\ -0 & -1 & 1 & 0 \\ -1 & 1 & 0 & -1 \\ 1 & 0 & -1 & 1 \end{bmatrix} \begin{pmatrix} .4 \\ 0 \\ .4 \\ .2 \end{pmatrix} = \begin{pmatrix} 0.2 \\ 0.4 \\ -0.6 \\ 0.2 \end{pmatrix}$$

The row player should always play the second row, that is, play 2 fingers.

28. Consider the variation on four-finger Morra in Exercise 17 of Section 6.2. If the row player decided to use the mix $p = (0 \quad .1 \quad .4 \quad .5)$, how should the column player respond?

Solution:

$$pA = (0 \quad .1 \quad .4 \quad .5)\begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 2 & -2 & 2 \\ 1 & -2 & 3 & -3 \\ -1 & 2 & -3 & 4 \end{bmatrix} = (-0.2 \quad 0.4 \quad -0.5 \quad 1.0)$$

The column player should always play the third column, that is, play 3 fingers.

29. Consider the finite version of the market game Antares vs. Bellatrix given in Exercise 7 of Section 6.1. Suppose the market for 400,000 chips per year will extend into the foreseeable future, and both companies will reset their prices annually. Suppose that $\frac{1}{3}$ of the time, Antares sets their price at \$80, and the remaining $\frac{2}{3}$ of the time, they set their price at \$110. Bellatrix is aware of this proportion and has reason to believe that Antares will continue this practice into the future. How should Bellatrix set their price?

Solution:

$$pA = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \end{pmatrix}\begin{bmatrix} 66 & 60 & 57 \\ 48 & 60 & 66 \end{bmatrix} = (54 \quad 60 \quad 63)$$

Thus, Bellatrix should always play the first column – that is, they should always set their price at \$80.

30. Consider the variable-sum game in Example 6.4 of Section 6.1, the Saturday chores at the Anderson household, version II. In the text we determined that the payoff matrix is as follows:

		Laurie			
		No chore	*Dishes*	*Raking*	*Both*
Jon	*No chore*	(10,10)	(12,6)	(8,15)	(0,21)
	Dishes	(8,15)	(10,10)	(8,15)	(4,18)
	Raking	(12,6)	(12,6)	(10,10)	(6,13)
	Both	(20,0)	(16,3)	(14,7)	(10,10)

a) Suppose that Jon decides the use the mix $p = (.1 \quad .3 \quad .4 \quad .2)$ and Laure decides to use the mix $q = \begin{pmatrix} 0 \\ .3 \\ .6 \\ .1 \end{pmatrix}$. With these mixes, what is the expected payoff to both children? [**Hint**: we have not formally worked with variable-sum games yet, but you can determine this. Just remember that the payoffs to Jon are in the first coordinates of the ordered pairs in A, and the payoffs to Laurie are the second coordinates. The formula $E = pAq$ still applies, but to each players payoffs separately.]

Solution: The expected payoffs for Jon:

$$E = pAq = (.1 \quad .3 \quad .4 \quad .2)\begin{bmatrix} 10 & 12 & 8 & 0 \\ 8 & 10 & 8 & 4 \\ 12 & 12 & 10 & 6 \\ 20 & 16 & 14 & 10 \end{bmatrix}\begin{pmatrix} 0 \\ .3 \\ .6 \\ .1 \end{pmatrix} = [10.22]$$

The expected payoffs for Laurie:

$$E = pAq = (.1 \quad .3 \quad .4 \quad .2)\begin{bmatrix} 10 & 6 & 15 & 21 \\ 15 & 10 & 15 & 18 \\ 6 & 6 & 10 & 13 \\ 0 & 3 & 7 & 10 \end{bmatrix}\begin{pmatrix} 0 \\ .3 \\ .6 \\ .1 \end{pmatrix} = [10.29]$$

So, the expected payoffs for $(Jon, Laurie)$ are $(\$10.22, \$10.29)$.

b) Suppose that Jon becomes aware that Laurie is using the mix given in part (a), and has reason to believe she will continue to do so, regardless of what he does. How should he respond? [**Hint**: Apply the expected value principle, just using Jon's payoffs.]

Solution: Following the hint:

$$Aq = \begin{bmatrix} 10 & 12 & 8 & 0 \\ 8 & 10 & 8 & 4 \\ 12 & 12 & 10 & 6 \\ 20 & 16 & 14 & 10 \end{bmatrix}\begin{pmatrix} 0 \\ .3 \\ .6 \\ .1 \end{pmatrix} = \begin{pmatrix} 8.4 \\ 8.2 \\ 10.2 \\ 11.2 \end{pmatrix}$$

His best response is to always choose the fourth row; that is, always choose both jobs.

31. In this exercise, we consider the derived game of A in the case when A is itself strictly determined. Consider a zero-sum game with payoff matrix $A = \begin{bmatrix} 0 & 1 \\ -1 & d \end{bmatrix}$, where d is not specified.

a) Show that no matter what value is assigned to d, the matrix A has a saddle point so represents a strictly determined game. What is the optimal solution of this game, as predicted by the minimax theorem?

Solution: The 0 entry in the a_{11} position is a saddle point for any choice of d. The row player should play the first row, the column player should play the first column, and the value is $v = 0$ (a fair game.)

b) Consider the derived game of A. Find $E(x, y)$ in standard form, and show that it is a saddle surface if $d \neq 0$. However, show that the saddle point is outside the range $0 \leq x \leq 1$ and $0 \leq y \leq 1$, so it does not represent the optimal solution of the game.

Solution: The surface $E(x, y)$ has equation:

$$E(x,y) = (x \quad 1-x)\begin{bmatrix} 0 & 1 \\ -1 & d \end{bmatrix}\begin{pmatrix} y \\ 1-y \end{pmatrix}$$
$$= y(x-1) - \big(x - d(x-1)\big)(y-1)$$
$$= dxy - dx - dy + x - y + d$$

Assume $d \neq 0$, so this equation does represent a saddle surface. In standard form, we have:

$$E(x,y) = d\left(xy - \frac{d-1}{d}x - \frac{d+1}{d}y\right) + d$$
$$= d\left(x\left(y - \frac{d-1}{d}\right) - \frac{d+1}{d}\left(y - \frac{d-1}{d}\right) - \frac{d^2-1}{d^2}\right) + d$$
$$= d\left(x\left(y - \frac{d-1}{d}\right) - \frac{d+1}{d}\left(y - \frac{d-1}{d}\right)\right) - \frac{d^2-1}{d} + d$$
$$= d\left(x - \frac{d+1}{d}\right)\left(y - \frac{d-1}{d}\right) + \frac{1}{d}$$

This is a saddle surface with saddle point located at $\left(\frac{d+1}{d}, \frac{d-1}{d}\right)$, and the value of E at that point is $\frac{1}{d}$. However, the saddle point is NOT in the correct range $0 \leq x \leq 1$ and $0 \leq y \leq 1$. Indeed, if $d > 0$, then $x = \frac{d+1}{d} > 1$, so x cannot represent a probability. On the other hand, if $d < 0$, then $d - 1 < d < 0$, so $y = \frac{d-1}{d}$ is a positive number; however, it is greater than 1 since $|d-1| > |d|$ so y cannot be interpreted as a probability. In other words, the location of the global saddle point for the surface which is the derived game does not agree with the solution of the game.

c) What happens if $d = 0$?

Solution: In this case, the formula for $E(x,y)$ reduces to simply $E = x - y$. This is not a saddle surface (in fact it is a plane). In this case, since the row player wants to maximize E, and they control only x, they should choose x as large as possible; that is, $x = 1$. By the same argument, the column player wants to minimize $E = x - y$, and they only control y so they should choose y as large as possible; that is $y = 1$. But this leads to the correct solution since both players are choosing their first strategies, resulting in the saddle point of A with $v = E = x - y = 1 - 1 = 0$.

32. Consider the same matrix A as in Exercise 31. Instead of using the derived game, try so solve it using the method of equalizing expectations. Show that you never get the correct answer, no matter what d is.

Solution:

$$(x \quad 1-x)\begin{bmatrix} 0 & 1 \\ -1 & d \end{bmatrix} = (x-1 \quad x-d(x-1))$$

Equalizing:

$$\begin{gathered} x-1 = x-d(x-1) = x-dx+d \\ -1 = -dx+d \\ dx = d+1 \\ x = \frac{d+1}{d}, \end{gathered}$$

assuming $d \neq 0$.

Also

$$\begin{bmatrix} 0 & 1 \\ -1 & d \end{bmatrix}\begin{pmatrix} y \\ 1-y \end{pmatrix} = \begin{pmatrix} 1-y \\ -y-d(y-1) \end{pmatrix}$$

Equalizing:

$$\begin{gathered} 1-y = -y-dy+d \\ 1 = -dy+d \\ dy = d-1 \\ y = \frac{d-1}{d}, \end{gathered}$$

assuming $d \neq 0$.

Thus, we obtain $(x,y) = \left(\frac{d+1}{d}, \frac{d-1}{d}\right)$, which is the same thing we obtained using the derived game approach in Exercise 31, which we know leads to a wrong answer since either x or y cannot be interpreted as a probability depending on the sign of d.
Finally, if $d = 0$, the equations reduce to

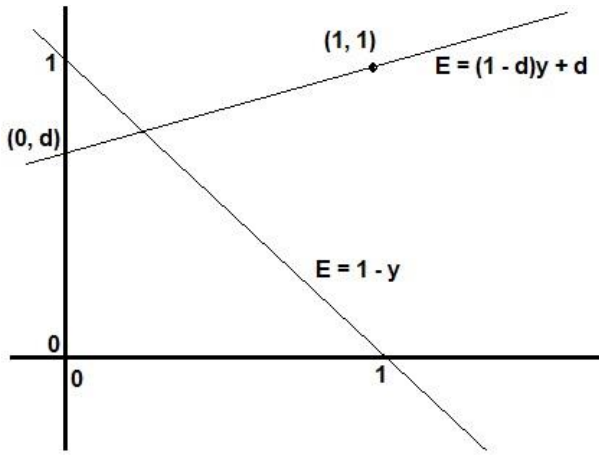
$$\begin{gathered} x-1 = x \\ 1-y = -y \end{gathered}$$

Neither of these has a solution. Thus, the method of equalizing expectation fails to give the correct answer if A has a saddle point (as remarked above the same thig always happens when there is a saddle point.)

An Alternate Approach: Consider the graphs of the expectation lines E_i for the column player (working with the row player would also lead to the same conclusion).

$$\begin{gathered} E_1 = 1-y \\ E_2 = y-d(y-1) = (1-d)y+d \end{gathered}$$

Observe that E_2 always passes through the point $(y,E) = (1,1)$ regardless of d, and that d is in fact the y-intercept of that line. The plot looks like:



We have just drawn one case for a specific value of d. Clearly, as long as $d < 1$, the point of intersection of the two lines has $0 < x, y < 1$, so represents a mixed strategy and so is NOT the solution to the game. If $2 > d > 1$ then the point of intersection is in the second quadrant, so has $x < 0$ (and, clearly, $y > 1$) and so neither can be interpreted as a probability. If $d = 2$, the lines are parallel and have no point of intersection. If $d > 2$, the intersection point is in the fourth quadrant, so $y < 0$ (and, clearly, $x > 1$) and so neither can be interpreted as a probability. The only remaining case is when $d = 1$, in which case the y-intercept $(0,1)$ is the point of intersection. Now, this point DOES represent a pure strategy – however, it is the WRONG one! With $x = 0$, the pure strategy is $(x, 1 - x) = (0,1)$, which means the row player chooses the second row. But the saddle point of this matrix is not in the second row. Thus, in every case, the method of equalizing expectations gives the incorrect solution to a strictly-determined game.

Chapter 7. More Game Theory

Section 7.1 Solving Larger Constant-sum Games

7.1.1 Games with Dominated Strategies

No music references or exercises in this subsection.

7.1.2 The $2 \times n$ and $m \times 2$ Case

No music references or exercises in this subsection.

7.1.3 A 3×3 example: The Square Subgame Theorem

Solutions to exercises:

Solve the games with the following payoff matrices. Assume the game is zero-sum.

1.

$$A = \begin{bmatrix} 1 & -2 & 3 \\ -3 & 1 & -1 \end{bmatrix}$$

Solution: Column 3 is dominated by column 1, so we can delete it. The resulting game (which is not strictly determined) is

$$\begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}$$

We can solve this 2×2 game by the derived game approach, equalizing expectation, or the method of oddments. We illustrate this example with the derived game approach. We have:

$$E(x, y) = [x \quad 1-x] \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} y \\ 1-y \end{bmatrix} = [7xy - 4y - 3x + 1]$$

$$= 7\left(xy - \frac{4}{7}y - \frac{3}{7}x\right) + 1$$

$$= 7\left(y\left(x - \frac{4}{7}\right) - \frac{3}{7}\left(x - \frac{4}{7}\right) - \frac{12}{49}\right) + 1$$

$$= 7\left(x - \frac{4}{7}\right)\left(y - \frac{3}{7}\right) - \frac{12}{7} + 1 = 7\left(x - \frac{4}{7}\right)\left(y - \frac{3}{7}\right) - \frac{5}{7}$$

Thus, the saddle point is located at $(x, y) = \left(\frac{4}{7}, \frac{3}{7}\right)$, with value $-\frac{5}{7}$. From here we determine the optimal solution to the 2×2 subgame, and then add a 0 to correspond to the inactive (dominated) strategy for the column player to obtain the optimal solution for the original 2×3 game:

$$\hat{p} = \left(\frac{4}{7}, \frac{3}{7}\right)$$

$$\hat{q} = \begin{pmatrix} \frac{3}{7} \\ \frac{4}{7} \\ 0 \end{pmatrix}$$

$$v = -\frac{5}{7}$$

2.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Solution: The entry $a_{31} = 5$ is a saddle point. Thus the game is strictly determined with optimal solution:

$$\hat{p} = (0 \quad 0 \quad 1)$$

$$\hat{q} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$v = 5$$

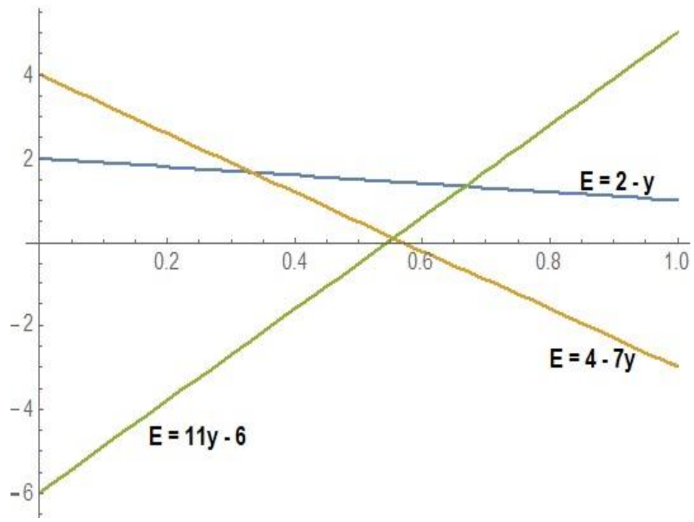
3.

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \\ 5 & -6 \end{bmatrix}$$

Solution: In this game there is no saddle point and no instances of dominance. Thus, we focus on the column player first:

$$A\hat{q} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \\ 5 & -6 \end{bmatrix} \begin{bmatrix} y \\ 1-y \end{bmatrix} = \begin{bmatrix} 2-y \\ 4-7y \\ 11y-6 \end{bmatrix}$$

Plot these Expectation lines in the same coordinate system:



For the column player, the worst case scenarios are on the top edge of these lines, so the best of the worst case scenarios is the lowest point on the top edge, which is the point where the first and third lines meet. To find the coordinates of this point, we equalize these expectation lines:

$$2 - y = 11y - 6$$
$$8 = 12y$$
$$y = \frac{2}{3}$$
$$v = E = 2 - \left(\frac{2}{3}\right) = \frac{4}{3}$$

This determines what the column player should do. Now, as discussed in the text, the line not used represents an inactive strategy for the other player, so we delete it to obtain a 2×2 subgame. We obtain:

$$\begin{bmatrix} 1 & 2 \\ 5 & -6 \end{bmatrix}$$

We now solve this game for the row player. Again, we can use the method of the derived games, equalizing expectation, or oddments. This time, we illustrate the technique of equalizing expectation:

$$\hat{p}A = (x \quad 1 - x)\begin{bmatrix} 1 & 2 \\ 5 & -6 \end{bmatrix} = [5 - 4x \quad 8x - 6]$$

Equalizing:

$$5 - 4x = 8x - 6$$

$$11 = 12x$$

$$x = \frac{11}{12}$$

So, the row player's optimal strategy to the subgame is $\begin{pmatrix} \frac{11}{12} & \frac{1}{12} \end{pmatrix}$ and the value is $v = E = 5 - 4\left(\frac{11}{12}\right) = \frac{4}{3}$ (which we already knew from the column's player's calculation above.) Putting in a 0 for the inactive strategy, the solution to the original 3×2 game is

$$\hat{p} = \begin{pmatrix} \frac{11}{12} & 0 & \frac{1}{12} \end{pmatrix}$$

$$q = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$$

$$v = \frac{4}{3}$$

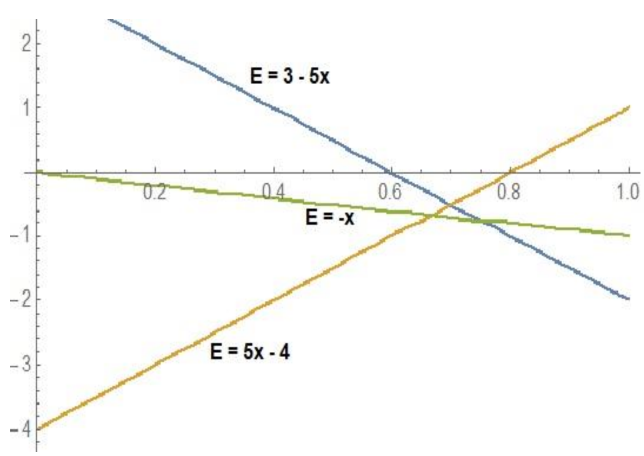
4.

$$A = \begin{bmatrix} -2 & 1 & -1 \\ 3 & -4 & 0 \end{bmatrix}$$

Solution: This game has no saddle points or dominated strategies, so we focus first on the row player:

$$\hat{p}A = (x \quad 1-x)\begin{bmatrix} -2 & 1 & -1 \\ 3 & -4 & 0 \end{bmatrix} = [3-5x \quad 5x-4 \quad -x]$$

Plot these Expectation lines in the same coordinate system:



For the row player, the best of the worst-case scenarios is the highest point along the bottom edge of this graph, which is the intersection of the second and third lines:

$$5x - 4 = -x$$
$$6x = 4$$
$$x = \frac{2}{3}$$
$$v = -x = -\frac{2}{3}$$

So, the first strategy is inactive for the column player. Deleting it yields the subgame:

$$\begin{bmatrix} 1 & -1 \\ -4 & 0 \end{bmatrix}$$

This time, we illustrate finding the column player's optimal strategy using oddments. The column oddments are 5,1 in that order, so $\hat{q} = \begin{pmatrix} \frac{1}{6} \\ \frac{5}{6} \end{pmatrix}$. For the original 2×3 game, the optimal solution is then:

$$\hat{p} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

$$\hat{q} = \begin{pmatrix} 0 \\ \frac{1}{6} \\ \frac{5}{6} \end{pmatrix}$$

$$v = -\frac{2}{3}$$

5.

$$A = \begin{bmatrix} 1 & 4 & -2 \\ 0 & 3 & -3 \\ -3 & -2 & 2 \end{bmatrix}$$

Solution: Row 1 dominates row 2, so delete row 2 to obtain a subgame with payoff matrix:

$$\begin{bmatrix} 1 & 4 & -2 \\ -3 & -2 & 2 \end{bmatrix}$$

Now, the first column dominates the second column, so delete the second to obtain a further subgame which is 2×2:

$$\begin{bmatrix} 1 & -2 \\ -3 & 2 \end{bmatrix}$$

By oddments, we find the solution to the subgame. The row oddments are 3,5, so $\hat{p} = \begin{pmatrix} \frac{5}{8} & \frac{3}{8} \end{pmatrix}$. The column oddments are 4,4, so $\hat{q} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$. The value is $v = \frac{detA}{8} = \frac{-4}{8} = -\frac{1}{2}$. Thus, the solution to the original 3×3 game is:

$$\hat{p} = \begin{pmatrix} \frac{5}{8} & 0 & \frac{3}{8} \end{pmatrix}$$

$$\hat{q} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}$$

$$v = -\frac{1}{2}$$

6. a)

$$A = \begin{bmatrix} 6 & -7 & -7 \\ -3 & 5 & 5 \\ 1 & 1 & 1 \end{bmatrix}$$

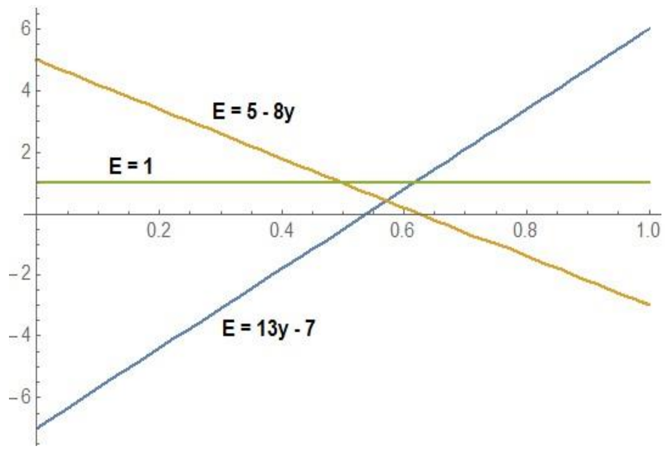
Solution: With identical payoffs, columns 2 and 3 dominate each other, so either one could be considered inactive. For example, if we delete the third column, we obtain the subgame with payoff matrix:

$$\begin{bmatrix} 6 & -7 \\ -3 & 5 \\ 1 & 1 \end{bmatrix}$$

There is no further dominance, so we focus on the column player first:

$$A\hat{q} = \begin{bmatrix} 6 & -7 \\ -3 & 5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y \\ 1-y \end{bmatrix} = \begin{bmatrix} 13y-7 \\ 5-8y \\ 1 \end{bmatrix}$$

Plotting the expectation lines:



The best of the worst case scenarios are on the lowest part of the top edge of edge lines, so on the horizontal segment between the lines with nonzero slope. The intersection of the line $E = 1$ with $E = 13y - 7$ is $y = \frac{8}{13} \approx 0.61538$. The intersection of the line $E = 1$ with $E = 5 - 8y$ is $y = \frac{1}{2}$. Thus, we have a range of values that work (all of which yield $E = 1$): $.5 \leq y \leq .615$.

If the column player chooses $y = .5$, then the first row is inactive for the row player. Deleting it leads to the subgame:

$$\begin{bmatrix} -3 & 5 \\ 1 & 1 \end{bmatrix}$$

But this matrix has a saddle point (with value $v = 1$. So the optimal strategy for the row player is to always choose the second row – a pure strategy. At the other extreme, if the column player chooses $y = \frac{8}{13}$, then the second row is inactive, and deleting it leads to the subgame:

$$\begin{bmatrix} 6 & -7 \\ 1 & 1 \end{bmatrix}$$

This matrix also has a saddle point with value $v = 1$ (although it is not in the same location as the previous case.) So again, the optimal strategy for the row player is to always choose the second row – a pure strategy.

But the column player could conceivably choose a value of y between these extreme cases, in which case it may not be clear which, if any, of the rows is inactive.

The way out of this seeming impasse is to act as if none of the rows is inactive and see if we can still determine $\hat{p}$ even though it has 3 coordinates, not two. (This is similar to what we did in the text in the 3×3 example which we solved via equalizing expectation.) So, suppose $\hat{p} = (x \quad y \quad 1-x-y)$. Then

$$\hat{p}A = (x \quad y \quad 1-x-y)\begin{bmatrix} 6 & -7 \\ -3 & 5 \\ 1 & 1 \end{bmatrix} = (5x - 4y + 1 \quad 4y - 8x + 1)$$

Each of these equations represents a plane in the (x, y, E) -space. Equalizing, we obtain:

$$5x - 4y + 1 = 4y - 8x + 1$$

$$13x = 8y$$

$$y = \frac{13}{8}x$$

This describes the $(x\ , y)$ values on the line where the two planes meet, but we need to know the E values along this line:

$$E = 5x - 4y + 1$$

$$= 5x - 4\left(\frac{13}{8}x\right) + 1$$

$$= 1 - \frac{3}{2}x$$

But the row player controls x and wants E to be as large as possible, so clearly, we must have $x = 0$, whence $y = \frac{13}{8}x = 0$ also. It follows that $\hat{p} = (0 \quad 0 \quad 1)$, so it turns out that BOTH row 1 and row 2 are inactive (and, surprisingly, the solution for the row player is a pure strategy even though there is no saddle point in the original matrix!). So one possible solution is:

$$\hat{p} = (0 \quad 0 \quad 1)$$

$$q = \begin{pmatrix} .5 \\ .5 \\ 0 \end{pmatrix}$$

$$v = 1$$

b) This game has more than one optimal solution. Find all of the solutions.

Solution: The solution we listed in part (a) was obtained by deleting the third column and making the choice $y = 0.5$. But we have already observed that other values of y work. Thus, the solution in part (a) can be generalized (without changing $\hat{p}$ or v) to:

$$\hat{p} = (0 \quad 0 \quad 1)$$

$$q = \begin{pmatrix} r \\ 1-r \\ 0 \end{pmatrix}, \quad \frac{1}{2} \leq r \leq \frac{8}{13}$$

$$v = 1$$

However, everything we did in part (a) also works if we delete the second column instead of the third, in which case we obtain a different set of solutions:

$$\hat{p} = (0 \quad 0 \quad 1)$$

$$q = \begin{pmatrix} r \\ 0 \\ 1-r \end{pmatrix}, \quad \frac{1}{2} \le r \le \frac{8}{13}$$

$$v = 1$$

In fact, since the second and third columns are identical, we could take any mix of them which sums to $1 - r$, so the complete set of optimal solutions looks like:

$$\hat{p} = (0 \quad 0 \quad 1)$$

$$q = \begin{pmatrix} r \\ s \\ 1-r-s \end{pmatrix}, \quad \frac{1}{2} \le r \le \frac{8}{13}, 0 \le s \le 1-r$$

$$v = 1$$

So not only is the solution not unique, but is a two-parameter infinite family of solutions. One of the parameters (r) arose due to a tie in the worst-case scenario payoffs for the column player, the other (s) arose because there were two identical columns. The example is also interesting since the optimal strategy for one of the players is a pure strategy even though the payoff matrix has no saddle point.

7.

$$A = \begin{bmatrix} 1 & 2 & -1 & -2 \\ -3 & 0 & 1 & 3 \\ -6 & -5 & 0 & 2 \end{bmatrix}$$

Solution: The first column dominates the second, so deleting the second leads to a subgame:

$$\begin{bmatrix} 1 & -1 & -2 \\ -3 & 1 & 3 \\ -6 & 0 & 2 \end{bmatrix}$$

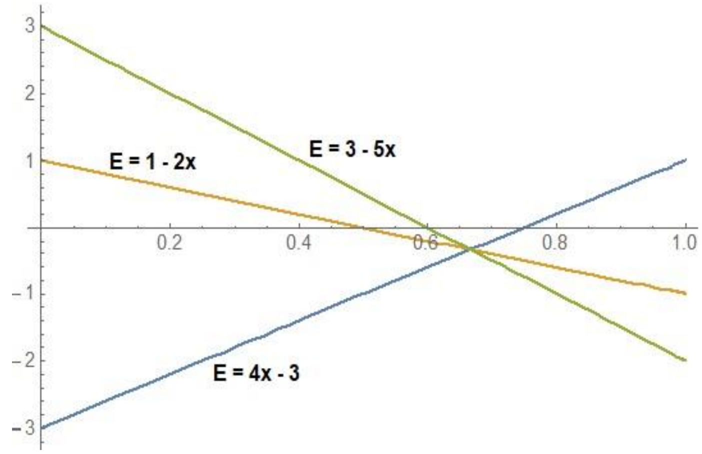
Now the second row dominates the third (this was true in the original matrix as well), so we can delete the third row to obtain the subgame:

$$\begin{bmatrix} 1 & -1 & -2 \\ -3 & 1 & 3 \end{bmatrix}$$

There is no further dominance, so we now focus on the row player:

$$\hat{p}A = (x \quad 1-x)\begin{bmatrix} 1 & -1 & -2 \\ -3 & 1 & 3 \end{bmatrix} = (4x-3 \quad 1-2x \quad 3-5x)$$

Plotting the expectation lines:



IT turns out that all three lines are concurrent – they all pass through the point $\left(\frac{2}{3}, -\frac{1}{3}\right)$, so $\hat{p} = \left(\frac{2}{3}, \frac{1}{3}\right)$ and $v = E = -\frac{1}{3}$ for this subgame. What about the column player? Well, as in the previous exercise, we have a choice as to which strategy is inactive.

If we delete the middle line (second column), we obtain the subgame:

$$\begin{bmatrix} 1 & -2 \\ -3 & 3 \end{bmatrix}$$

By oddments, the column player should play $\begin{pmatrix} \frac{5}{9} \\ \frac{4}{9} \end{pmatrix}$ in the subgame. This means, in the original game, by including 0's for the inactive strategies, we obtain:

$$\hat{p} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix}$$

$$\hat{q} = \begin{pmatrix} \frac{5}{9} \\ 0 \\ 0 \\ \frac{4}{9} \end{pmatrix}$$

$$v = -\frac{1}{3}$$

On the other hand, if we delete the last column instead, we obtain the subgame:

$$\begin{bmatrix} 1 & -1 \\ -3 & 1 \end{bmatrix}$$

By oddments (and then inserting 0's for the inactive strategies), we obtain:

$$\hat{p} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix}$$

$$\hat{q} = \begin{pmatrix} \frac{1}{3} \\ 0 \\ \frac{2}{3} \\ 0 \end{pmatrix}$$

$$v = -\frac{1}{3}$$

Thus, we have two distinct optimal solutions for this game.

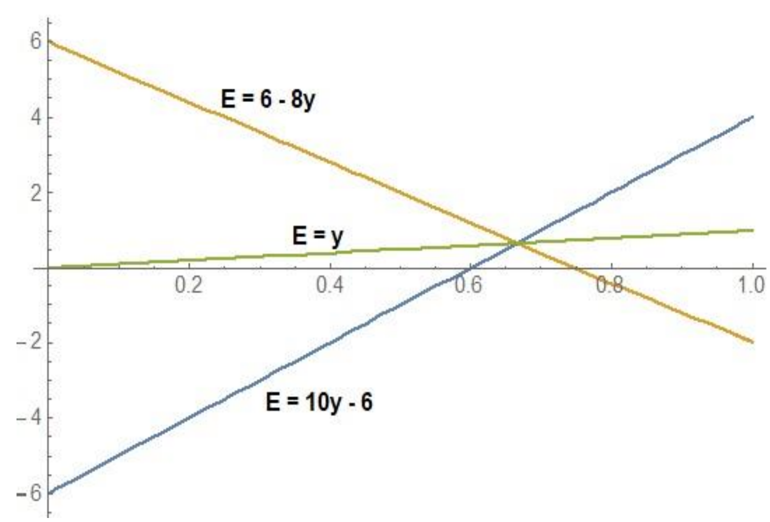
8. The following game has more than one solution given by two distinct square subgames. Find the solutions and determine which 2×2 subgames give the correct solution. Why doesn't the third 2×2 subgame give a correct solution?

$$A = \begin{bmatrix} 4 & -6 \\ -2 & 6 \\ 1 & 0 \end{bmatrix}$$

Solution:

$$A\hat{q} = \begin{bmatrix} 4 & -6 \\ -2 & 6 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix} = \begin{pmatrix} 10y - 6 \\ 6 - 8y \\ y \end{pmatrix}$$

Plotting the expectation lines:



As in the previous exercise, the three lines are concurrent, this time at the point $(y, E) = \left(\frac{2}{3}, \frac{2}{3}\right)$. This determines the column player's optimal strategy $\hat{q}$ and the value $v = E = \frac{2}{3}$.

To find the row player's optimal strategy, we again have a choice of which strategy is inactive. However, if we make the second row inactive, we are left with the subgame:

$$\begin{bmatrix} 4 & -6 \\ 1 & 0 \end{bmatrix}$$

But notice that the 0 is a saddler point of this matrix. Thus, the optimal solution for this subgame would involve both players choosing a pure strategy, which is clearly NOT the solution to the original game (for which we already know $\hat{q} = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$.) One way to see why this happens is to look in the above graph. When we delete the only line with the negative slope, we are left with two increasing lines, and the point where they meet at $y = \frac{2}{3}$ is no longer the lowest point on the top edge! The lowest point is now the origin, corresponding to $y = 0$ which gives us the pure strategy corresponding to the saddle point in the subgame.

However, if we make either one of the other two rows inactive, then the remaining lines have opposite sign slopes, so the point where they cross at $y = \frac{2}{3}$ is the lowest point along the top edge of the graph for the subgame (which will not have a saddle point), and gives the correct minimax solution to both the subgame and the entire game.

Indeed, if the first row is inactive, we obtain the subgame:

$$\begin{bmatrix} -2 & 6 \\ 1 & 0 \end{bmatrix}$$

We solve this by oddments, then inset the 0 for the inactive strategy, giving an optimal solution to the original game:

$$\hat{p} = \begin{pmatrix} 0 & \frac{1}{9} & \frac{8}{9} \end{pmatrix}$$

$$\hat{q} = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$$

$$v = \frac{2}{3}$$

If instead we make the third row inactive, we obtain this subgame:

$$\begin{bmatrix} 4 & -6 \\ -2 & 6 \end{bmatrix}$$

Again, solving via oddments and remembering the inactive strategy, we obtain the optimal solution:

$$\hat{p} = \begin{pmatrix} \frac{4}{9} & \frac{5}{9} & 0 \end{pmatrix}$$

$$\hat{q} = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$$

$$v = \frac{2}{3}$$

In summary, the game has a non-unique optimal solution; in fact, two solutions coming from 2 of the 3 possible 2×2 subgames.

9. Consider a variation on three-finger Morra. Each player simultaneously puts out either 1,2, or 3 fingers. If the total number of fingers revealed is even, the column player pays the row player, while of the total is odd, the row player pays the column player. However, instead of the payment being $1, the number of dollars is will be the (non-negative) difference between the number of fingers they show. Find the optimal solution by equalizing expectation.

Solution: Here is the payoff matrix:

$$A = \begin{bmatrix} 0 & -1 & 2 \\ -1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

There is no dominance (nor saddle point) in this matrix. Perhaps, then, all three strategies are active., in which case, the square subgame which determines the solution is the entire game. So, like the rock-paper-scissors example we did in the text, we will try t solve this assuming all three strategies are possibly active. Focus on the row player first:

$$\hat{p}A = (x \quad y \quad 1-x-y)\begin{bmatrix} 0 & -1 & 2 \\ -1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix} = (2-3y-2x \quad y-1 \quad 2x-y)$$

Each of these equations is a plane in (x, y, E) – space. Equalizing:

$$y - 1 = 2x - y$$
$$2y - 1 = 2x$$

Now combined with the first equation we have:

$$E = 2 - 3y - 2x = 2 - 3y - (2y - 1) = 3 - 5y$$

Equalizing this with the second equation:

$$3 - 5y = y - 1$$
$$4 - 6y$$
$$y = \frac{2}{3}$$

Also,

$$v = E = y - 1 = \tfrac{2}{3} - 1 = -\tfrac{1}{3}.$$

Finally,

$$2y - 1 = 2x$$
$$\frac{4}{3} - 1 = 2x$$
$$x = \frac{1}{6}$$

For the column player,

$$A\hat{q} = \begin{bmatrix} 0 & -1 & 2 \\ -1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 - x - y \end{pmatrix} = \begin{pmatrix} 2 - 3y - 2x \\ y - 1 \\ 2x - y \end{pmatrix}$$

This is the same set of equations as for the row player (because the matrix is symmetric), so has the same solution. So indeed, all three strategies are active (and we need not search other square subgames). The optimal solution is therefore:

$$\hat{p} = \begin{pmatrix} \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{pmatrix}$$

$$\hat{q} = \begin{pmatrix} \frac{1}{6} \\ \frac{2}{3} \\ \frac{1}{6} \end{pmatrix}$$

$$v = -\frac{1}{3}$$

Notice that, unlike rock-paper-scissors, the value of this game is not zero (it's not a fair game!), even though both players have the same optimal strategy. Remember - a symmetric payoff matrix does not mean a 'symmetric game'. For a symmetric game, we should have $A^T = -A$ instead of $A^T = A$. That is the payoff matrix is antisymmetric. (See also Exercise 5 in Section 7.2.2.)

10. Consider the usual version of three-finger Morra. Each player simultaneously puts out either 1,2, or 3 fingers. If the total number of fingers revealed is even, the column player pays the row player \$1, while of the total is odd, the row player pays the column player \$1. Find the optimal solution by equalizing expectation. What is unusual about this problem?

Solution: Here's the payoff matrix:

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

Again, we try to find a solution by first assuming all three strategies are active. Since the matrix is symmetric, we would obtain the same optimal mix for both players (but, like Exercise 9, the game is not symmetric so we should not expect it to be fair unless by a numerical accident.)

$$\hat{p}A = (x \quad y \quad 1-x-y)\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = (1-2y \quad 2y-1 \quad 1-2y)$$

Equalizing:

$$1-2y = 2y-1$$
$$2 = 4y$$
$$y = \frac{1}{2}$$
$$v = E = 1-2(y) = 1-2\left(\frac{1}{2}\right) = 0$$

So, the game turned out to be fair after all. However, notice that the solution does not depend on x! This is what is unusual about this problem. Thus, we have infinitely many optimal solutions:

$$\hat{p} = \left(x \quad \frac{1}{2} \quad \frac{1}{2}-x\right), \quad 0 \le x \le \frac{1}{2}$$
$$\hat{q} = \begin{pmatrix} y \\ \frac{1}{2} \\ \frac{1}{2}-y \end{pmatrix}, \quad 0 \le y \le \frac{1}{2}$$
$$v = 0$$

The alert reader will observe that if $x = 0$ or $x = \frac{1}{2}$, and, simultaneously $y = 0$ or $y = \frac{1}{2}$, then both players are simultaneously making one strategy inactive and so not every solution has all three strategies active. (This does not invalidate our calculations, because by writing $\hat{p} = (x \quad y \quad 1-x-y)$ we are assuming only that all three strategies are possibly inactive.) Indeed, had we noticed, there are two identical rows (and two identical columns) in the payoff matrix. So, like in Exercise 6 above, either row dominates the other (and the same for the identical columns). Thus, if we delete the dominated row and column (first or third, in both cases), we arrive at a subgame with the 2×2 payoff matrix

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

or its negative. For example, we obtain this matrix if both players delete the same strategy. We can then easily solve this to obtain $p = q^T = \left(\frac{1}{2}, \frac{1}{2}\right)$ After accounting for the inactive strategies, the solution involves $\hat{p} = \hat{q}^T = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$ or $\left(0, \frac{1}{2}, \frac{1}{2}\right)$, depending on which strategy is inactive. Just as in Exercise 6, because the rows are identical, we don't need one to be inactive, we only need the mix of the identical rows so that they're played together half the time. This would lead to

$$\hat{p} = \hat{q}^T = \left(x, \frac{1}{2}, \frac{1}{2}-x\right), \quad 0 \le x \le \frac{1}{2},$$

Which is, of course, the same solution we obtained above.

Section 7.2. Solving Constant-Sum Games with Linear Programming.

7.2.1 The $m \times n$ Case and the General Minimax Theorem

No musical references or exercises in this subsection.

7.2.2 The Square Subgame Theorem Revisited

Solutions to exercises:

1. Consider a zero-sum game with payoff matrix

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{5} & \mathbf{-3} \\ \mathbf{-3} & \mathbf{1} \end{bmatrix}$$

a) Write down the linear programming problems for both the row player and the column player.

Solution: First translate the game so all the payoffs are positive. Adding 4 to each payoff yields

$$B = \begin{bmatrix} 9 & 1 \\ 1 & 5 \end{bmatrix}$$

The column player's problem:

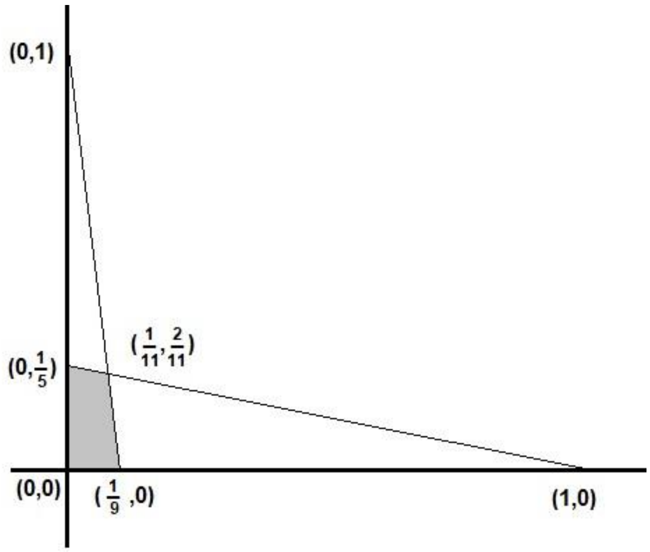
$$\begin{gathered} \text{Maximize } z = u_1 + u_2 \\ \text{Subject to:} \\ 9u_1 + u_2 \leq 1 \\ u_1 + 5u_2 \leq 1 \\ u_1 \geq 0, \;\; u_2 \geq 0 \end{gathered}$$

The row player's problem is the dual of this:

$$\begin{gathered} \text{Minimize } w = t_1 + t_2 \\ \text{Subject to:} \\ 9t_1 + t_2 \geq 1 \\ t_1 + 5t_2 \geq 1 \\ t_1 \geq 0, \;\; t_2 \geq 0 \end{gathered}$$

b) Solve the problems from part (a) by graphing in the decision space, and use your answers to solve the game.

Solution: For the column player, here is the feasible set:

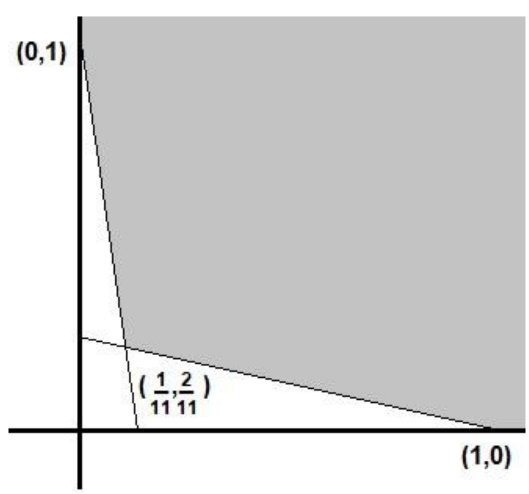


Evaluating the objective function at the corners:

(u_1, u_2)	$z = u_1 + u_2$
$(0,0)$	0
$\left(0, \frac{1}{5}\right)$	$\frac{1}{5}$
$\left(\frac{1}{9}, 0\right)$	$\frac{1}{9}$
$\left(\frac{1}{11}, \frac{2}{11}\right)$	$\frac{3}{11}$

The optimal point is the last row, thus, $v_B = \frac{1}{z} = \frac{11}{3}$ and $\hat{q} = \begin{pmatrix} vu_1 \\ vu_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$.

For the row player, here is the feasible set:



Evaluating the objective function at the corners:

(t_1, t_2)	$w = t_1 + t_2$
$(0, 1)$	1
$(1, 0)$	1
$\left(\frac{1}{11}, \frac{2}{11}\right)$	$\frac{3}{11}$

The optimal point is the last row. So $v_B = \frac{1}{w} = \frac{11}{3}$ and $\hat{p} = (vt_1, vt_2) = \left(\frac{1}{3}, \frac{2}{3}\right)$. Thus, the complete solution to the game is:

$$\hat{p} = \left(\frac{1}{3}, \frac{2}{3}\right)$$

$$q = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$$

$$v_A = v_B - 4 = \frac{11}{3} - 4 = -\frac{1}{3}$$

c. Instead, solve the column player's problem using the simplex algorithm and use your answer to solve the game for both players.

Solution: Pivoting the tableau for the problem the column player must solve:

	u_1	u_2	S_1	S_2	$Capacity$
S_1	9	1	1	0	1
S_2	1	5	0	1	1
z	-1	-1	0	0	0

(The pivot row and column are highlighted all the way across the table, into the row labels and column labels in the top and leftmost row and column. In the initial tableau above, those are the u_1 column and the S_1 row, for example.)

	u_1	u_2	S_1	S_2	$Capacity$
u_1	1	$\frac{1}{9}$	$\frac{1}{9}$	0	$\frac{1}{9}$
S_2	0	$\frac{44}{9}$	$-\frac{1}{9}$	1	$\frac{8}{9}$
z	0	$-\frac{8}{9}$	$\frac{1}{9}$	0	$\frac{1}{9}$

	u_1	u_2	S_1	S_2	$Capacity$
u_1	1	0	$\frac{5}{44}$	$-\frac{1}{44}$	$\frac{1}{11}$
u_2	0	1	$-\frac{1}{44}$	$\frac{9}{44}$	$\frac{2}{11}$
z	0	0	$\frac{1}{11}$	$\frac{2}{11}$	$\frac{3}{11}$

The solution to both the linear programming problems are:

Row player (dual)	**Column player (primal)**
$t_1 = \frac{1}{11}$	$u_1 = \frac{1}{11}$
$t_2 = \frac{2}{11}$	$u_2 = \frac{2}{11}$
$w = \frac{3}{11}$ (Minimized)	$z = \frac{3}{11}$ (Maximized)

These are the same solutions we obtained graphically in part (b). Thus, the solution to the game is exactly as in part (b).

d) Solve the game by another method and verify that your solutions in parts (b) and (c) are correct.

Solution: The choices of methods are the derived game approach, by equalizing expectation, or the method of oddments. For these methods, there is no need to translate the game first to make the payoffs positive; you can work directly with the matrix A. The reader can check than any of these methods leads to the same answer as in parts (b) and (c).

2. Consider a zero-sum game with payoff matrix

$$A = \begin{bmatrix} -2 & -1 \\ -3 & 2 \\ -5 & 5 \end{bmatrix}$$

a) Write down the linear programming problems for both the row player and the column player.

Solution: First translate the game by adding 6 to all the payoffs (or some other number to make all the payoffs positive.)

$$B = \begin{bmatrix} 4 & 5 \\ 3 & 8 \\ 1 & 11 \end{bmatrix}$$

For the column player:

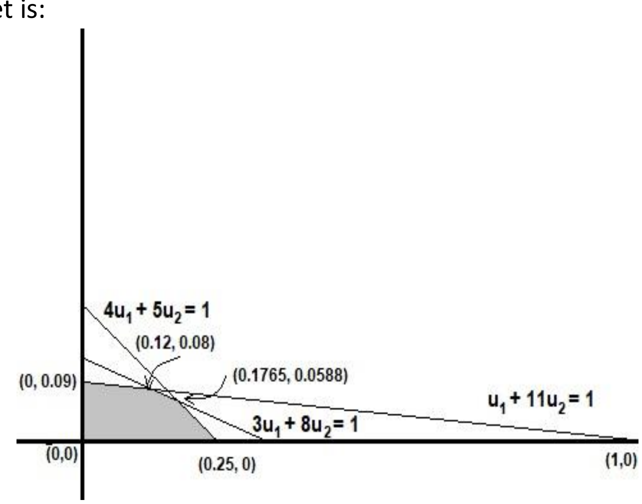
$$\text{Maximize } z = u_1 + u_2$$
Subject to:
$$4u_1 + 5u_2 \le 1$$
$$3u_1 + 8u_2 \le 1$$
$$u_1 + 11u_2 \le 1$$
$$u_1 \ge 0, \quad u_2 \ge 0$$

For the row player:

$$\text{Minimize } w = t_1 + t_2 + t_3$$
Subject to:
$$4t_1 + 3t_2 + t_3 \ge 1$$
$$5t_1 + 8t_2 + 11t_3 \ge 1$$
$$t_1 \ge 0, \quad t_2 \ge 0, \quad t_3 \ge 0$$

b) Your problem for the column player should have only two decision variables. Solve it by using graphing in the decision space. Did you obtain a pure strategy as optimal? Explain.

Solution: The feasible set is:



Evaluate the objective function at the corners:

$(\boldsymbol{u_1, u_2})$	$\boldsymbol{z = u_1 + u_2}$
$(\mathbf{0}, \mathbf{0})$	0
$\left(\mathbf{0}, \frac{\mathbf{1}}{\mathbf{11}}\right) \approx (\mathbf{0}, \mathbf{0.09})$	$\frac{1}{11} \approx 0.09091$
$\left(\frac{\mathbf{3}}{\mathbf{25}}, \frac{\mathbf{2}}{\mathbf{25}}\right) \approx (\mathbf{0.12}, \mathbf{0.08})$	$\frac{1}{5} = 0.2$
$\left(\frac{\mathbf{3}}{\mathbf{17}}, \frac{\mathbf{1}}{\mathbf{17}}\right) \approx (\mathbf{0.1765}, \mathbf{0.0588})$	$\frac{4}{17} \approx 0.2353$
$\left(\frac{\mathbf{1}}{\mathbf{4}}, \mathbf{0}\right)$	$\frac{1}{4} = 0.25$

The optimal point is the last row. The solution is $(u_1, u_2) = \left(\frac{1}{4}, 0\right)$ **and** $\boldsymbol{z} = \frac{\mathbf{1}}{\mathbf{4}}$. Thus, $v = \frac{1}{z} = 4$, and the solution for the column player is:

$$\hat{q} = \binom{vu_1}{vu_2} = \binom{1}{0},$$

and $v_A = v_B - 6 = 4 - 6 = -2$.

Yes, the solution we obtained is a pure strategy (because the game has a saddle point at $a_{11} = -2$.)

c) Your problem for the row player should have three decision variables, so you can't solve it easily by graphing in the decision space. However, based on your answer from part (a), you should be able to tell which square subgame is equivalent to the original game. Based on that, solve the game for the row player as well.

Solution: The solution to the linear programming problem for the column player did not list the slack variables. However, by looking at the graph of the feasible set, it is clear that only the first constraint is binding at the optimal point. Therefore, S_2 and S_3 are both nonzero. By complementary slackness, the two marginal values M_2 and M_3 are both 0. But these are the decision variables t_2 and t_3 in the dual problem, which is the problem for the row player. Thus, the second and third rows of A are inactive, as well as the second column for the column player. Deleting the inactive strategies, we arrive at the 1×1 subgame with payoff matrix $[a_{11}] = [-2]$.
Or, simply note that since a_{11} is a saddle point, we already know the game reduces to that 1×1 subgame. Either way, the solution to the row player is $\hat{p} = (1 \quad 0 \quad 0)$.

d) Solve the game for the column player via the simplex algorithm and thereby completely solve the original game for both players. Your answer should agree with what you got in parts (b) and (c).

Solution: We arrive at the optimal tableau in only one pivot:

	u_1	u_2	S_1	S_2	S_3	$Capacity$
S_1	4	5	1	0	0	1
S_2	3	8	0	1	0	1
S_3	1	11	0	0	1	1
z	-1	-1	0	0	0	0

	u_1	u_2	S_1	S_2	S_3	$Capacity$
u_1	1	$\frac{5}{4}$	$\frac{1}{4}$	0	0	$\frac{1}{4}$
S_2	0	$\frac{17}{4}$	$-\frac{3}{4}$	1	0	$\frac{1}{4}$
S_3	0	$\frac{39}{4}$	$-\frac{1}{4}$	0	1	$\frac{3}{4}$
z	0	$\frac{1}{4}$	$\frac{1}{4}$	0	0	$\frac{1}{4}$

The solution is

Row player (dual problem)	**Column player (primal problem)**
$\boldsymbol{t_1 = \frac{1}{4}}$	$u_1 = \frac{1}{4}$
$\boldsymbol{t_2 = 0}$	$u_2 = 0$
$\boldsymbol{t_3 = 0}$	$z = \frac{1}{4}$ (maximized)
$\boldsymbol{w = \frac{1}{4}}$ **(minimized)**	

Thus, $v_B = \frac{1}{z} = 4$ and the solution to the game is same as above. Namely,

$$\hat{p} = (vt_1 \quad vt_2 \quad vt_3) = (1 \quad 0 \quad 0)$$
$$\hat{q} = \begin{pmatrix} vu_1 \\ vu_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$v_A = v_B - 6 = 4 - 6 = -2$$

3. Consider a zero-sum game with the payoff matrix

$$A = \begin{bmatrix} -20 & 19 \\ 4 & -6 \end{bmatrix}$$

Write down the linear programming problem that both players must solve, and solve the game using the simplex algorithm.

Solution: Translate by 21 to make the payoffs positive:

$$B = \begin{bmatrix} 1 & 40 \\ 25 & 15 \end{bmatrix}$$

For the column player:

$$\text{Maximize } z = u_1 + u_2$$
$$\text{Subject to:}$$
$$u_1 + 40u_2 \leq 1$$
$$25u_1 + 15u_2 \leq 1$$
$$u_1 \geq 0, \;\; u_2 \geq 0$$

For the row player:

$$\text{Minimize } w = t_1 + t_2$$
$$\text{Subject to:}$$
$$t_1 + 25t_2 \geq 1$$
$$40t_1 + 15t_2 \geq 1$$
$$t_1 \geq 0, \;\; t_2 \geq 0$$

Pivoting:

	u_1	u_2	S_1	S_2	$Capacity$
S_1	1	40	1	0	1
S_2	25	15	0	1	1
z	-1	-1	0	0	0

	u_1	u_2	S_1	S_2	$Capacity$
S_1	0	$\frac{197}{5}$	1	$-\frac{1}{25}$	$\frac{24}{25}$
u_1	1	$\frac{3}{5}$	0	$\frac{1}{25}$	$\frac{1}{25}$
z	0	$-\frac{2}{5}$	0	$\frac{1}{25}$	$\frac{1}{25}$

	u_1	u_2	S_1	S_2	$Capacity$
u_2	0	1	$\frac{5}{197}$	$-\frac{1}{985}$	$\frac{24}{985}$
u_1	1	0	$-\frac{3}{197}$	$\frac{8}{197}$	$\frac{5}{197}$
z	0	0	$\frac{2}{197}$	$\frac{39}{985}$	$\frac{49}{985}$

The solution to both problems:

Row player (dual problem)	Column player (primal problem)
$\boldsymbol{t_1 = \frac{2}{197} = \frac{10}{985}}$	$u_1 = \frac{5}{197} = \frac{25}{985}$
$\boldsymbol{t_2 = \frac{39}{985}}$	$u_2 = \frac{24}{985}$
$\boldsymbol{w = \frac{49}{985}}$ **(minimized)**	$z = \frac{49}{985}$ (maximized)

So $v_B = \frac{1}{z} = \frac{985}{49}$, and the solution to the game is:

$$\hat{p} = (vt_1 \quad vt_2) = \begin{pmatrix} \frac{10}{49} & \frac{39}{49} \end{pmatrix}$$

$$\hat{q} = \begin{pmatrix} vu_1 \\ vu_2 \end{pmatrix} = \begin{pmatrix} \frac{25}{49} \\ \frac{24}{49} \end{pmatrix}$$

$$v_A = v_B - 21 = \frac{985}{49} - 21 = -\frac{44}{49}$$

4. Consider a zero-sum game with the following payoff matrix:

$$A = \begin{bmatrix} 1 & -2 & 3 & -4 \\ -5 & 1 & -6 & 2 \end{bmatrix}$$

a) Write down the linear programming problems that both players must solve.

Solution: Translate by 7 to obtain the payoff matrix

$$B = \begin{bmatrix} 8 & 5 & 10 & 3 \\ 2 & 8 & 1 & 9 \end{bmatrix}$$

For the column player:

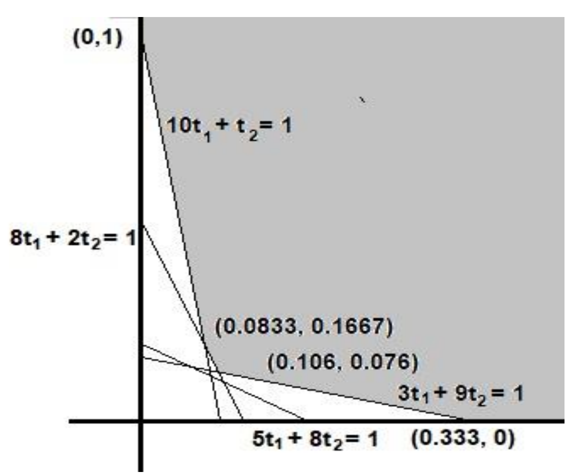
$$\begin{gathered} \text{Maximize } z = u_1 + u_2 + u_3 + u_4 \\ \text{Subject to:} \\ 8u_1 + 5u_2 + 10u_3 + 3u_4 \leq 1 \\ 2u_1 + 8u_2 + u_3 + 9u_4 \leq 1 \\ u_1 \geq 0, \quad u_2 \geq 0, \quad u_3 \geq 0, \quad u_4 \geq 0 \end{gathered}$$

For the row player:

$$\begin{gathered} \text{Minimize } w = t_1 + t_2 \\ \text{Subject to:} \\ 8t_1 + 2t_2 \geq 1 \\ 5t_1 + 8t_2 \geq 1 \\ 10t_1 + t_2 \geq 1 \\ 3t_1 + 9t_2 \geq 1 \\ t_1 \geq 0, \quad t_2 \geq 0 \end{gathered}$$

b) One of your two problems has just two decision variables. Solve that linear programming problem by graphing in the decision space.

Solution: The row player's problem has just two decision variables. Here is the feasible set:



Evaluating the objective function at the corners:

(t_1, t_2)	$w = t_1 + t_1$
$(0, 1)$	1
$\left(\frac{1}{12}, \frac{1}{6}\right) \approx (0.0833, 0.1667)$	$\frac{1}{4} = 0.25$
$\left(\frac{7}{66}, \frac{5}{66}\right) \approx (0.106, 0.076)$	$\frac{2}{11} \approx 0.18182$
$\left(\frac{1}{3}, 0\right)$	$\frac{1}{3} \approx 0.333$

The third point is optimal. Thus, $(t_1, t_2) = \left(\frac{7}{66}, \frac{5}{66}\right)$ and $w = \frac{2}{11}$. Thus, $v_B = \frac{1}{w} = \frac{11}{2}$. The optimal strategy for the row player is $\hat{p} = \left(\frac{7}{12}, \frac{5}{12}\right)$, and $v_A = v_B - 7 = \frac{11}{2} - 7 = -\frac{3}{2}$.

c). Use your answer to determine which 2×2 subgame is equivalent to the original game, and solve the game for both players (you may use any method you like on the 2×2 subgame.)

Solution: From the graph of the feasible set, we see that the second and the third constraint are not binding at the optimal point. Therefore (by complementary slackness), $u_2 = 0 = u_3$, so these two columns represent inactive strategies. Therefore, we can delete these columns from A to obtain the equivalent 2×2 subgame:

$$\begin{bmatrix} 1 & -4 \\ -5 & 2 \end{bmatrix}$$

Solving this for the column player, using oddments, we see $\hat{q} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$ for this subgame. Therefore, accounting for the inactive strategies, the solution to the original game is

$$\hat{p} = \left(\frac{7}{12}, \frac{5}{12}\right)$$

$$\hat{q} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}$$

$$v_A = -\frac{3}{2}$$

d) Solve the game using the simplex algorithm instead.

Solution: Pivoting the column player's tableau:

	u_1	u_2	u_3	u_4	S_1	S_2	$Capacity$
S_1	8	5	10	3	1	0	1
S_2	2	8	1	9	0	1	1
z	-1	-1	-1	-1	0	0	0

	u_1	u_2	u_3	u_4	S_1	S_2	$Capacity$
u_1	1	$\frac{5}{8}$	$\frac{5}{4}$	$\frac{3}{8}$	$\frac{1}{8}$	0	$\frac{1}{8}$
S_2	0	$\frac{27}{4}$	$-\frac{3}{2}$	$\frac{33}{4}$	$-\frac{1}{4}$	1	$\frac{3}{4}$
z	0	$-\frac{3}{8}$	$\frac{1}{4}$	$-\frac{5}{8}$	$\frac{1}{8}$	0	$\frac{1}{8}$

	u_1	u_2	u_3	u_4	S_1	S_2	$Capacity$
u_1	1	$\frac{7}{22}$	$\frac{29}{22}$	0	$\frac{3}{22}$	$-\frac{1}{22}$	$\frac{1}{11}$
u_4	0	$\frac{9}{11}$	$-\frac{2}{11}$	1	$-\frac{1}{33}$	$\frac{4}{33}$	$\frac{1}{11}$
z	0	$\frac{3}{22}$	$\frac{3}{22}$	0	$\frac{7}{66}$	$\frac{5}{66}$	$\frac{2}{11}$

Thus, $v_B = \frac{1}{z} = \frac{11}{2}$, and the solution to the game is:

$$\hat{p} = \left(\frac{7}{12}, \frac{5}{12}\right)$$

$$\hat{q} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}$$

$$v_A = -\frac{3}{2}$$

As we saw earlier.

5. Recall that if a zero-sum game has a payoff matrix such that $A^T = -A$, the game is said to be ***symmetric***. (This is a bit of a misnomer because the matrix itself is said to be *antisymmetric*, but don't worry about that.) An example was the game rock-paper-scissors. Prove that a symmetric game is always a fair game; that is, it has value $v = 0$. [**Hint**: Just as we did in rock-paper-scissors, argue that in any symmetric game, we must have $\hat{p} = \hat{q}^T$, that is, both players have the same optimal strategy. Then apply part 3 of the minimax theorem, and take the transpose of both sides.]

Solution: There may be other ways to see $\hat{p} = \hat{q}^T$, but one approach is to just use the minimax theorem. That way you can also show $v = 0$ at the same time. Indeed, part 1 of the theorem says $\hat{p}$ is characterized by the property that for some real number v, we have:

$$\hat{p}Aq \geq v$$

Negating both sides, we obtain:

$$-(\hat{p}Aq) \leq -v$$

But the matrix A is antisymmetric, so $-A = A^T$. Combined with the above, this says

$$\hat{p}A^T q \leq -v$$

Take transpose of both sides, and note that any 1×1 matrix is symmetric:

$$(\hat{p}A^T q)^T \leq (-v)^T = -v$$
$$q^T A \hat{p}^T \leq -v, \text{ for all } q. \quad (*)$$

Now part 2 of the minimax theorem says $\hat{q}$ is characterized by the existence of a real number w such that

$$pA\hat{q} \leq w, \quad \text{for all } p$$

But that is exactly what the previous line (*) says for $\hat{p}^T$ in place of $\hat{q}$, where $w = -v$, and where q^T is playing the role of the arbitrary p (as q is chosen arbitrarily in (*)).

Therefore, $\hat{q} = \hat{p}^T$, and taking transposes gives $\hat{p} = \hat{q}^T$.
Finally, apply part 3 of the minimax theorem since we know $w = -v$:

$$-v = w = \hat{p}A\hat{q} = v$$

Therefore, $2v = 0$, and so $v = 0$ and the game is fair.

6. We mentioned in the text that it does not matter what your translate the payoff matrix by, as long as the result has all positive entries. To illustrate this, re-solve the game in Exercise 1 above, but use a different constant than you used in Exercise 1 to translate the matrix. (For example, if you added 4 in Exercise 1, try adding 5 or 6 instead.)

Solution: The payoff matrix is

$$A = \begin{bmatrix} 5 & -3 \\ -3 & 1 \end{bmatrix}$$

In our solution to Exercise 1, we translated by 4. Let' illustrate here the case if we translate by 5 instead. We obtain the payoff matrix

$$B = \begin{bmatrix} 10 & 2 \\ 2 & 6 \end{bmatrix}$$

We leave it to the reader to write down the explicit linear programming problems that the players must solve. We'll also skip the graphical method and just use the simplex algorithm:

	u_1	u_2	S_1	S_2	$Capacity$
S_1	10	2	1	0	1
S_2	2	6	0	1	1
z	-1	-1	0	0	0

	u_1	u_2	S_1	S_2	$Capacity$
u_1	1	$\frac{1}{5}$	$\frac{1}{10}$	0	$\frac{1}{10}$
S_2	0	$\frac{28}{5}$	$-\frac{1}{5}$	1	$\frac{4}{5}$
z	0	$-\frac{4}{5}$	$\frac{1}{10}$	0	$\frac{1}{10}$

	u_1	u_2	S_1	S_2	$Capacity$
u_1	1	0	$\frac{3}{28}$	$-\frac{1}{28}$	$\frac{1}{14}$
u_2	0	1	$-\frac{1}{28}$	$\frac{5}{28}$	$\frac{1}{7}$
z	0	0	$\frac{1}{14}$	$\frac{1}{7}$	$\frac{3}{14}$

The solution to both problems:

Row player (dual problem)	Column player (primal problem)
$\boldsymbol{t_1 = \frac{1}{14}}$	$u_1 = \frac{1}{14}$
$\boldsymbol{t_2 = \frac{1}{7}}$	$u_2 = \frac{1}{7}$
$\boldsymbol{w = \frac{3}{14}}$ (minimized)	$z = \frac{3}{14}$ (maximized)

So, $v_B = \frac{1}{z} = \frac{14}{3}$, and the solution to the game is:

$$\hat{p} = (vt_1 \quad vt_2) = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

$$\hat{q} = \begin{pmatrix} vu_1 \\ vu_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$$

$$v_A = v_B - 5 = \frac{14}{3} - 5 = -\frac{1}{3}$$

This agrees with our solution in Exercise 1.

7. In Section 6.1, in Exercise 6, we considered a finite version of the Antares vs. Bellatrix game, where each company was deciding between two prices. Solve that game using linear programming.

Solution: The payoff matrix is

$$A = \begin{bmatrix} 63 & 57 \\ 54 & 66 \end{bmatrix}$$

Since the payoffs are already positive, it is not necessary to translate the game first. On the other hand, the pivoting is less tedious with smaller numbers, so why not translate by subtracting a positive constant – just be sure to keep all the entries positive. For example, if we translate by -50, we obtain the matrix

$$\begin{bmatrix} 13 & 7 \\ 4 & 16 \end{bmatrix}$$

We leave it to the reader to write down both linear programming problems, and get right to the pivoting:

	$\boldsymbol{u_1}$	$\boldsymbol{u_2}$	$\boldsymbol{S_1}$	$\boldsymbol{S_2}$	$\boldsymbol{Capacity}$
$\boldsymbol{S_1}$	13	7	1	0	1
$\boldsymbol{S_2}$	4	16	0	1	1
$\boldsymbol{z}$	-1	-1	0	0	0

	u_1	u_2	S_1	S_2	*Capacity*
u_1	1	$\frac{7}{13}$	$\frac{1}{13}$	0	$\frac{1}{13}$
S_2	0	$\frac{180}{13}$	$-\frac{4}{13}$	1	$\frac{9}{13}$
z	0	$-\frac{6}{13}$	$\frac{1}{13}$	0	$\frac{1}{13}$

	u_1	u_2	S_1	S_2	*Capacity*
u_1	1	0	$\frac{4}{45}$	$-\frac{7}{180}$	$\frac{1}{20}$
u_2	0	1	$-\frac{1}{45}$	$\frac{13}{180}$	$\frac{1}{20}$
z	0	0	$\frac{1}{15}$	$\frac{1}{30}$	$\frac{1}{10}$

We'll skip writing out the solution to both linear programming problems and go right to the solution of the game. We have $v_B = \frac{1}{z} = 10$. Thus:

$$\hat{p} = (vt_1 \quad vt_2) = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

$$\hat{q} = \begin{pmatrix} vu_1 \\ vu_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$v_A = v_B + 50 = 10 + 50 = 60$$

It's interesting that the value of this game is 60, which is the same value as the full, infinite continuous version of this game in the text.

8. In Section 6.1, in Exercise 7, we consider a finite version of the Antares vs. Bellatrix game, where one of the companies was deciding between two prices and the other between three prices. Solve the game using linear programming.

Solution: The payoff matrix is

$$A = \begin{bmatrix} 66 & 60 & 57 \\ 48 & 60 & 66 \end{bmatrix}$$

As in the previous problem, it is not necessary to translate the game, but we will for convenience lower the payoffs: Subtracting 45 from everything yields:

$$B = \begin{bmatrix} 21 & 15 & 12 \\ 3 & 15 & 21 \end{bmatrix}$$

In fact, since each payoff is a multiple 3 we can further simplify by scaling the payoffs by a factor of $\frac{1}{3}$ to obtain:

$$C = \begin{bmatrix} 7 & 5 & 4 \\ 1 & 5 & 7 \end{bmatrix}$$

Pivoting:

	u_1	u_2	u_3	S_1	S_2	*Capacity*
S_1	7	5	4	1	0	1
S_2	1	5	7	0	1	1
z	-1	-1	-1	0	0	0

	u_1	u_2	u_3	S_1	S_2	*Capacity*
u_1	1	$\frac{5}{7}$	$\frac{4}{7}$	$\frac{1}{7}$	0	$\frac{1}{7}$
S_2	0	$\frac{30}{7}$	$\frac{45}{7}$	$-\frac{1}{7}$	1	$\frac{6}{7}$
z	0	$-\frac{2}{7}$	$-\frac{3}{7}$	$\frac{1}{7}$	0	$\frac{1}{7}$

	u_1	u_2	u_3	S_1	S_2	*Capacity*
u_1	1	$\frac{1}{3}$	0	$\frac{7}{45}$	$-\frac{4}{45}$	$\frac{1}{15}$
u_3	0	$\frac{2}{3}$	1	$-\frac{1}{45}$	$\frac{7}{45}$	$\frac{2}{15}$
z	0	0	0	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{1}{5}$

We'll skip writing the solution to the linear programming problems and go tight to the solution of the game:

$$\hat{p} = (vt_1 \quad vt_2) = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

$$\hat{q} = \begin{pmatrix} vu_1 \\ vu_2 \\ vu_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ 0 \\ \frac{2}{3} \end{pmatrix}$$

$$v_A = v_B + 45 = 3(v_C) + 45 = 3(5) + 45 = 60.$$

Section 7.3. Using a Computer to Solve Constant-Sum Games.

Solutions to exercises:

Use Mathematica or Excel to solve the following constant-sum games. Some of these games appeared in earlier problem sets.

Remark: In our solutions, we will use Mathematica to illustrate. We leave the Excel solutions to the reader.

1. The game of two-finger Morra (Example 6.1) can be extended to three-finger Morra. The same rules apply – if the total number of fingers revealed is even, the column player pays the row player $1, while of the total is odd, the row player pays the column player $1.

Solution: The payoff matrix is

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

Translate by 2 to obtain a matrix with positive payoffs:

$$B = \begin{bmatrix} 3 & 1 & 3 \\ 1 & 3 & 1 \\ 3 & 1 & 3 \end{bmatrix}$$

We write out the problem that the row player must solve (because that is a minimization problem and we intend to use Mathematica.) Remember to read the coefficients down the columns to form the constraints since the row player's problem is the dual (although, since this is a symmetric matrix, we'd get the same numbers reading across the rows.) We obtain:

$$\begin{gathered} \text{Minimize } w = t_1 + t_2 + t_3 \\ \text{Subject to:} \\ 3t_1 + t_2 + 3t_3 \geq 1 \\ t_1 + 3t_2 + t_3 \geq 1 \\ 3t_1 + t_2 + 3t_3 \geq 1 \\ t_1 \geq 0,\ \ t_2 \geq 0,\ \ t_3 \geq 0 \end{gathered}$$

There are two identical constraints because there are two identical columns. The correct Mathematica command is:

DualLinearProgramming$[\{1,1,1\}, \{\{3,1,3\}, \{1,3,1\}, \{3.1.3\}\}, \{1,1,1\}$

Mathematica returns the output

$$\left\{\left\{\frac{1}{4},\frac{1}{4},0\right\},\left\{\frac{1}{4},\frac{1}{4},0\right\},\{0,0,0\},\{0,0,0\}\right\}$$

Thus, $t_1 = t_2 = u_1 = u_2 = \frac{1}{4}$, and $t_3 = u_3 = 0$. So, $w = z = \frac{1}{4} + \frac{1}{4} + 0 = \frac{1}{2}$, and $v_B = \frac{1}{z} = 2$. Therefore, the solution to the game is

$$\hat{p} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

$$q = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}$$

$$v_A = v_B - 2 = 2 - 2 = 0$$

In short, neither player uses the third finger option so the entire game reduces to two-finger Morra! On the other hand, note that this game has other optimal solutions (see Exercise 10 from Section 7.1.3), so Mathematica does not necessarily find all the optimal solutions to a game (or, more generally, to a linear programming problem) with this command.

2. The game of two-finger Morra (Example 6.1) can be extended to four-finger Morra. The same rules apply – if the total number of fingers revealed is even, the column player pays the row player \$1, while of the total is odd, the row player pays the column player \$1.

Solution: The payoff matrix is

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}$$

Translate by 2 to obtain a payoff matrix with positive entries:

$$B = \begin{bmatrix} 3 & 1 & 3 & 1 \\ 1 & 3 & 1 & 3 \\ 3 & 1 & 3 & 1 \\ 1 & 3 & 1 & 3 \end{bmatrix}$$

We leave it for the reader to write out the linear programming problem the players must solve. As in Exercise 1, we use Mathematica on the row player's problem. Again, the fact that there are duplicate rows and columns indicates that there are multiple optimal solutions, and Mathematica finds just one of them. The command is

$$\textbf{DualLinearProgramming}[\{1,1,1,1\},\{\{3,1,3,1\},\{1,3,1,3\},\{3,1,3,1\},\{1,3,1,3\}\},\{1,1,1,1\}]$$

Mathematica returns

$$\left\{\left\{\frac{1}{4},\frac{1}{4},0,0\right\},\left\{\frac{1}{4},\frac{1}{4},0,0\right\},\{0,0,0,0\},\{0,0,0,0\}\right\}$$

This translates to the following optimal solution:

$$\hat{p} = \begin{pmatrix}\frac{1}{2} & \frac{1}{2} & 0 & 0\end{pmatrix}$$

$$\hat{q} = \begin{pmatrix}\frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0\end{pmatrix}$$

$$v_A = v_B - 2 = 2 - 2 = 0.$$

3. A variation on four-finger Morra. Each player simultaneously puts out either 1,2,3 or 4 fingers. Again, if the total number of fingers revealed is even, the column player pays the row player, while of the total number is odd, the row player pays the column player. However, instead of the payment being $1, the number of dollars will be the (non-negative) difference between the number of fingers they show.

Solution: The payoff matrix is

$$A = \begin{bmatrix} 0 & -1 & 2 & -3 \\ -1 & 0 & -1 & 2 \\ 2 & -1 & 0 & -1 \\ -3 & 2 & -1 & 0 \end{bmatrix}$$

Translate by 4 to obtain:

$$B = \begin{bmatrix} 4 & 3 & 6 & 1 \\ 3 & 4 & 3 & 6 \\ 6 & 3 & 4 & 3 \\ 1 & 6 & 3 & 4 \end{bmatrix}$$

The Mathematica command is

$$\textbf{DualLinearProgramming}[\{1,1,1,1\},\{\{4,3,6,1\},\{3,4,3,6\},\{6,3,4,3\},\{1,6,4,3\}\},\{1,1,1,1\}]$$

Mathematica returns

$$\left\{\left\{\frac{1}{23},\frac{2}{23},\frac{2}{23},\frac{1}{23}\right\},\left\{\frac{1}{23},\frac{2}{23},\frac{2}{23},\frac{1}{23}\right\},\{0,0,0,0\},\{0,0,0,0\}\right\}$$

Thus, $w = \frac{1}{23} + \frac{2}{23} + \frac{2}{23} + \frac{1}{23} = \frac{6}{23}$, and $v_B = \frac{1}{w} = \frac{23}{6}$. The solution to the game is

$$\hat{p} = \begin{pmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{pmatrix}$$

$$\hat{q} = \begin{pmatrix} \frac{1}{6} \\ \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{6} \end{pmatrix}$$

$$v_A = v_B - 4 = \frac{23}{6} - 4 = -\frac{1}{6}$$

4. A guessing game. Bill secretly puts $\$x$ in his hand, where $x = 0,1,3$, or 5. Ann must guess how many dollars Bill is holding. If she guesses correctly, Bill pays her twice the number of dollars he is holding, plus one dollar more; that is, he pays her $\$(2x + 1)$. If she guesses incorrectly, she must pay Bill the (positive) difference between her guess and the actual number, plus a $2 penalty. Assume Ann is the row player.

Solution: We found the payoff matrix in Exercise 10 of Section 6.1. It is

$$A = \begin{bmatrix} 1 & -3 & -5 & -7 \\ -3 & 3 & -4 & -6 \\ -5 & -4 & 7 & -4 \\ -7 & -6 & -4 & 11 \end{bmatrix}$$

Translate by 8 to obtain a matrix with positive payoffs:

$$B = \begin{bmatrix} 9 & 5 & 3 & 1 \\ 5 & 11 & 4 & 2 \\ 3 & 4 & 15 & 4 \\ 1 & 2 & 4 & 19 \end{bmatrix}$$

The Mathematica command is:

DualLinearProgramming$[\{1,1,1,1\},\{\{9,5,3,1\},\{5,11,4,2\},\{3,4,15,4\},\{1,2,4,19\}\},\{1,1,1,1\}]$

Mathematica returns

$$\left\{\left\{\frac{661}{8804},\frac{339}{8804},\frac{275}{8804},\frac{335}{8804}\right\},\left\{\frac{661}{8804},\frac{339}{8804},\frac{275}{8804},\frac{335}{8804}\right\},\{0,0,0,0\},\{0,0,0,0\}\right\}$$

Thus, $w = \frac{661}{8804} + \frac{339}{8804} + \frac{275}{8804} + \frac{335}{8804} = \frac{805}{4402}$; $v_B = \frac{1}{w} = \frac{4402}{805}$ The solution to the game is:

$$\hat{p} = \begin{pmatrix}\frac{661}{1610} & \frac{339}{1610} & \frac{275}{1610} & \frac{335}{1610}\end{pmatrix} \approx \begin{pmatrix}0.411 & 0.211 & 0.171 & 0.208\end{pmatrix}$$

$$\hat{q} = \begin{pmatrix}\frac{661}{1610}\\ \frac{339}{1610}\\ \frac{275}{1610}\\ \frac{335}{1610}\end{pmatrix} \approx \begin{pmatrix}0.411\\ 0.211\\ 0.171\\ 0.208\end{pmatrix}$$

$$v_A = v_B - 8 = \frac{4402}{1610} - 8 = -\frac{8478}{1610} \approx -5.27$$

This game strongly favors Bill!

5. Consider the marketing game Antares vs. Bellatrix of Example .5. Construct a finite version of this game as follows: Instead of allowing the set process to be any number in the interval [80,120], assume the companies are debating between just a few specific choices (all of which lie within this interval.) Assume both companies are deciding between the prices \$80, \$86, \$96, \$106, and \$120.

Solution: The payoff matrix is

$$A = \begin{bmatrix} 66 & 64.2 & 61.2 & 58.2 & 54 \\ 62.4 & 61.68 & 60.48 & 59.28 & 57.6 \\ 56.4 & 57.48 & 59.28 & 61.08 & 63.6 \\ 50.4 & 53.28 & 58.08 & 62.88 & 69.6 \\ 42 & 47.4 & 56.4 & 65.4 & 78 \end{bmatrix}$$

Since the payoffs are positive, wo do not translate the matrix. The Mathematica command is:

$$\textbf{DualLinearProgramming}\left[\begin{array}{c}\{1,1,1,1,1\},\\ \left\{\begin{array}{c}\{66,64.2,61.2,58.2,54\},\{62.4,61.68,60.48,59.28,57.6\},\\ \{56.4,57.48,59.28,61.08,63.6\},\{50.4,53.28,58.08,62.88.69.6\},\\ \{42,47.4,56.4,65.4,78\}\end{array}\right\},\\ \{1,1,1,1,1\}\end{array}\right]$$

(In Mathematica, type in one line when inputting the command.)

Mathematica returns:

$$\{\{0.0125, 0, 0, 0, 0.004167\}, \{0.0083, 0, 0, 0, 0.0083\}, \{\ldots\}, \{\ldots\}\}$$

What's in the ellipses is irrelevant for the solution to a standard form minimization, so we did not fill in these values. (The missing values are extremely close to 0, and we suspect that they are, in fact, equal to 0 (as is the previous examples), with the discrepancy being caused by roundoff error of some kind, possibly due to the fact that the decimal entries in the payoff matrix caused Mathematica to automatically switch from the simplex algorithm to an interior point method, in which a tolerance is specified – the missing values are likely within that tolerance to 0, so the algorithm halted.)

Thus, $w = 0.0125 + 0 + 0 + 0 + 0.004167 = 0.0166667 = \frac{1}{60}$, and so $v = \frac{1}{w} = 60$. The solution to the game is

$$\hat{p} = \begin{pmatrix} \frac{3}{4} & 0 & 0 & 0 & \frac{1}{4} \end{pmatrix}$$

$$q = \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}$$

$$v = 60$$

6. Solve the game with the following payoff matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & -2 \\ -3 & 0 & 1 & 3 \\ -6 & -5 & 0 & 2 \end{bmatrix}$$

Solution: Translate by 7 to obtain positive payoffs:

$$B = \begin{bmatrix} 8 & 9 & 6 & 5 \\ 4 & 7 & 8 & 10 \\ 1 & 2 & 7 & 9 \end{bmatrix}$$

The Mathematica command is

DualLinearProgramming$[\{1,1,1,1\}, \{\{8,4,1\}, \{9,7,2\}, \{6,8,7\}, \{5,10,9\}\}, \{1,1,1,1\}]$

Mathematica returns

$$\left\{\left\{\frac{1}{10},\frac{1}{20},0\right\},\left\{\frac{1}{12},0,0,\frac{1}{15}\right\},\left\{0,0,\frac{19}{60}\right\},\{0,0,0\}\right\}$$

Thus, $w=\frac{1}{10}+\frac{1}{20}+0=\frac{3}{20}$ and $v_B=\frac{1}{w}=\frac{20}{3}$. The solution to the game is

$$\hat{p}=\begin{pmatrix}\frac{2}{3} & \frac{1}{3} & 0\end{pmatrix}$$

$$\hat{q}=\begin{pmatrix}\frac{5}{9}\\ 0\\ 0\\ \frac{4}{9}\end{pmatrix}$$

$$v_A=v_B-7=\frac{20}{3}-7=-\frac{1}{3}$$

7. Solve the game with the following payoff matrix:

$$A=\begin{bmatrix}-2 & -5\\ -3 & 2\\ -5 & 5\end{bmatrix}$$

Solution: Translate by 6 to obtain positive payoffs:

$$B=\begin{bmatrix}4 & 1\\ 3 & 8\\ 1 & 11\end{bmatrix}$$

The Mathematica command is

DualLinearProgramming$[\{1,1,1\},\{\{4,3,1\},\{1,8,11\}\},\{1,1\}]$

Mathematica returns

$$\left\{\left\{\frac{5}{29},\frac{3}{29},0\right\},\left\{\frac{7}{29},\frac{1}{29}\right\},\left\{0,0,\frac{11}{29}\right\},\{0,0,0\}\right\}$$

Thus, $w=\frac{8}{29}$ and $v_B=\frac{1}{w}=\frac{29}{8}$. The solution to the game is

$$\hat{p}=\begin{pmatrix}\frac{5}{8} & \frac{3}{8} & 0\end{pmatrix}$$

$$q = \begin{pmatrix} \frac{7}{8} \\ \frac{1}{8} \end{pmatrix}$$

$$v_A = v_B - 6 = \frac{29}{8} - 6 = -\frac{19}{8}$$

8. Solve the game with the following payoff matrix:

$$A = \begin{bmatrix} 0 & -1 & 3 & -4 & 7 \\ -2 & 1 & 0 & -1 & 3 \\ 1 & -3 & -4 & 5 & -1 \\ 3 & 3 & -2 & 0 & -5 \\ -4 & -3 & 6 & -2 & 8 \end{bmatrix}$$

Solution: Translate by 6 to obtain

$$B = \begin{bmatrix} 6 & 5 & 9 & 2 & 13 \\ 4 & 7 & 6 & 5 & 9 \\ 7 & 3 & 2 & 11 & 5 \\ 9 & 9 & 4 & 6 & 1 \\ 2 & 3 & 12 & 4 & 14 \end{bmatrix}$$

The Mathematica command is

DualLinearProgramming$\left[\begin{matrix}\{1,1,1,1,1\}, \{\{6,4,7,9,2\}, \{5,7,3,9,3\}, \{9,6,2,4,12\}, \{2,5,11,6,4\}, \{13,9,5,1,14\}\}, \\ \{1,1,1,1,1\}\end{matrix}\right]$

Mathematica returns

$$\left\{\left\{0, \frac{67}{3928}, \frac{103}{3928}, \frac{145}{1964}, \frac{45}{982}\right\}, \left\{0, \frac{25}{491}, \frac{17}{491}, \frac{32}{491}, \frac{6}{491}\right\}, \left\{\frac{71}{491}, 0,0,0,0\right\}, \{0,0,0,0,0\}\right\}$$

Thus, $w = 0 + \frac{67}{3928} + \frac{103}{3928} + \frac{145}{1964} + \frac{45}{982} = \frac{80}{491}$, and $v_B = \frac{1}{w} = \frac{491}{80}$.

The solution to the game is

$$\hat{p} = \begin{pmatrix} 0 & \frac{67}{640} & \frac{103}{640} & \frac{145}{320} & \frac{45}{160} \end{pmatrix}$$

$$\hat{q} = \begin{pmatrix} 0 \\ \frac{25}{80} \\ \frac{17}{80} \\ \frac{32}{80} \\ \frac{6}{80} \end{pmatrix}$$

$$v_A = v_B - 6 = \frac{491}{80} - 6 = \frac{11}{80}$$

Section 7.4. Variable-Sum Games

7.4.1 Dominance and Nash Equilibrium Points.

No exercises in this subsection.

7.4.2 Mixed Strategies, Payoff Polygons, And Pareto Efficiency

No exercises in this subsection.

7.4.3 Goals of Play, the Relative Zero-Sum Game, and Strictly Determined Games

Musical Reference in the Text: The final example of this section (page 346) involving Botany students pays homage to American rock band the Doors. The students' names, Robby and Jim, refer to Robbie Krieger (guitarist) and Jim Morrison (vocalist) of the band. The soil nutrients are named for drummer John Densmore, and combining the brand names for the pesticides produces keyboardist Ray Manzarek. Several Doors songs are also alluded tom including 'Hyacinth House', 'Spanish Caravan', 'The Unknown Soldier', and 'Indian Summer'. (Website: https://thedoors.com)

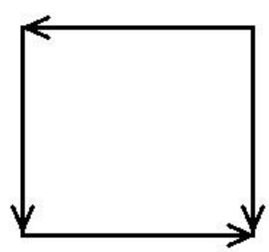
Solutions to exercises:

1. Consider the following payoff matrix:

$$A = \begin{bmatrix} (0,5) & (4,2) \\ (3,-1) & (5,4) \end{bmatrix}$$

Draw the movement diagram. Determine all instances of dominance, and determine if there is a pure strategy Nash equilibrium.

Solution: The movement diagram:

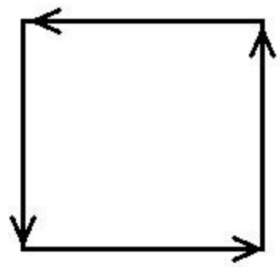


The second row dominates the first. Neither column dominates the other. The outcome with payoff $(5,4)$ is a Nash equilibrium.

2. Same directions as Exercise 1 for the payoff matrix:

$$B = \begin{bmatrix} (1,3) & (4,2) \\ (2,1) & (3,4) \end{bmatrix}$$

Solution: The movement diagram:

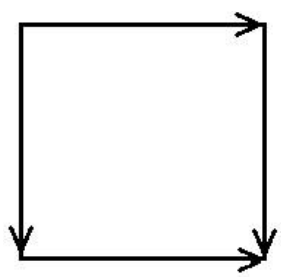


There are no dominant strategies, nor are there any Nash equilibria in pure strategies.

3. Same directions as Exercise 1 for the following payoff matrix:

$$C = \begin{bmatrix} (1,3) & (3,4) \\ (2,1) & (4,2) \end{bmatrix}$$

Solution: The movement diagram:

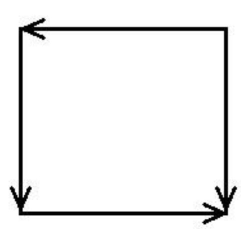


The second row dominates the first, and the second column dominates the first. The outcome with payoff $(4,2)$ is a Nash equilibrium in pure strategies.

4. Same directions as Exercise 1 for the following payoff matrix:

$$D = \begin{bmatrix} (1,3) & (2,1) \\ (4,2) & (3,4) \end{bmatrix}$$

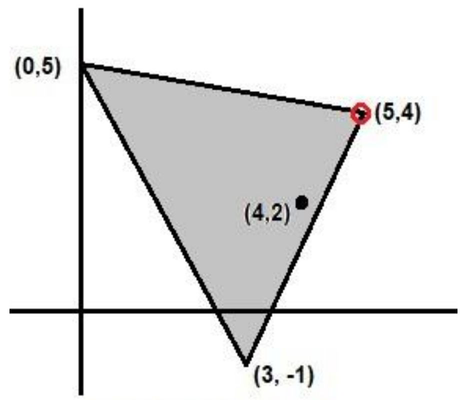
Solution: The movement diagram:



The movement diagram is the same as in Exercise 1. So, the second row dominates the first, neither column dominates the other, and the outcome with payoff $(3,4)$ is a Nash equilibrium in pure strategies.

5. Draw the payoff polygons for the payoff matrices $A - D$ of Exercises 1-4. In each case, determine if the game is strictly determined. Notice that different payoff matrices can have the same payoff polygon!

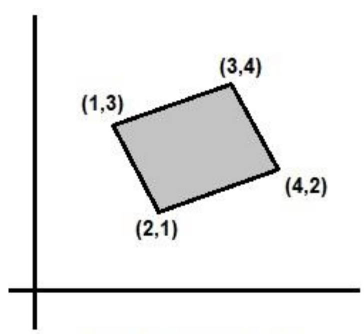
Solution: In each diagram, any pure strategy Nash equilibria are circled.
For A:



Payoff polygon for A

The outcome with payoff (5,4) is a Nash equilibrium. The Pareto efficient points are those along the line segment connecting $(0,5)$ to $(5,4)$, including the Nash equilibrium. Thus, the game is strictly determined.

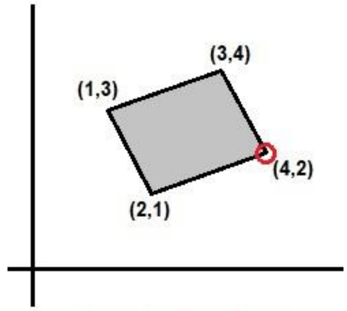
For B:



Payoff polygon for B

The Pareto efficient points are those along the segment connecting $(3,4)$ to $(4,2)$. Since none of these points is a Nash equilibrium, the game is not strictly determined.

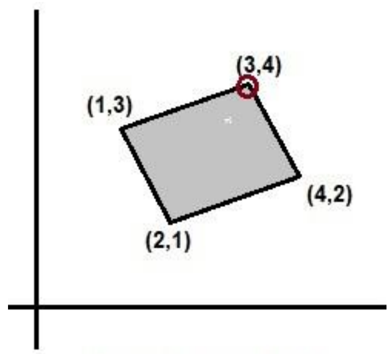
For C:



Payoff polygon for C

The polygon is the same as for B, with the same Pareto efficient points. However, in the matrix C, there is a Nash equilibrium at $(4,2)$, which is the unique Pareto efficient one, so the game is strictly determined.

For D:



Payoff polygon for D

The payoff polygon is the same as for B and C, with the same Pareto efficient points. In D, there is a unique Pareto efficient Nash equilibrium at $(3,4)$, so the game is strictly determined.

6. Among those games in Exercises 1-4 which are strictly determined, does the solution satisfy the first condition to benefit the group (that the outcome has the highest possible sum of payoffs)?

Solution: We have noted that A, C, and D are strictly determined. In A and D, the solution does represent the highest possible total payoff, so they do satisfy the first condition for the benefit of the group. In C, the condition fails, as the sum of the payoffs at the solution is 6, but there is another outcome where the payoffs sum to 7.

7. Consider the game from subsection 7.4.2 with payoff matrix

$$A = \begin{bmatrix} (3,5) & (6,1) \\ (7,2) & (5,7) \end{bmatrix}$$

a) Verify that if both players ply using a $50/50$ mix, so $p = q^T$, then the expected payoff is $(E_R, E_C) = (5.25, 3.75)$ as claimed in the text.

Solution: For the row player:

$$E_R = (.5 \quad .5)\begin{bmatrix} 3 & 6 \\ 7 & 5 \end{bmatrix}\begin{pmatrix} .5 \\ .5 \end{pmatrix} = [5.25]$$

And for the column player:

$$E_C = (.5 \quad .5)\begin{bmatrix} 5 & 1 \\ 2 & 7 \end{bmatrix}\begin{pmatrix} .5 \\ .5 \end{pmatrix} = [3.75]$$

As claimed.

b) Verify that if $p = (.3 \quad .7)$ and $q = \begin{pmatrix} .6 \\ .4 \end{pmatrix}$, then the expected payoff is $(E_R, E_C) = (5.6, 3.82)$ as claimed in the text.

Solution: For the row player:

$$E_R = (.3 \quad .7)\begin{bmatrix} 3 & 6 \\ 7 & 5 \end{bmatrix}\begin{pmatrix} .6 \\ .4 \end{pmatrix} = [5.6]$$

And for the column player:

$$E_C = (.3 \quad .7)\begin{bmatrix} 5 & 1 \\ 2 & 7 \end{bmatrix}\begin{pmatrix} .6 \\ .4 \end{pmatrix} = [3.82]$$

As claimed.

c) Verify that if both players play their prudential strategies (determined in the text), then the payoffs will be $(E_R, E_C) = (5.4, 3.6667)$.

Solution: The row player's prudential strategy is (.4,.6), and the column player's prudential strategy is $\begin{pmatrix} .667 \\ .333 \end{pmatrix}$, Thus for the row player:

$$E_R = \begin{pmatrix} \frac{2}{5} & \frac{3}{5} \end{pmatrix}\begin{bmatrix} 3 & 6 \\ 7 & 5 \end{bmatrix}\begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} = \left[\frac{27}{5}\right] = [5.4]$$

And for the column player:

$$E_C = \begin{pmatrix} \frac{2}{5} & \frac{3}{5} \end{pmatrix}\begin{bmatrix} 5 & 1 \\ 2 & 7 \end{bmatrix}\begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} = \left[\frac{11}{3}\right] = [3.6667]$$

As claimed.

d) Verify the solution to the relative zero-sum game is what is claimed in the text.

Solution: The payoff matrix for the relative zero-sum game is

$$\begin{bmatrix} -2 & 5 \\ 5 & -2 \end{bmatrix}$$

By oddments, $\hat{p} = \hat{q}^T = (.5 \quad .5)$, and then

$$v = \hat{p}A\hat{q} = (.5 \quad .5)\begin{bmatrix} -2 & 5 \\ 5 & -2 \end{bmatrix}\begin{pmatrix} .5 \\ .5 \end{pmatrix} = [1.5],$$

which verifies the claim in the text.

8. Continuing with the game in Exercise 7, we intend to see what happens in the derived game when the players try various strategies. We have already determined in the text what the payoffs are if both players use the equalizing strategy (and we know this outcome is a Nash equilibrium.) Also, in Exercise 7 (c), we determined the payoffs if both players use their prudential strategies.

a) Determine the payoffs if the row player uses the equalizing strategy, and the column player plays prudentially.

Solution: For the row player:

$$E_R = \begin{pmatrix} \frac{5}{9} & \frac{4}{9} \end{pmatrix}\begin{bmatrix} 3 & 6 \\ 7 & 5 \end{bmatrix}\begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} = \left[\frac{136}{27}\right] \approx [5.0370]$$

For the column player:

$$E_C = \begin{pmatrix} \frac{5}{9} & \frac{4}{9} \end{pmatrix}\begin{bmatrix} 5 & 1 \\ 2 & 7 \end{bmatrix}\begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} = \left[\frac{11}{3}\right] \approx [3.6667]$$

Thus, $(E_R, E_C) = (5.0370 \quad 3.6667)$

b) Determine the payoffs if the row player uses the prudential strategy, and the column player uses the equalizing strategy.

Solution: We have

$$E_R = \begin{pmatrix} \frac{2}{5} & \frac{3}{5} \end{pmatrix}\begin{bmatrix} 3 & 6 \\ 7 & 5 \end{bmatrix}\begin{pmatrix} \frac{1}{5} \\ \frac{4}{5} \end{pmatrix} = \left[\frac{135}{25}\right] = [5.4]$$

And

$$E_C = \begin{pmatrix} \frac{2}{5} & \frac{3}{5} \end{pmatrix} \begin{bmatrix} 5 & 1 \\ 2 & 7 \end{bmatrix} \begin{pmatrix} \frac{1}{5} \\ \frac{4}{5} \end{pmatrix} = \begin{bmatrix} \frac{108}{25} \end{bmatrix} = [4.32]$$

Thus, $(E_R, E_C) = (5.4 \quad 4.32)$

c) Determine the payoffs if the row player uses the prudential strategy, and the column player (guessing correctly that the row player is playing prudentially), uses the expected value principle to respond with a pure strategy.

Solution: Using the column player's payoffs:

$$E_C = \begin{pmatrix} \frac{2}{5} & \frac{3}{5} \end{pmatrix} \begin{bmatrix} 5 & 1 \\ 2 & 7 \end{bmatrix} = \begin{pmatrix} \frac{16}{5} & \frac{23}{5} \end{pmatrix}$$

The expected value principle says the column player should choose the second column; that is, the counter-prudential strategy is $q = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then $E_C = \frac{23}{5} = 4.6$. Also

$$E_R = \begin{pmatrix} \frac{2}{5} & \frac{3}{5} \end{pmatrix} \begin{bmatrix} 3 & 6 \\ 7 & 5 \end{bmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{bmatrix} \frac{27}{25} \end{bmatrix} = [1.08]$$

Thus, $(E_R, E_C) = (1.08 \quad 4.6)$

d) Determine if the row player uses the equalizing strategy, and the column player (guessing incorrectly that the row player is playing prudentially), uses the expected value principle to respond to the prudential strategy with a pure strategy.

Solution: As in part (c), the counter-prudential strategy is $q = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then

$$E_R = \begin{pmatrix} \frac{5}{9} & \frac{4}{9} \end{pmatrix} \begin{bmatrix} 3 & 6 \\ 7 & 5 \end{bmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{bmatrix} \frac{50}{9} \end{bmatrix} \approx [5.5556]$$

And

$$E_C = \begin{pmatrix} \frac{5}{9} & \frac{4}{9} \end{pmatrix} \begin{bmatrix} 5 & 1 \\ 2 & 7 \end{bmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{11}{3} \end{pmatrix} \approx [3.6667]$$

Thus, $(E_R, E_C) = (5.5556 \quad 3.6667)$.

e) Determine the payoffs if each player thinks the other is playing prudentially, and uses the expected value principle to respond.

Solution: We already know the counter-prudential strategy for the column player is $q = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. For the row player,

$$\begin{bmatrix} 3 & 6 \\ 7 & 5 \end{bmatrix} \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 4 \\ \frac{19}{3} \end{pmatrix}$$

Since $\frac{19}{3} > 4$, the counter-prudential strategy for the row player is $p = (0 \quad 1)$. Thus, both players choose their second strategy, and we end up at the outcome $(E_R, E_C) = (5,7)$.

f) Suppose the row player is using $p = (x, 1 - x)$ and the column player is using $q = \begin{pmatrix} y \\ 1 - y \end{pmatrix}$. find a formula for the expected payoffs to each player (E_R, E_C). Your solution should be an ordered pair with a saddle surface equation of crossed type in each coordinate.

Solution: For the row player:

$$E_R = (x \quad 1 - x) \begin{bmatrix} 3 & 6 \\ 7 & 5 \end{bmatrix} \begin{pmatrix} y \\ 1 - y \end{pmatrix} = [x + 2y - 5xy + 5]$$

And for the column player:

$$E_C = (x \quad 1 - x) \begin{bmatrix} 5 & 1 \\ 2 & 7 \end{bmatrix} \begin{pmatrix} y \\ 1 - y \end{pmatrix} = [9xy - 6x - 5y + 7]$$

Thus, the payoff at a general point of the derived game is

$$(E_R, E_C) = (-5xy + x + 2y + 5, 9xy - 6x - 5y + 7)$$

These formulas can be used to find all of the above payoffs in parts (a) - (e) by plugging in appropriate values of x and y.

9. Consider the four games in Exercises 1-4, with payoff matrices A, B, C, and D. For each game, find the relative zero-sum game, and solve it. Find an example of a strictly determined variable-sum game, where the relative zero-sum game is also strictly determined, but with a different solution than predicted by the original variable-sum game. Also, find an example of a strictly determined variable-sum game whose relative zero-sum game is not strictly determined. These examples show (in two different ways) that the question posed at the end of the section has a negative answer.

Solution: For

$$A = \begin{bmatrix} (0,5) & (4,2) \\ (3,-1) & (5,4) \end{bmatrix}$$

The relative zero-sum game has payoff matrix

$$A_0 = \begin{bmatrix} -5 & 2 \\ 4 & 1 \end{bmatrix}$$

There is no saddle point, so we solve by oddments. We obtain:

$$\hat{p} = \begin{pmatrix} \frac{3}{10} & \frac{7}{10} \end{pmatrix}$$
$$\hat{q} = \begin{pmatrix} \frac{1}{10} \\ \frac{9}{10} \end{pmatrix}$$

$$v = \frac{-detA_0}{10} = \frac{13}{10}$$

This is an example of a variable-sum game which is strictly determined, but the relative zero-sum game is not.

For

$$B = \begin{bmatrix} (1,3) & (4,2) \\ (2,1) & (3,4) \end{bmatrix}$$

The relative zero-sum game has payoff matrix

$$B_0 = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}$$

By oddments, the solution is

$$\hat{p} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$
$$\hat{q} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$v = \frac{-detB_0}{6} = 0$$

In this example, neither B nor B_0 is strictly determined.

For

$$C = \begin{bmatrix} (1,3) & (3,4) \\ (2,1) & (4,2) \end{bmatrix}$$

The relative zero-sum game is

$$C_0 = \begin{bmatrix} -2 & -1 \\ 1 & 2 \end{bmatrix}$$

In this game, $a_{21} = 1$ is a saddle point, so the solution is

$$\hat{p} = \begin{pmatrix} 0 & 1 \end{pmatrix}$$
$$\hat{q} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$v = 1$$

In this game, both C and C_0 are strictly determined. However, the solutions are not the same. In the variable-sum game, the predicted solution has

$$\hat{q} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

For

$$D = \begin{bmatrix} (1,3) & (2,1) \\ (4,2) & (3,4) \end{bmatrix}$$

The relative zero-sum game is

$$D_0 = \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix}$$

This is another example where the variable-sum game is strictly determined, but the relative zero-sum game is not. By oddments, the solution to the zero-sum game is

$$\hat{p} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$
$$\hat{q} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$$

$$v = \frac{-detD_0}{6} = 0$$

10. For any of the four matrices A, B, C, D of Exercises 1-4 which do not have a pure strategy Nash equilibrium, find the equalizing strategies that yield a mixed strategy Nash equilibrium, and the corresponding payoffs.

Solution: The only one which did not have a Nash equilibrium in pure strategies was

$$B = \begin{bmatrix} (1,3) & (4,2) \\ (2,1) & (3,4) \end{bmatrix}$$

To find the Nash equilibrium in mixed strategies, each player must equalize their opponent's payoffs. So, for the row player, we use the column player's payoffs:

$$\begin{pmatrix} x & 1-x \end{pmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 2x+1 & 4-2x \end{bmatrix}$$

Equalizing

$$\begin{gathered} 2x + 1 = 4 - 2x \\ 4x = 3 \\ x = \frac{3}{4} \end{gathered}$$

Thus, $p = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \end{pmatrix}$. For the column player, we use the row player's payoffs:

$$\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{pmatrix} y \\ 1-y \end{pmatrix} = \begin{pmatrix} 4-3y \\ 3-y \end{pmatrix}$$

Equalizing:

$$\begin{gathered} 4 - 3y = 3 - y \\ 1 = 2y \\ y = \frac{1}{2} \end{gathered}$$

Thus, $q = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$. If each player uses these strategies, we obtain a Nash equilibrium with payoffs:

$$E_R = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \end{pmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{bmatrix} \frac{5}{2} \end{bmatrix}$$

$$E_C = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \end{pmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{bmatrix} \frac{5}{2} \end{bmatrix}$$

11. a) Find a Nash equilibrium in the game with payoff matrix:

$$\begin{bmatrix} (4,-1) & (1,6) \\ (-2,3) & (2,2) \end{bmatrix}$$

Solution: There is no Nash equilibrium in pure strategies, so we use the equalizing mixed strategies:

$$\begin{bmatrix} x & 1-x \end{bmatrix}\begin{bmatrix} -1 & 6 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 3-4x & 4x+2 \end{bmatrix}$$

Equalizing:

$$\begin{gathered} 3-4x = 4x+2 \\ 1 = 8x \\ x = \frac{1}{8} \end{gathered}$$

This determines the equalizing strategy for the row player. For the column player:

$$\begin{bmatrix} 4 & 1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} y \\ 1-y \end{bmatrix} = \begin{bmatrix} 3y+1 \\ 2-4y \end{bmatrix}$$

Equalizing:

$$\begin{gathered} 3y+1 = 2-4y \\ 7y = 1 \\ y = \frac{1}{7} \end{gathered}$$

Thus, a Nash equilibrium occurs when

$$p = \begin{pmatrix} \frac{1}{8} & \frac{7}{8} \end{pmatrix}$$

$$q = \begin{pmatrix} \frac{1}{7} \\ \frac{6}{7} \end{pmatrix}$$

$$E_R = \begin{pmatrix} \frac{1}{8} & \frac{7}{8} \end{pmatrix}\begin{bmatrix} 4 & 1 \\ -2 & 2 \end{bmatrix}\begin{pmatrix} \frac{1}{7} \\ \frac{6}{7} \end{pmatrix} = \frac{10}{7}$$

$$E_C = \begin{pmatrix} \frac{1}{8} & \frac{7}{8} \end{pmatrix}\begin{bmatrix} -1 & 6 \\ 3 & 2 \end{bmatrix}\begin{pmatrix} \frac{1}{7} \\ \frac{6}{7} \end{pmatrix} = \frac{5}{2}$$

So $(E_R, E_C) = \left(\frac{10}{7}, \frac{5}{2}\right)$

b) Solve the relative zero-sum game. If both players use their optimal strategies suggested by the relative zero-sum game, what are their expected payoffs?

Solution: The relative zero-sum game has payoff matrix:

$$A_0 = \begin{bmatrix} 5 & -5 \\ -5 & 0 \end{bmatrix}$$

By oddments, the solution is

$$\hat{p} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$
$$\hat{q} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$$
$$v = \frac{detA_0}{15} = \frac{-25}{15} = -\frac{5}{3}$$

The expected payoffs in the original game if the players use these strategies:

$$E_R = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{bmatrix} 4 & 1 \\ -2 & 2 \end{bmatrix} \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} = \frac{10}{9}$$
$$E_C = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{bmatrix} -1 & 6 \\ 3 & 2 \end{bmatrix} \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} = \frac{25}{9}$$

Thus, $(E_R, E_C) = \left(\frac{10}{9}, \frac{25}{9}\right)$. Note that $E_R - E_C = \frac{10}{9} - \frac{25}{9} = -\frac{5}{3}$, as predicted by the relative zero-sum game solution.

c) Find the expected payoffs if each player plays prudentially.

Solution: To find the prudential strategies, each player uses his own payoffs and solves as if the game were zero-sum. For the row player:

$$\begin{bmatrix} 4 & 1 \\ -2 & 2 \end{bmatrix}$$

Solving for the row player by oddments or by equalizing expectation leads to $p = \left(\frac{4}{7}, \frac{3}{7}\right)$. To find the security level for the row player, $v = \frac{detA}{7} = \frac{10}{7}$. For the column player:

$$\begin{bmatrix} -1 & 6 \\ 3 & 2 \end{bmatrix}$$

From the column oddments we obtain $q = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \end{pmatrix}$, with security level $v = -\frac{detA}{8} = \frac{5}{2}$. If both player play prudentially, then the payoffs are

$$E_R = \begin{pmatrix} \frac{4}{7} & \frac{3}{7} \end{pmatrix} \begin{bmatrix} 4 & 1 \\ -2 & 2 \end{bmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \frac{10}{7}$$

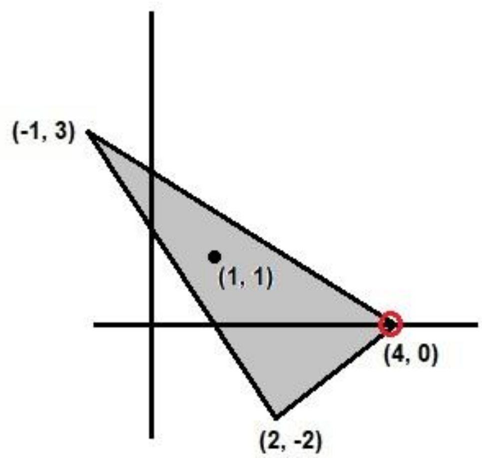
$$E_C = \begin{pmatrix} \frac{4}{7} & \frac{3}{7} \end{pmatrix} \begin{bmatrix} -1 & 6 \\ 3 & 2 \end{bmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \frac{5}{2}$$

Thus, in this case, if both players play prudentially, they each receive exactly their security levels. Furthermore, the payoff $\left(\frac{10}{7}, \frac{5}{2}\right)$ is exactly the same as at the Nash equilibrium in part (a) (which was obtained by a completely different set of mixed strategies.)

12. a) Show that the following game is strictly determined:

$$\begin{bmatrix} (1,1) & (-1,3) \\ (2,-2) & (4,0) \end{bmatrix}$$

Solution: The second row and the second column are dominant, so the point of intersection $(4,0)$ is necessarily a Nash equilibrium in pure strategies. It is clearly the only one in pure strategies, so we only need check to see if it is Pareto efficient. The payoff polygon is:



The Nash equilibrium at $(4,0)$ is circled. The Pareto efficient points all lie on the edge connecting $(-1,3)$ to $(4,0)$, so it includes the Nash equilibrium. With a unique Pareto efficient Nash equilibrium, this game is strictly determined.

b) Solve the relative zero-sum game, and show it is also strictly determined.

Solution: The relative zero-sum game has payoff matrix

$$A_0 = \begin{bmatrix} 0 & -4 \\ 4 & 4 \end{bmatrix}$$

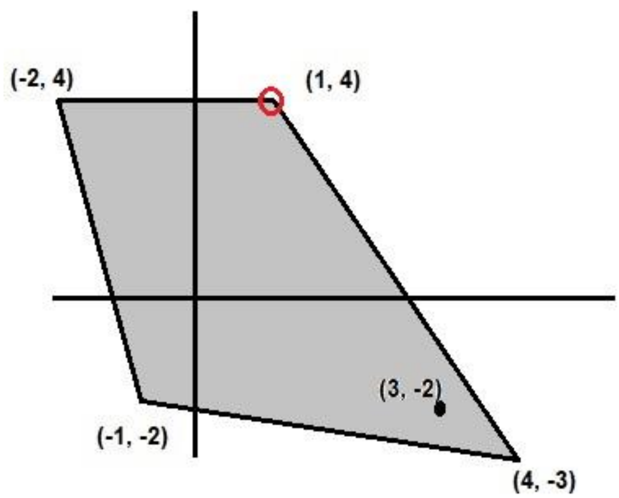
The entry $a_{22} = 4$ is a saddle point (and, like in the original game, it I the intersection of two dominant strategies.)

13. Consider the following payoff matrix:

$$\begin{bmatrix} (3,-2) & (1,4) & (3,-2) \\ (-2,4) & (-1,-2) & (4,-3) \end{bmatrix}$$

Draw the payoff polygon, and show that the game is strictly determined.

Solution: The outcome with payoff $(1,4)$ is a Nash equilibrium, and it is the only one in pure strategies (draw the movement diagram.) The payoff polygon (with the Nash equilibrium circled) is:



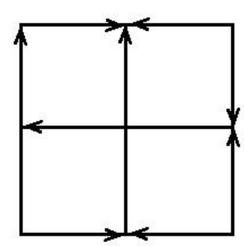
The Pareto efficient outcomes are those along the edge connecting $(1,4)$ to $(4,-3)$. This game has a unique Pareto efficient Nash equilibrium, so it is strictly determined.

14. Consider the payoff matrix

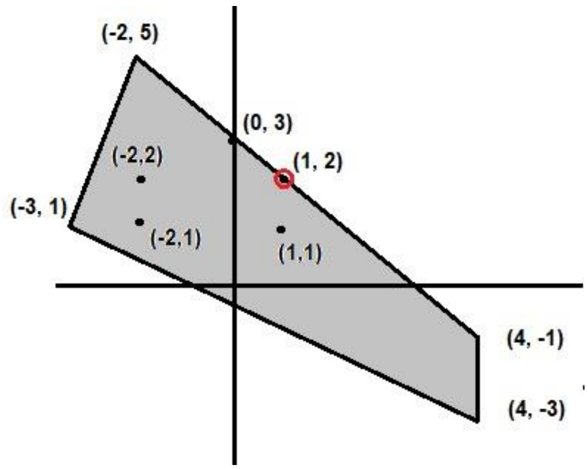
$$\begin{bmatrix} (4,-1) & (1,2) & (-2,1) \\ (0,3) & (-2,2) & (4,-3) \\ (-3,1) & (-2,5) & (1,1) \end{bmatrix}$$

Draw the payoff polygon and show that the game is strictly determined. Show also that, in this game, every Pareto efficient outcome has the same sum of the payoffs for both players. In particular, the solution also satisfies the first condition which benefits the group.

Solution: First, we draw the movement diagram:



The diagram shows there is a unique Nash equilibrium in pure strategies at the outcome with payoff $(1,2)$. The payoff polygon is:



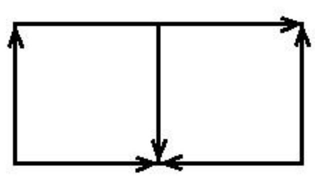
The Pareto efficient points all lie on the edge connecting $(-2,5)$ to $(4,-1)$, and includes the Nash equilibrium at $(1,2)$. All these points lie on the line with equation $x + y = 3$. Therefore, with a unique Pareto efficient Nash equilibrium, this game is strictly determined. Also, because all the points on the Pareto efficient edge have the same total payoffs ($x + y = 3$), which is clearly the largest possible sum of the payoffs, the solution also satisfies the stronger condition that the total payoffs are as large as possible.

15. Consider the following payoff matrix:

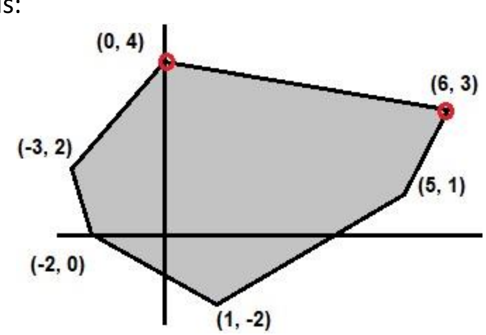
$$\begin{bmatrix} (5,1) & (-3,2) & (6,3) \\ (-2,0) & (0,4) & (1,-2) \end{bmatrix}$$

a) Draw the movement diagram and the payoff polygon, and show that the game is not strictly determined.

Solution: The movement diagram is:



The diagram reveals two different Nash equilibria in pure strategies at the outcomes with payoffs $(6,3)$ and $(0,4)$. The payoff polygon is:



The Pareto efficient points lie on the edge connecting $(0{,}4)$ to $(6{,}3)$, and includes both Nash equilibria. Since we have two Nash equilibria which are both Pareto efficient, but which are not equivalent or interchangeable, the game is not strictly determined.

b) If each player tries for his best Nash equilibrium point, what is the outcome of the game?

Solution: The row player prefers the $(6{,}3)$, so chooses the first row. The column player prefers the $(0{,}4)$, so chooses the second column. The result is $(-3{,}2)$. Not only is that a poor payoff for both players, but the fact that the outcome is not a Nash equilibrium verifies the claim made in part (a) that the Nash equilibria are not interchangeable.

c) With only a minor change to the payoff matrix, the game can have quite different dynamics. Suppose that instead of $(0{,}4)$, the payoff to the column player changes to c, so that $(0, c)$ becomes the payoff in the a_{22} position of the matrix. Show that, if $c > 0$, then there is no change to the movement diagram at all, so there are still two Nash equilibria (even if $c = 0$, there are still two Nash equilibria in two strategies), but if $0 \leq c \leq \frac{7}{3}$, the game becomes strictly determined.

Solution: If $c \geq 0$, there is still an upward pointing arrow in the second column, so the movement diagram does not change and $(0, c)$ is a Nash equilibrium. However, if $0 \leq c \leq \frac{7}{3}$, the point $(0, c)$ becomes an interior point of the polygon, so is no longer Pareto efficient. In fact, even if $\frac{7}{3} \leq c \leq 3$, when $(0, c)$ is still a corner, the point $(0, c)$ is no longer Pareto efficient because the slope of the segment joining $(0, c)$ to $(6{,}3)$ becomes positive if $c \leq 3$. The upshot is that for $0 \leq c \leq 3$, the Nash equilibrium $(0, c)$ is not Pareto efficient, so $(6{,}3)$ becomes the unique Pareto efficient Nash equilibrium for these range of values of c, and the game becomes strictly determined.

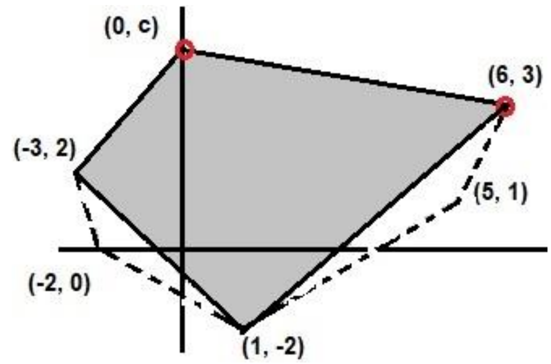
(Remark: If this problem calls to mind the 'stable ranges' we saw in linear programming, that's because in this exercise, you *are* performing a sensitivity analysis on the game. For the range $0 \leq c \leq 3$, the game is strictly determined, but suddenly becomes otherwise when $c > 3$ – a sort of stable range for that payoff of the game. Each coordinate of each Nash equilibrium of a strictly determined game would have a similarly defined stable range over which the game remains strictly determined.)

d) Notice also that if $c \geq 0$, the second column dominates the first column. Re-analyze the game by deleting the dominated column, in the original case when $c = 4$, and in the case when $0 \leq c \leq \frac{7}{3}$. You should notice that despite the fact that the payoff polygon changes, the predictions for the outcomes remains the same. (However, this is *not* always true! See Chapter 11, Exercise 4 of [Straffin, 1993] for an example where deleting the dominated strategies does affect the predicted outcome.)

Solution: If you delete the dominated column, the payoff matrix becomes

$$\begin{bmatrix} (-3{,}2) & (6{,}3) \\ (0, c) & (1, -2) \end{bmatrix}$$

The outcomes $(0, c)$ and $(6{,}3)$ are still Nash equilibria. The payoff polygon changes to a 'sub"-polygon:



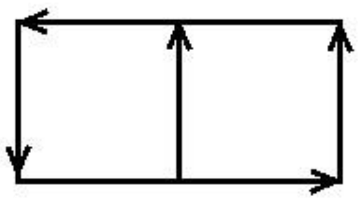
The same conclusions hold for the subgame as for the whole game: If $0 \leq c \leq 3$, then $(0, c)$ is a Nash equilibrium, but not Pareto efficient, so the game is strictly determined with $(6,3)$ being the unique Pareto efficient Nash equilibrium (and, in fact, the unique Preto efficient point.) If $c > 3$, then $(0, c)$ becomes Pareto efficient (and remains a Nash equilibrium), so the game now has two, inequivalent Pareto efficient Nash equilibria, so is not strictly determined, just as in the entire game.

16. Consider the variable-sum game with payoff matrix

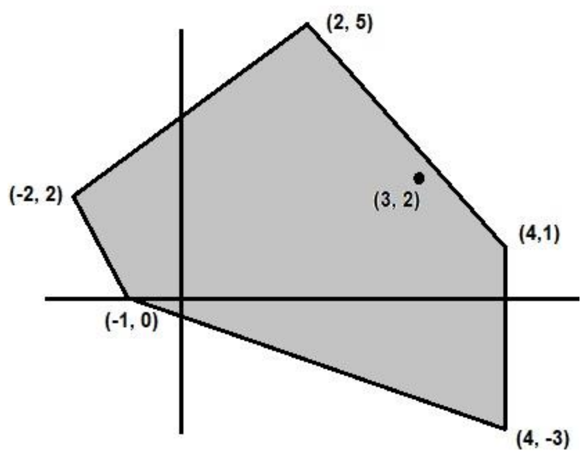
$$A = \begin{bmatrix} (3,2) & (-1,0) & (4,-3) \\ (4,1) & (-2,2) & (2,5) \end{bmatrix}$$

a) Show there is no dominance in this game. Also show there is no pure strategy Nash equilibrium. Draw the payoff polygon.

Solution: Draw the movement diagram first:



From the movement diagram it is clear there is no dominance and no Nash equilibria in pure strategies. The payoff polygon is:



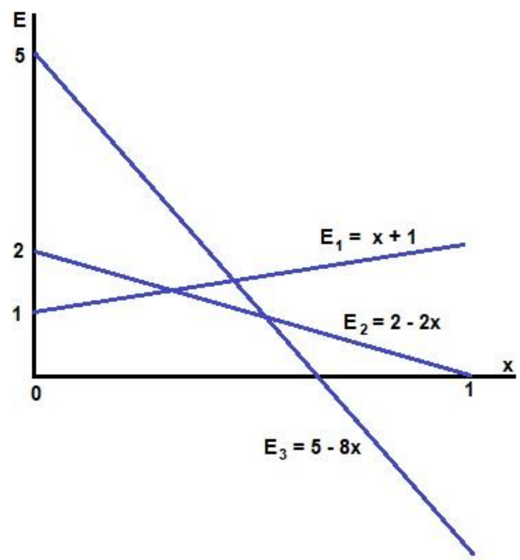
The Pareto efficient outcomes are located along the edge connecting $(2,5)$ to $(1,4)$.

b) In the remainder of this exercise, we will find a mixed strategy Nash equilibrium. Suppose the row player plays $p = (x, 1-x)$. Using the column player's payoffs compute $pA = (E_1 \quad E_2 \quad E_3)$. Each of the E_i represents the expected payoff to the column player if he uses the i'th pure strategy. Plot all three lines in the x, E-plane as we did in Section 7.1.2 for constant-sum games. We will show that we seek a solution which is the point of intersection of two of the lines, just as in the constant-sum case. However, remember now that the column player wants E to be as large as possible, since we are working with his payoffs, not the row player's. The top edge of the graph represents the best-case scenarios for the column player. Find the point which is the lowest point on this top edge (the minimax point.)

Solution:

$$pA = (x \quad 1-x)\begin{bmatrix} 2 & 0 & -3 \\ 1 & 2 & 5 \end{bmatrix} = (x+1 \quad 2-2x \quad 5-8x)$$

Plotting the lines:



The point which is lowest along the top edge is the intersection of the lines E_1 and E_3. The coordinates are:

$$x + 1 = 5 - 8x$$
$$9x = 4$$
$$x = \frac{4}{9}$$

Thus

$$p = \left(\frac{4}{9}, \frac{5}{9}\right)$$

c) Compute $pA = (E_1 \quad E_2 \quad E_3)$ using the specific p that you found in part (b). Observe that two of the coordinates are equal (because we equalized two of the payoffs in part (b)), while the third is smaller (because we chose the minimax in part (b).) Now suppose the column player uses the mix

$$q = \begin{pmatrix} x \\ y \\ 1-x-y \end{pmatrix}$$

Compute pAq, the expected payoff for the column player. You should obtain the expression $pAq = \frac{13}{9} - \frac{1}{3}y$. Since the column player wants to maximize his expected payoff, it is clear the maximum occurs when $y = 0$. Conclude that the column player should never play the second column. It becomes an inactive strategy, just as in the constant-sum case.

Solution:

$$pA = \left(\frac{4}{9}, \frac{5}{9}\right)\begin{bmatrix} 2 & 0 & -3 \\ 1 & 2 & 5 \end{bmatrix} = \begin{pmatrix} \frac{13}{9} & \frac{10}{9} & \frac{13}{9} \end{pmatrix}$$

Indeed, the E_2 coordinate is smaller than the remaining two coordinates, which are equal. Now the expected payoff to the column player is:

$$pAq = \left(\frac{4}{9}, \frac{5}{9}\right)\begin{bmatrix} 2 & 0 & -3 \\ 1 & 2 & 5 \end{bmatrix}\begin{pmatrix} x \\ y \\ 1-x-y \end{pmatrix} = \left[\frac{13}{9} - \frac{1}{3}y\right]$$

In order for the column player to maximizer this, he should choose $y = 0$, which means the second column is an inactive strategy.

d) Now, since $y = 0$, the column player should play $q = \begin{pmatrix} x \\ 0 \\ 1-x \end{pmatrix}$. Compute $Aq = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$, this time using the row players payoffs in A, to find the value of x which equalizes the expected payoffs $E_1 = E_2$ for the row player.

Solution: So, the column player will play the q indicated with $y = 0$. The payoffs to the row player are then

$$Aq = \begin{bmatrix} 3 & -1 & 4 \\ 4 & -2 & 2 \end{bmatrix}\begin{pmatrix} x \\ 0 \\ 1-x \end{pmatrix} = \begin{pmatrix} 4-x \\ 2x+2 \end{pmatrix}$$

Equalizing:

$$4 - x = 2x + 2$$
$$2 = 3x$$
$$x = \frac{2}{3}$$

Thus

$$q = \begin{pmatrix} \frac{2}{3} \\ 0 \\ \frac{1}{3} \end{pmatrix}$$

e) Explain why the p you found in part (b), together with the q you found in part (d), yields a Nash equilibrium, find the corresponding payoff (E_R, E_C), and plot it on the payoff polygon you found in part (a). Is the game strictly determined?

Solution: When the row player plays p, consider what happens if the column player unilaterally deviates from q. If the second coordinate of q remains 0, then we know any value of x will produce the same expected payoff, $\frac{13}{9}$, for the column player. If the second coordinate becomes positive, then the expected payoff for the column player is some weighted average of $\frac{13}{9}$ and $\frac{10}{9}$, and any such average must be strictly smaller than $\frac{13}{9}$. Either way, there is no incentive for the column player to change either x or y in q.

Similarly, if the column player plays q, then the second column is inactive, and the game is the same as the 2×2 subgame obtained by deleting that column:

$$\begin{bmatrix} (3,2) & (4,-3) \\ (4,1) & (2,5) \end{bmatrix}$$

However, we already know how to find a Nash equilibrium in a 2×2 game. Each player must choose a mix to equalize the other player's payoffs. Thus, q must be chosen to equalize the row player's payoffs:

$$\begin{bmatrix} 3 & 4 \\ 4 & 2 \end{bmatrix} \begin{pmatrix} x \\ 1-x \end{pmatrix} = \begin{pmatrix} 4-x \\ 2x+2 \end{pmatrix}$$

That is, we must solve $4 - x = 2x + 2$. But this is exactly the calculation we did in part (d) to find x. Thus, there is no incentive for the row player to unilaterally deviate from p. Since neither player stands to gain from a unilateral deviation from these strategies, the outcome is by definition a Nash equilibrium. The payoffs should come out to be $\frac{10}{3}$ for the row player (the common value of E_1 and E_2 in part (d) when $x = \frac{2}{3}$), and $\frac{13}{9}$ for the column player. We verify this:

$$E_R = \left(\frac{4}{9}, \frac{5}{9}\right) \begin{bmatrix} 3 & -1 & 4 \\ 4 & -2 & 2 \end{bmatrix} \begin{pmatrix} \frac{2}{3} \\ 0 \\ \frac{1}{3} \end{pmatrix} = \left[\frac{10}{3}\right]$$

$$E_C = \left(\frac{4}{9}, \frac{5}{9}\right) \begin{bmatrix} 2 & 0 & -3 \\ 1 & 2 & 5 \end{bmatrix} \begin{pmatrix} \frac{2}{3} \\ 0 \\ \frac{1}{3} \end{pmatrix} = \left[\frac{13}{9}\right]$$

Thus, $(E_R, E_C) = \left(\frac{10}{3}, \frac{13}{9}\right) \approx (3.33,\ 1.44)$. In the payoff polygon, this is an interior point. Located roughly between (3,2) and (4,1). Although this is close to the Pareto efficient edge, it is still an interior point, so not Pareto efficient. Therefore, this game is not strictly determined.

Remark: The techniques of Exercise 16 should work on any $m \times 2$ or $2 \times n$ variable-sum game which is not strictly determined, and yield a Nash equilibrium which is the solution to some 2×2 square subgame, just as in the constant-sum case. This suggests that a version of the square subgame theorem should hold for variable-sum games. We simply rephrase the theorem to remove the word 'solution' to the game. We state the theorem as 'A Nash equilibrium in a variable-sum game agrees with the Nash equilibrium in some square subgame.' However, this Nash equilibrium is not generally Pareto efficient so it may not be worth the effort to compute it.

Chapter 8. Sensitivity Analysis, Ordinal Games, and n-Person Games

Section 8.1 Sensitivity Analysis in Game Theory

8.1.1 What if Play is Not Simultaneous?

No music references or exercises in this subsection.

8.1.2 Game Trees and Reverse Induction

Music references in the text: Example 8.1, the (fictitious) board game Without Frontiers is a musical homage to Peter Gabriel. His 1980 album Peter Gabriel 3 included the song 'Games Without Frontiers.' Two of the protagonists of the song are Hans and Enrico. (Website: https://petergabriel.com/)

Example 8.2 with Mr. Brooker pays musical homage to he band Procol Harum. The vocalist for Procol Harum was Gary Brooker. The Salty Dog Tavern a reference to their 1969 album A Salty Dog, while the Pale White Whale Corporation is a nod to both their song 'Whaling stories' from their 1970 album Home, as well as their massive hit single from 1967, 'A Whiter Shade of Pale'. Finally, in PH Royal Zonophone, the PH stands for Procol Harum and their 1967 UK record label was named Regal Zonophone. (Website: https://www.procolharum.com/)

No exercises in this subsection.

8.1.3 What if Communication is Allowed?

No exercises in this subsubsection.

8.1.4 What if Your Opponent is Indifferent? Games Against Nature.

Music reference in the text: The example about Mr. Holly deciding whether or not to carry an umbrella pays homage to the 1966 song 'Bus Stop' by the Hollies. (Website: https://www.thehollies official.com/)

Solutions to exercises and more music references:

1. The following payoff matrices are for zero-sum games. In each case, predict the outcome if the row player players first, and repeat the exercises if the column player goes first. Give reasons for your answers.

a)

$$\begin{bmatrix} 6 & 1 & 0 & -2 \\ 3 & 5 & 2 & 7 \\ -2 & -3 & -1 & 6 \end{bmatrix}$$

Solution: Note that row two dominates row three, so there would never be any reason for the row player to play row three. In fact, if she chooses the second row, the column player would respond by choosing the third column to obtain the 2 payoff (by the expected value principle.) If she chooses the first row, the column player would respond by choosing the fourth column to obtain the -2 payoff. Since 2 is better than -2 for the row player, the row player would decide to choose the second row, and the predicted outcome would be the 2 payoff in the a_{23} position.

If the column player goes first, and chooses any column, the row player would respond with the row that contains the highest payoff in that column, by the expected value principle. Looking at the column maxima, they are 6,5,2,7. The column player knows that the payoff will be one of these numbers, and prefers the smallest one, 2, that comes form selecting the third column. Thus, the outcome is again $a_{23} = 2$.

In fact, this point is a saddle point in the matrix, and we mentioned in the text that sequential play would not affect the outcome of any strictly determined game. This example illustrates that.

b)

$$\begin{bmatrix} -4 & 3 & 1 \\ 2 & 0 & 4 \\ 1 & -1 & -3 \end{bmatrix}$$

Solution: Unlike pat (a), this game is not strictly determined. If the row player went first, she knows the column player will choose the column with the row minimum. Thus, she should choose the row that has the highest possible row minimum, which is the second row. The column player responds by choosing the second column, leading to the outcome $a_{22} = 0$.

If the column player moves first, he should choose the column with the smallest column maximum, which is the first column. The row player responds with the second row leading to the outcome $a_{21} = 2$.

c)

$$\begin{bmatrix} -2 & -1 & 0 & 1 & 2 \\ 10 & 8 & 5 & -1 & -9 \\ 5 & -9 & 3 & -8 & 7 \end{bmatrix}$$

Solution: This game is also not strictly determined. If the row player plays first, she should choose the row with the largest row minimum, which is the first row. Then the column player responds by choosing the first column, and the payoff is -2.

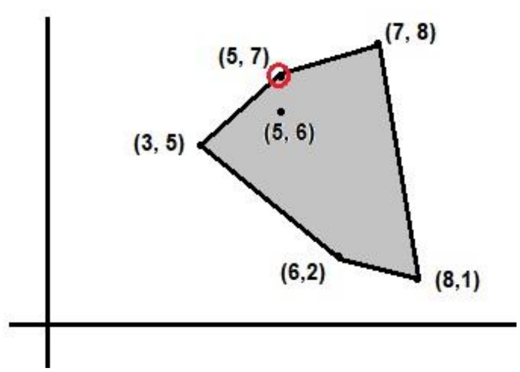
If the column player plays first, he should choose the column with the smallest row maximum, which is the fourth column. The row player responds by choosing the first row for a payoff of 1.

2. In the text we considered the variable-sum game with payoff matrix:

$$\boldsymbol{A} = \begin{bmatrix} (5,7) & (6,2) & (8,1) \\ (3,5) & (5,6) & (7,8) \end{bmatrix}$$

Draw the payoff polygon and verify that the pure strategy Nash equilibrium is not Pareto efficient, so that the game is not strictly determined. Verify that if the column player plays first, the predicted outcome is the Nash equilibrium, but f the row player goes first, both players can improve on their outcomes.

Solution: The outcome with payoff $(5,7)$ is a Nash equilibrium in pure strategies. It is circled in the payoff polygon:



The Pareto efficient outcomes lie along the edge connecting $(7,8)$ to $(8,1)$, and since the Nash equilibrium is not among them, this game is not strictly determined.

Suppose the column player goes first. Notice the first row dominates the second row. So, no matter what column is selected, the row player will respond by choosing the first row, which is always better for him. The column player knows this, so knows the outcome will be $(5,7)$, $(6,2)$, or $(8,1)$. Therefore, he would choose the first column to obtain the 7 payoff, the best of the three, for himself. Thus, the outcome is the Nash equilibrium at $(5,7)$.

Suppose the row player goes first. If she chooses the first row, then the column player chooses the first column so the outcome is again the Nash equilibrium. However, if she chooses the second row, the column plyer now has a choice between $(3,5)$, $(5,6)$, or $(7,8)$. The column player will respond by choosing the third row, so the outcome is $(7,8)$. This is better for both players than the Nash equilibrium. Also, $(7,8)$ is Pareto efficient. It even satisfies the stronger condition that the total payoffs are the highest possible, since every other point in the payoff polygon lies below the line $x + y = 15$.

3. In the text we considered the variable-sum game with payoff matrix:

$$\begin{bmatrix} (9,5) & (5,7) \\ (4,4) & (4,6) \\ (0,1) & (2,3) \end{bmatrix}$$

Verify the claims made in the text. That is, both players have dominant strategies, the game is strictly determined, and if either player plays first, the outcome is the same as in simultaneous play.

Solution: The first row dominates both of the other rows, and the second column dominate the first. The $(5,7)$ at the intersection of these two dominant strategies is a Nash equilibrium, and the predicted outcome under simultaneous play. If either player plays first, they would choose their dominant strategy, and the best response to that would e for the other player to play their dominant strategy (since the payoff is always the best regardless of what the other player chose.) Thus, we arrive at the Nash equilibrium in sequential play as well.

4. Consider the payoff matrix in Exercise 3. In the text, we suggested that if we allowed communication between the players, and the column player went first, that the row player could obtain the $(9,5)$ outcome, even though it is not an equilibrium point. We investigate this here.

a) Suppose that the column player moves first, but before he moves, the row player says to him "If you choose the second column, I will choose the third row." Assume that the column player believes her. What is the resulting outcome of the game? Explain.

Solution: Assuming the column player believes the row player, he can expect the outcome of $(2,3)$ if he chooses his dominant strategy. However, he reasons, if he chooses the first column instead, the row player will obviously choose the first row to get the 9 payoff. But then, the payoff is $(9,5)$, which is better than the 3 he would get if he chooses his dominant column. So, he chooses the first row, and the payoff obtained is $(9,5)$.

b) Suppose the column player moves first, but before he moves, the row player says to him "If you choose the second column, I will choose the second row." What is the resulting outcome, again, assuming the column player believes her. Explain.

Solution: This time, the column player will not be swayed, and will stick to the second column, figuring the outcome will be $(4,6)$ as opposed to the $(9,5)$, and he'd rather have the 6 than the 5.

c) These types of remarks that the row player makes to the column player can be called *threats*. A threat is a statement about action you will take, depending on what your opponent does; that is, it is contingent on your opponent's move. ('If you do this, I'll do that in response'…) Furthermore, your response should certainly hurt your opponent. Observe that in both part (a) and part (b), if the column player ignores your threat, then his payoffs go down from the predicted outcome without communication. Does a threat hurt or help the row player? However, a threat does not always work. What would be a necessary condition on the payoffs in the second column in order for there to be a threat which would have the desired effect?

Solution: If the person making the threat follows through, then their own payoffs, like their opponents, will go down. That is, a threat hurts the person making the threat. This must be the case, since, otherwise, the person making the threat would be making a rational choice (if it helped his payoffs, that would be rational!), and therefore, being the rational choice, the opponent would expect it, so there is no reason to state aloud your intentions. Otherwise put, if your threat did not hurt you (as well as hurt your opponent), there would be no point in making the 'threat' at all, since it is exactly what rational behavior would predict that you would do. You only need to state aloud your intentions when your actions would deviate from what rational behavior dictates.

As in part (b), we see that a threat does not always work. In order for it to work, the outcome which would be chosen if the threat were ignored must be of the form (a, b), where $b < 5$ (since he could obtain the 5 payoff if he changed his strategy based on the threat), and where $a < 5$ (otherwise the row player is not actually making a threat since that behavior would be rational.) Also, $b < 7$ must hold so that, if ignored, the threat would actually hurt the column player. But this is automatic in this case since we already noted $b < 5$.

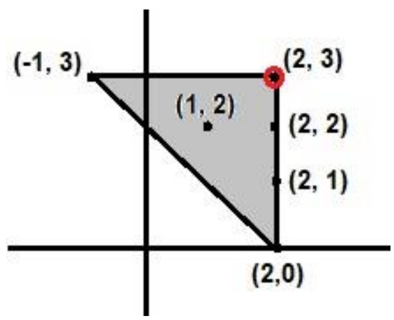
Remark: In a game where a threat will not help the row player, it is possible that a *promise* might, or perhaps a combination of a threat and a promise. For more information on the difference between a threat and a promise, and more examples, see Chapter 14 of Straffin, (1993).

5. Consider the round of the game Without Frontiers that we introduced in the text. Verify that the game in normal form is a strictly-determined variable sum game, and determine the solution.

Solution: The reader can refer back to the example in the text to see what the strategies are, but the payoff matrix we derived there is:

$$A = \begin{bmatrix} (1,2) & (1,2) & (-1,3) & (-1,3) \\ (2,3) & (2,0) & (2,1) & (2,2) \end{bmatrix}$$

The second row dominates the first. Knowing this, the column player would choose the first column. The resulting outcome with payoff $(2,3)$ is the unique Nash equilibrium in pure strategies. The payoff polygon is:



The Nash equilibrium is circled and is clearly Pareto Efficient (in fact, it is the only Pareto efficient outcome!), so the game is strictly determined. Thus, the predicted outcome is that Hans chooses the strategy of first building attack ships, but then colonizing the Terran planet if Enrico does. Enrico chooses the strategy of colonizing the Terran planet. So, they both end up colonizing the Terran planet and the payoff is $(2,3)$.

6. a) If we vary the rules of Without Frontiers, so that Enrico knows what decision Hans makes on the first step, then the number of strategies open to Enrico doubles from two to four. Write down Enrico's four strategies.

Solution: Enrico's action can now be made contingent on what Hans does in the first step. So the strategies are:

A – Attack factory base if Hans builds attack ships and attack factory base if Hans fortifies the base (always attack.)

B – Attack the factory base if Hans builds attack ships and colonize the Terran planet of Hans fortifies the base.

C – Colonize the Terran planet of Hans builds attack ships and attack the factory base if Hans fortifies the base.

D – Colonize the Terran planet if Hans builds attack ship and colonize the Terran planet of Hans fortifies the base (always colonize.)

b) Convert the game to normal form, assuming Enrico knows what decision Hans makes on the first step. Solve the game.

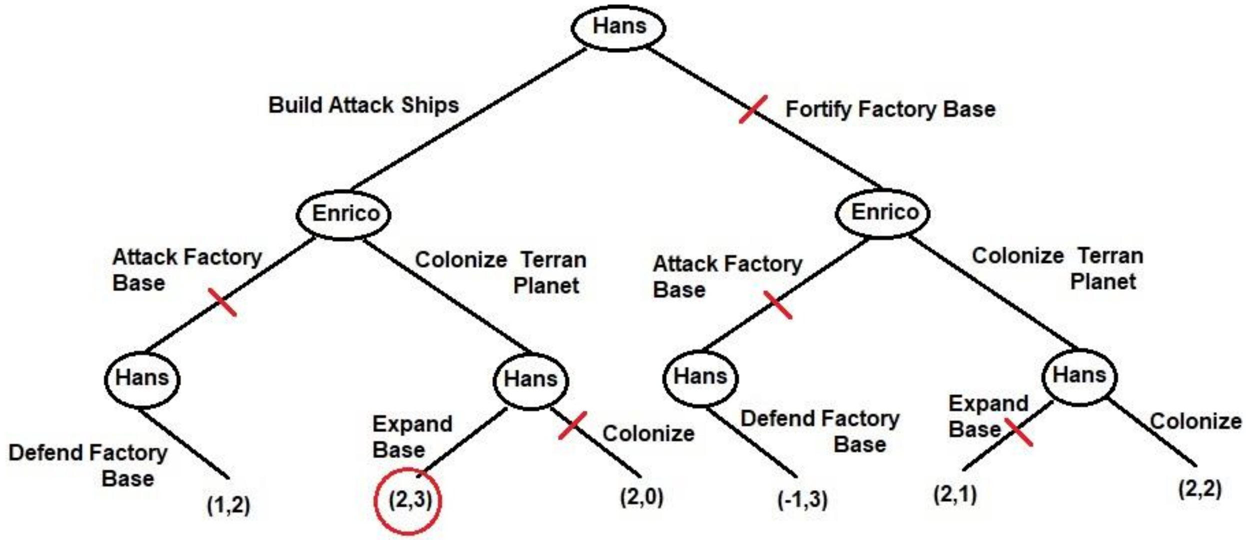
Solution: The payoff matrix is (refer to the text for Hans' strategies):

		Hans			
		AE	AC	FE	FC
Enrico	A	$(1,2)$	$(1,2)$	$(-1,3)$	$(-1,3)$
	B	$(1,2)$	$(1,2)$	$(2,1)$	$(2,1)$
	C	$(2,3)$	$(2,0)$	$(-1,3)$	$(-1,3)$
	D	$(2,3)$	$(2,0)$	$(2,1)$	$(2,2)$

Note that D dominates all other strategies for Enrico. Given that, Hans will choose AE (expecting Enrico to choose D), with the resulting payoff of a_{41} =(2,3) represents a Nash equilibrium in pure strategies. Notice that $a_{31} = (2,3)$ is also a Nash equilibrium in pure strategies. Both of these Nash equilibria are Pareto efficient. However, they are equivalent (since they both have the same value $(2,3)$) and interchangeable (since they both appear in the same column, the result of Enrico choosing C or D still results in the equilibrium outcome.) The game tree is the same as in Exercise 5 – except that this time, the circled point represents two distinct Nash equilibria which are Pareto efficient. Thus, the game is strictly determined, with either Nash equilibrium with payoff $(2,3)$ as an acceptable solution (although the one where Enrico chooses D is probably more likely because of the dominant row.)

c) When Enrico knows Hans' initial decision, we mentioned in the text that the game becomes a game of perfect information. In that case, analyze the game by performing reverse induction on the game tree. Your solution should agree with the one you got in part (b).

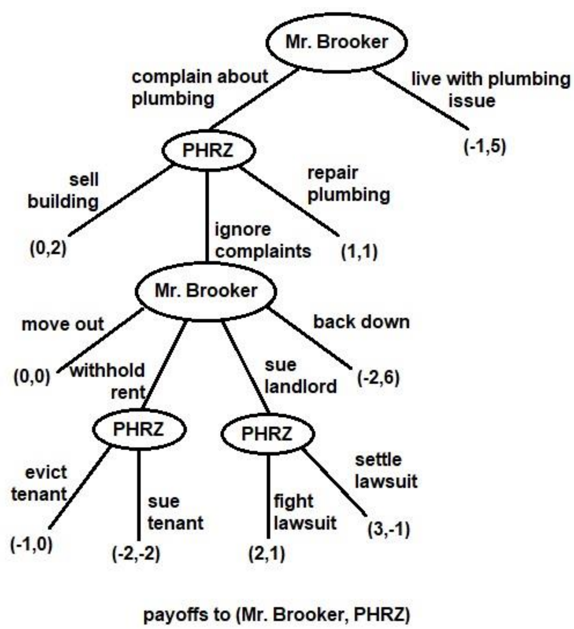
Solution: The game tree is shown with the appropriate branches crossed off, and the solution circled:



The predicted outcome agrees with part (b).

7. We solved the Salty Dog Tavern problem (Example 8.2) by analyzing the game tree. Instead, convert the game to normal form, and analyze it by considering the payoff matrix. Your answer should agree with the one we got in the text.

Solution: We first write down the possible strategies, coming from the tree diagram:



Let's make Mr. Brooker the row player. He has five strategies:

L – Live with the plumbing issue
CM – Complain about the plumbing; Move out if PHRZ ignores him
CW – Complain about the plumbing; Withhold rent if PHRZ ignores him
CS – Complain about the plumbing; Sue PHRZ if they ignore him
CB – Complain about the plumbing, Back down if they ignore him

On the other hand, PHRZ has six strategies:

S – Sell building
R – Repair plumbing
IEF – Ignore complaint, Evict tenant if they withhold rent, Fight if tenant sues
IES – Ignore complaint, Evict tenant if they withhold rent, Settle lawsuit if tenant sues
ISF – Ignore complaint, Sue tenant if they withhold rent, fight if tenant sues
ISS – Ignore complaint, Sue tenant if they withhold rent, settle lawsuit if tenant sues

This leads to the 5×6 payoff matrix:

		PHRZ					
		S	R	IEF	IES	ISF	ISS
Mr. Brooker	L	$(-1,6)$	$(-1,6)$	$(-1,6)$	$(-1,6)$	$(-1,6)$	$(-1,6)$
	CM	$(0,2)$	$(1,1)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
	CW	$(0,2)$	$(1,1)$	$(-1,0)$	$(-1,0)$	$(-2,-2)$	$(-2,-2)$
	CS	$(0,2)$	$(1,1)$	$(2,1)$	$(3,-1)$	$(2,1)$	$(3,-1)$
	CB	$(0,2)$	$(1,1)$	$(-2,6)$	$(-2,6)$	$(-2,6)$	$(-2,6)$

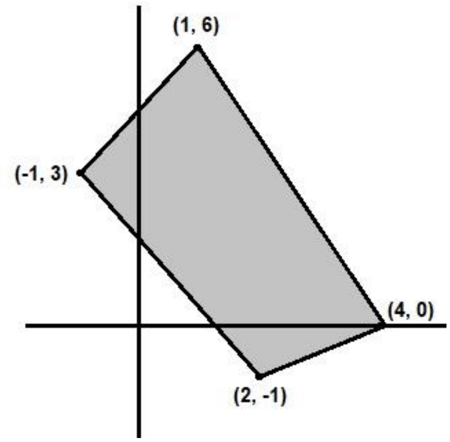
Notice that CS dominates all other strategies for Mr. Brooker. Knowing that, PHRZ will choose S by higher order dominance (or by the expected value principle.) Thus, Mr. Brooker complains about the plumbing, and is prepared to sue if his complaint is ignored. However, instead PHRZ sells the building. This is the exact same conclusion we came to in the text using the tree diagram.

8. Consider the variable-sum game with payoff matrix:

$$A = \begin{bmatrix} (4,0) & (1,6) \\ (-1,3) & (2,-1) \end{bmatrix}$$

a) Draw the payoff polygon. Verify that the game is not strictly determined.

Solution: A movement diagram shows no Nash equilibrium in pure strategies. The payoff polygon:



b) Find each player's prudential strategies and security levels. Mark the point with the security levels as $(v_R, v_C) = (x_0, y_0)$ on the payoff polygon. Determine the negotiation set, including the coordinates of the two endpoints.

Solution: For the row player,

$$A = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$$

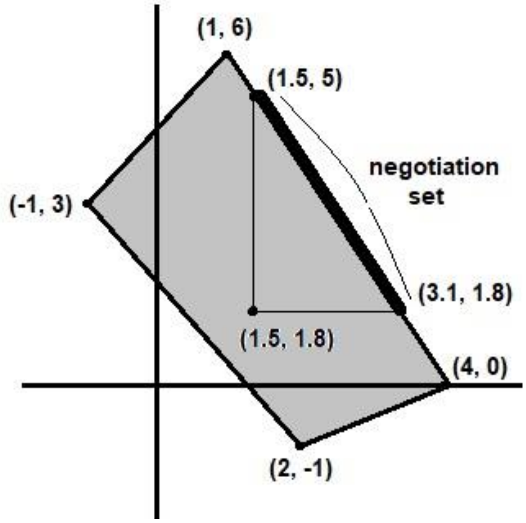
By oddments, the prudential strategy is $p = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $v_R = \frac{detA}{6} = \frac{3}{2}$.

For the column player,

$$A = \begin{bmatrix} 0 & 6 \\ 3 & -1 \end{bmatrix}$$

By oddments, the prudential strategy is $q = \begin{pmatrix} \frac{7}{10} \\ \frac{3}{10} \end{pmatrix}$ and $v_C = -\frac{\det(A^T)}{10} = \frac{9}{5}$.

Thus, $(v_R, v_C) = (x_0, y_0) = (1.5, 1.8)$. Extending vertically and horizontally from this point determines the negotiation seton the Pareto efficient points. The pareto efficient outcomes lie on the edge connecting $(1,6)$ to $(4,0)$. This edge has equation $y = 8 - 2x$. Thus, the negotiation set is the segment between $(1.5,5)$ and $(3.1, 1.8)$ on this edge, marked with a thick line in the picture:



c) One of the suggestions in the text for the negotiated solution point was the midpoint of the negotiation set. Find the coordinates of that point.

Solution: The midpoint has coordinates:

$$\left(\frac{1.5+3.1}{2}, \frac{5+1.8}{2}\right) = (2.3, 3.4)$$

d) Another suggestion in the text for the negotiated solution point was to preserve the proportions of $\frac{v_C}{v_R}$, which entails finding the point in the negotiation set which also lies on the radial line through (x_0, y_0) and the origin. Find the coordinates of that point.

Solution: The slope of the radial line through $(1.5, 1.8)$ is $\frac{1.8}{1.5} = 1.2$, so the equation of the radial line is $y = 1.2x$. This line meets the line $y = 8 - 2x$ when

$$1.2x = 8 - 2x$$
$$3.2x = 8$$
$$x = \frac{8}{3.2} = 2.5$$

Thus, the coordinates of the point in the negotiation set that we seek is $(2.5, 3)$

e) A third suggestion was to find the line through (x_0, y_0) with slope $m = 1$, and take the point where this line intersects the negotiation set. Find the coordinates of that point.

Solution: The line in question has equation $y = x + .3$, which meets the negotiation set when

$$x + .3 = 8 - 2x$$
$$3x = 7.7$$

$$x = \frac{7.7}{3} = 2.56667$$

The coordinates of the suggested solution are $(2.56667, 2.86667)$

9. In the text, we mentioned that John Nash discovered a unique arbitration scheme that satisfied certain axioms. Here is his method, which we present without discussion of the axioms involved, and without proof: The suggested negotiated solution point is the point (x, y) in the negotiation set which maximizes the product $(x - x_0)(y - y_0)$. We'll call that point the *Nash point*. Find the coordinates of the Nash point for the payoff matrix in Exercise 8.

Solution: We must maximize $(x - 1.5)(y - 1.8)$, subject to the constraint that (x, y) lies in the Negotiation set, which means $y = 8 - 2x$, and $1.5 \le x \le 3.1$ (see the diagram in Exercise 8b above.)

Thus,

$$\begin{aligned} P &= (x - 1.5)(y - 1.8) \\ &= (x - 1.5)(8 - 2x - 1.8) \\ &= (x - 1.5)(6.2 - 2x) \\ &= -2x^2 + 9.2x - 9.3 \end{aligned}$$

From Chapter 1, we recognize this as a downward opening parabola, so the maximum occurs at the vertex, which we find y completing the square to put it into standard form:

$$\begin{aligned} P &= -2(x^2 - 4.6x) - 9.3 \\ &= -2(x^2 - 4.6x + 5.29) + 10.58 - 9.3 \\ &= -2(x - 2.3)^2 + 1.28 \end{aligned}$$

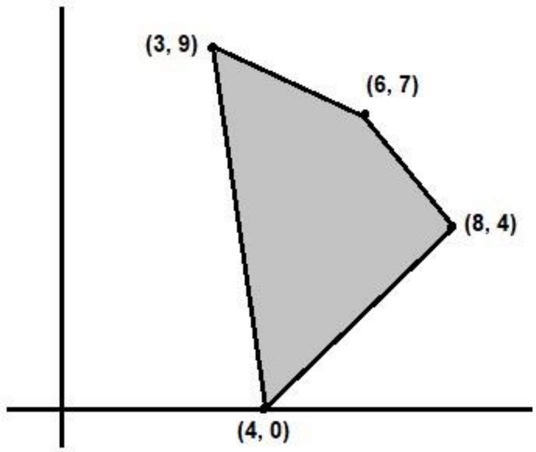
So, $x = 2.3$ and $y = 8 - 2x = 3.4$ (which does also satisfy $1.5 \le x \le 3.1$ as required.) So the Nash point is $(2.3, 3.4)$, which happens to be the midpoint of the negotiation set in Exercise 8c (but in general, the Nash point will not agree with the midpoint.)

10. Consider the variable-sum game with payoff matrix

$$A = \begin{bmatrix} (8,4) & (3,9) \\ (6,7) & (4,0) \end{bmatrix}$$

a) Verify that this game is not strictly determined, and draw the payoff polygon.

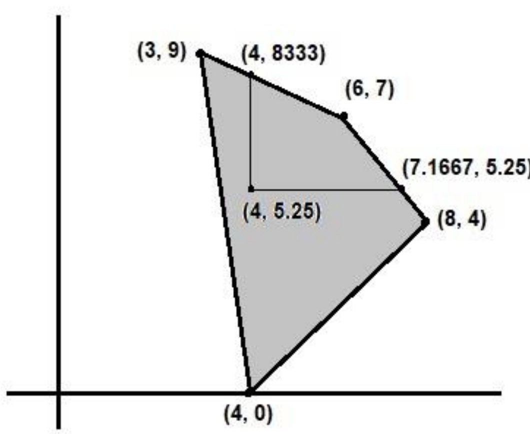
Solution: A movement diagram shows no Nash equilibrium in pure strategies, so there will not be any Nash equilibria which are Pareto efficient, and hence the game is not strictly determined. The payoff polygon is:



b) Determine the prudential strategies for each player and the corresponding security levels. Mark the point with the security levels as $(v_R, v_C) = (x_0, y_0)$ on the payoff polygon. Determine the negotiation set, including the coordinates of the two endpoints.

Solution: For the row player, the prudential strategy is $p = (0,1)$ and $v_R = 4$ because the row player's payoff matrix has a saddle point at $a_{22} = 4$.

For the column player, using oddments we find the prudential strategy is $q = \begin{pmatrix} \frac{3}{4} \\ \frac{1}{4} \end{pmatrix}$ and $v_C = \frac{63}{12} = \frac{21}{4} = 5.25$. Thus, the security levels point is $(v_R, v_C) = (4, 5.25)$. Extending horizontally and vertically from this point, we determine the negotiation set:



The negotiation set consists of points along the two upper edges (which have negative slopes). The line connecting $(3,9)$ to $(6,7)$ has equation $y = 11 - \frac{2}{3}x$, and the negotiation set includes those points between $\left(4, \frac{25}{3}\right) \approx (4, 8.333)$ and $(6,7)$. The line connecting $(6,7)$ to $(8,4)$ has equation $y = 16 - \frac{3}{2}x$, and the negotiation set includes those points between $(6,7)$ and $\left(\frac{43}{6}, \frac{21}{4}\right) \approx (7.16667, 5.25)$. In summary, the negotiation set is the union of those two line segments.

$$\text{Negotiation set: } \left\{ (x, y) \;\middle|\; \begin{array}{c} 4 \le x \le 6 \quad and\; y = 11 - \frac{2}{3}x \\ 6 \le x \le 7.1667 \;\; and\; y = 16 - \frac{3}{2}x \end{array} \right\}$$

c) Find the coordinates of the Nash point for the negotiated game.

Solution: The Nash point could be on either one of the two segments that make up the negotiation set. If it is on the segment with $4 \le x \le 6$, we must

$$\text{Maximize } P = (x - 4)(y - 5.25)$$
$$\text{Subject to}$$
$$4 \le x \le 6 \quad and\; y = 11 - \frac{2}{3}x$$

Therefore

$$P = (x-4)\left(11 - \frac{2}{3}x - 5.25\right) = (x-4)\left(5.75 - \frac{2}{3}x\right)$$
$$P = -\frac{2}{3}x^2 + \frac{101}{12}x - 23$$
$$= -\frac{2}{3}\left(x^2 - \frac{101}{8}x\right) - 23$$
$$= -\frac{2}{3}\left(x^2 - \frac{101}{8}x + \frac{10{,}201}{256}\right) + \frac{10{,}201}{384} - 23$$
$$P = -\frac{2}{3}\left(x - \frac{101}{16}\right)^2 + \frac{1369}{384}$$

This is a downward opening parabola, with vertex at $x = \frac{101}{16} \approx 6.3125$. However, this is outside the interval $4 \le x \le 6$, so it follows that the maximum value of P over this interval occurs at the endpoint closest to the vertex – that is, when $x = 6$.

If the Nash point is on the other segment, where $6 \le x \le \frac{43}{6} \approx 7.1667$, we must maximize the same product, but subject to the constraint $y = 16 - \frac{3}{2}x$. Again, we complete the square to obtain a downward opening parabola in standard form:

$$
\begin{aligned}
P = (x-4)(y-5.25) &= (x-4)\left(16-\frac{3}{2}x-5.25\right)\\
&= (x-4)\left(10.75-\frac{3}{2}x\right)\\
P &= -\frac{3}{2}x^2+16.75x-43\\
&= -\frac{3}{2}\left(x^2-\frac{67}{6}x\right)-43\\
&= -\frac{3}{2}\left(x-\frac{67}{12}\right)^2+\frac{13{,}467}{288}-43\\
P &= -\frac{3}{2}\left(x-\frac{67}{12}\right)^2+\frac{361}{96}
\end{aligned}
$$

The vertex is located at $x=\frac{67}{12}\approx 5.5833$. However, this is outside the interval $6\leq x\leq\frac{43}{6}$, so, again, the maximum occurs at the endpoint closest to the vertex, which is again $x=6$. It follows that when $x=6$, the value of P is maximized over both segments of the negotiation set, so therefore $(6,7)$ is the Nash point.

11. The following payoff matrix represents a game against Nature, where you are the row player.

$$
\begin{bmatrix} 1 & 2 & 3 & 0 & 1\\ 2 & 1 & 2 & 1 & 1\\ 7 & 0 & 0 & 2 & 1\\ 0 & 2 & 0 & 0 & 2 \end{bmatrix}
$$

a) Suppose you know that Nature is using the mixed strategy $q=\begin{pmatrix} .1\\ .3\\ .2\\ .3\\ .1 \end{pmatrix}$. What row should you choose?

Solution: By the expected value principle, we chose the row with the highest expected payoff:

$$
\begin{bmatrix} 1 & 2 & 3 & 0 & 1\\ 2 & 1 & 2 & 1 & 1\\ 7 & 0 & 0 & 2 & 1\\ 0 & 2 & 0 & 0 & 2 \end{bmatrix}\begin{pmatrix} .1\\ .3\\ .2\\ .3\\ .1 \end{pmatrix}=\begin{pmatrix} 1.4\\ 1.3\\ 1.4\\ 0.8 \end{pmatrix}
$$

So, either the first or the third row would be the best choice.

b) Suppose you know that Nature is using $q = \begin{pmatrix} 0 \\ .3 \\ 0 \\ .1 \\ .6 \end{pmatrix}$. What row should you choose?

Solution:

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 1 \\ 2 & 1 & 2 & 1 & 1 \\ 7 & 0 & 0 & 2 & 1 \\ 0 & 2 & 0 & 0 & 2 \end{bmatrix} \begin{pmatrix} 0 \\ .3 \\ 0 \\ .1 \\ .6 \end{pmatrix} = \begin{pmatrix} 1.2 \\ 1.0 \\ 0.8 \\ 1.8 \end{pmatrix}$$

The fourth row gives the highest payoff, so should be selected.

12. For the payoff matrix in Exercise 11, what row is recommended by:

a) The Laplace method?

Solution: The third row has the highest row sum (and therefore the highest average payoff.)

b) The Wald method?

Solution: The second row has the highest row minimum (best of the worst-case scenarios), so choose that row.

c) The lazy method.

Solution: The lazy method (which is not being proposed as a serious method! – it is just used for illustrative purposes) says always choose the first row.

d) The Hurwicz method (see Chapter 10 of Straffin, (1993).) This method is similar to Wald's method, but instead of looking at just the minimum payoff in each row, we take a weighted average of the minimum and the maximum in each row as follows. Choose a weight α between 0 and 1. For each row, compute the weighted average:

$$\alpha(row\ minimum) + (1 - \alpha)(row\ maximum)$$

Then choose the row with the highest weighted average. Notice that if $\alpha = 1$, this reduces to Wald's method, looking only at the worst that can happen in each row. At the other extreme, when $\alpha = 0$, you completely ignore the worst case and look only at the best-case scenario in each row. You are free to choose whatever value of α you want, but the closer to 0 your choice is, the more you are weighing the best that can happen, so this is a more optimistic outlook. That's why α is called the *coefficient of optimism*.

Determine which row this method recommends for the matrix in Exercise 11 if $\alpha = .9$. What about $\alpha = .7$?

Solution: The relevant calculations are in the table:

$\alpha(row\ minimum) + (1-\alpha)(row\ maximum)$	$\alpha = .9$	$\alpha = .7$
First row	$.9(0) + .1(3) = .3$	$.7(0) + .3(3) = .9$
Second Row	$.9(1) + .1(2) = 1.1$	$.7(1) + .3(2) = 1.3$
Third Row	$.9(0) + .1(7) = .7$	$.7(0) + .3(7) = 2.1$
Fourth Row	$.9(0) + .1(2) = .2$	$.7(0) + .3(2) = .6$

Thus, with $\alpha = .9$, choose the second row, while the more optimistic $\alpha = .7$ suggests choosing the third row.

13. Don Condo, a NASA scientist, is studying a newly discovered planet which has been named D'Rhonda. Planet D'Rhonda is close enough to Earth to make it feasible to send probes and rockets there if we want to. The probes could be robotic, or could be a human exploratory mission to set up a small scientific base there, or even a larger mission to set up a sustainable permanent human colony there, if the conditions are right. Of course, we could also be content to study D'Rhonda from a distance using telescopes and other remote instruments. The benefits to humankind depend on what we find there. D'Rhonda could be: (A) a barren planet with no water, no useful minerals, and no life, or (B) barren of water and life, but rich in minerals, or (C) a 'Terran' (Earthlike) planet with water, minerals, and possibly life, or (D) populated by an advanced civilization of benevolent aliens, or (E) populated by a race of hostile aliens. When NASA weighed the costs and possible benefits of each option, they came up with the following table of payoffs to humankind (don't worry about the units):

	A	B	C	D	E
Study remotely w/telescopes only	2	1	0	0	3
Send robotic probe	1	2	3	4	1
Human exploratory scientific base	0	1	5	7	-3
Human colony	-2	0	6	10	-5

Musical Homage to: Donald Fagen (of Steely Dan), who on his 2012 solo album Sunken Condos, released the song 'Planet D'Rhonda'. (Website: https://en.wikipedia.org/wiki/Donald_Fagen)

a) Suppose NASA estimates the probabilities of each of the five cases $A-E$ as

$$q = \begin{pmatrix} .4 \\ .55 \\ .03 \\ .01 \\ .01 \end{pmatrix}$$

If NASA is correct, what form of exploration should we undertake?

Solution: We compute the expected payoffs in each case:

$$Aq = \begin{bmatrix} 2 & 1 & 0 & 0 & 3 \\ 1 & 2 & 3 & 4 & 1 \\ 0 & 1 & 5 & 7 & -3 \\ -2 & 0 & 6 & 10 & -5 \end{bmatrix} \begin{pmatrix} .4 \\ .55 \\ .03 \\ .01 \\ .01 \end{pmatrix} = \begin{pmatrix} 1.38 \\ 1.64 \\ 0.74 \\ -0.57 \end{pmatrix}$$

If NASA is correct about their assessment of q, then we get the best expected payoff from row 2 the 1.64 that results form sending a robotic probe.

b) Suppose NASA has no idea what the probabilities of $A-E$ are. What form of exploration should we undertake according to Laplace?

Solution: The highest row sum (which implies the highest average payoff) is row two – send a robotic probe.

c) Same question for Wald's method.

Solution: The row with the highest minimum is again the second row – send a robotic probe.

d) Same question for Hurwicz method (see Exercise 12d) with $\alpha = .4$

Solution:

Study remotely w/telescopes only	$.4(0) + .6(3) = 1.8$
Send robotic probe	$.4(1) + .6(4) = 2.8$
Human exploratory scientific base	$.4(-3) + .6(7) = 3.0$
Human colony	$.4(-5) + .6(10) = 4.0$

Thus, the recommendation is for a human colony.

14. Consider the following hypothetical game of politics, which we model as a sequential game. The US President is a member of one party, say the "Coffee" Party, which controls the Senate. The House of Representatives is controlled by the opposite party, say the "Tea" Party. The President recently signed a Health Care bill, which is now federal law. Inexplicably, the Tea Party is angry about this law and would like to repeal it.

Currently, the federal government need to pass a budget for next year, and the budget contains money for social assistance programs such as Food Stamps and other programs which the Tea Party is against. The Tea Party's main goal is to repeal the Health Care law, and their secondary goal is to lower the budget allocations for food stamps and other programs.

The budget is now being considered by the House of Representatives. The Tea Party has three options. They can pass the budget as it is written, which means they get neither of their desires. This would be considered the end of the game (in a clear victory for the Coffee Party.) The second choice for the Tea Party is to compromise and accept the Health Care law, but try to negotiate a reduced budget for Food Stamps and other social programs. If they do this, the Coffee Party will agree to a slight reduction in the budget in order to get it passed. The Coffee Party would regard this as a victory even with the reduced budget, so this choice would also end the game. The third choice for the Tea Party is to take a hard line and demand that the Health Care law be repealed before they negotiate anything in the budget proposal. IF they choose this option, the game does not end, and the Coffee Party must respond.

The Coffee Party has two choices. The first is to give in to the Tea Party demands and repeal the Health Care Law in order to pass the budget. The Tea Party would regard this as a big victory since they got their highest priority, even if they ended up passing the Food Stamp budget. The other choice for the Coffee Party is to resist the Tea Party demands. That is, they will simply ignore the demands. After all, the Health Care law is already a law, so the time is past for negotiation about it. As federal law, everyone is required to follow it, and to demand to repeal it before negotiating a budget pus the Tea Party on shaky legal ground, plus it makes them look like impetuous bullies just trying to force their will in a battle they already lost, which is bad for their image and will probably lead to lost seats in the next election. Thus, if the Coffee Party resists, the ball is back in the court of the Tea Party.

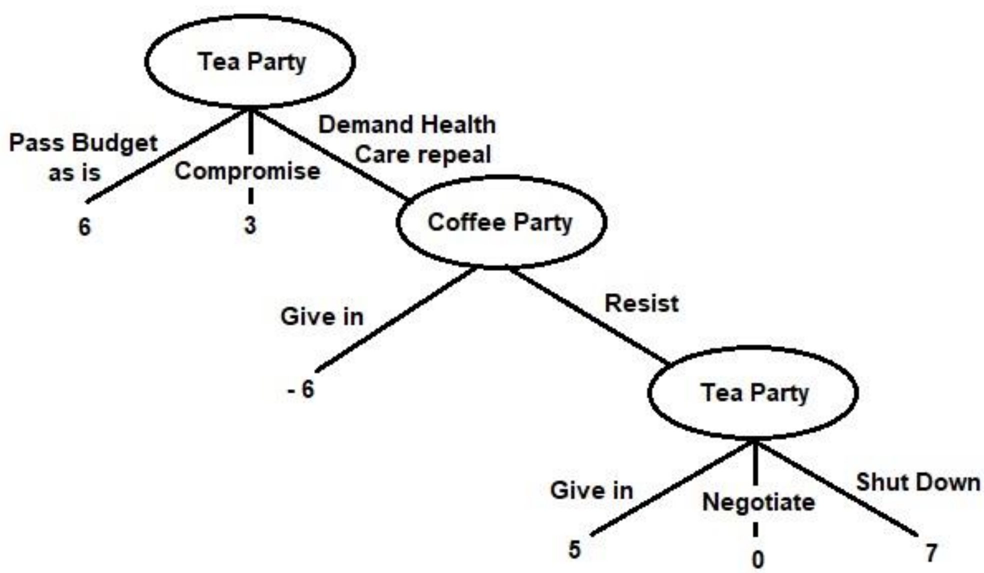
Now the Tea Party has three options. First, they can give up their demand to repeal the Health Care law. We call this option "give in", and it will end the game. A second option is to negotiate with the Coffee Party. If they do so, the Law will remain intact, but the Coffee Party will compromise and give the Tea Party a reduction in the Food Stamps budget (probably a larger reduction than if the Tea Party hadn't made their demand in the first place.) Their third option is to stick to their demand and refuse to do anything. In this case, the government shuts down because no budget is passed. Although the tree diagram might conceivably continue beyond this point, we'll assume that if the government shuts down, the game ends in order to keep the tree small.

In reality, this is not a zero-sum game. For example, if the government shuts down, everyone is hurt (including the Tea Party) due to the detrimental effect this would have on the economy. However, we would like to model this as a zero-sum game, and we do this by considering the payoff to be the number of congressional seats gained by either party in the next mid-term elections. If one party gains 3 seats, the other must lose 3 seats, so with this interpretation it is a zero-sum game.

According to studies of which both sides are aware, the payoffs in the next election are very likely to be as follows: If the Tea Party passes the budget as is, they lose 6 seats to the Coffee Party. If they compromise and negotiate a reduced Food Stamp budget, they lose 3 seats to the Coffee Party. If they demand a repeal to the Health Care law, and the Coffee Party gives in to this demand, the Coffee Party will lose 6 seats to the Tea Party. If the Coffee Party resists those demands, and the Tea Party gives in, the Tea Party will lose 5 seats to the Coffee Party. If the Coffee Party resists and the Tea Party then negotiates, then neither party loses seats to the other. Finally, f the government shuts down, the Tea Party will lose 7 seats to the Coffee Party.

a) Draw the game tree for this game, and label each terminal node with the payoff to the Coffee Party.

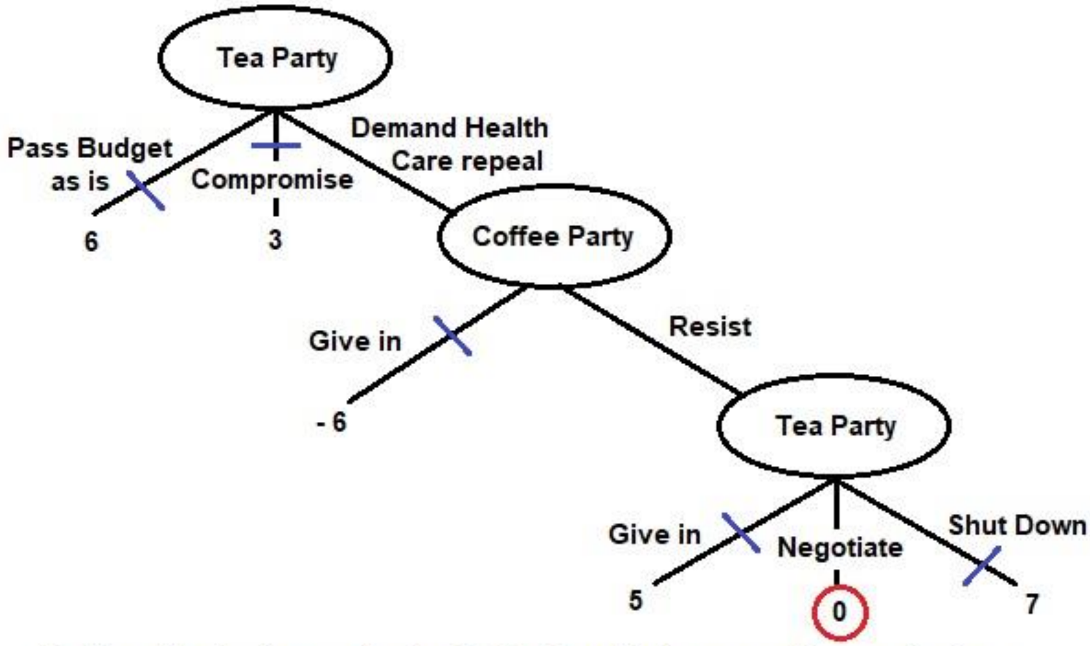
Solution: The game in extended form:



Projected gains in seats by the Coffee Party in the next mid-term elections

b) Notice this game has perfect information. Analyze the game by reverse induction, and determine the outcome.

Solution: The appropriate branches crossed off and the outcome circled:



Projected gains in seats by the Coffee Party in the next mid-term elections

The prediction is that the Tea Party demands the repeal of the Health Care law at first, the Coffee Party resists, then the Tea Party negotiates. The net change in congressional seats is 0.

c) Convert the game to normal form with the Coffee Party as the row player, and solve the game by considering the payoff matrix. (Your answer should agree with part (b).)

Solution: The Coffee Party has only two strategies:
G – Give in to Tea Parry demands to repeal the Health Care law
R – Resist Tea Party demands to repeal the Health Care law

The Tea Party has five strategies:

P – Pass budget proposal as is currently written
C – Compromise
DG – Demand repeal of health care; Give in if the Coffee Party resists
DN – Demand repeal of health care; Negotiate if Coffee Party resists
DS – Demand repeal of health care; shut down government of Coffee Party resists

This leads to the 2×5 payoff matrix:

$$\begin{array}{ll} & \text{Tea Party} \\ & \begin{matrix} P & C & DG & DN & DS \end{matrix} \\ \text{Coffee Party} \begin{matrix} G \\ R \end{matrix} & \begin{bmatrix} 6 & 3 & -6 & -6 & -6 \\ 6 & 3 & 5 & 0 & 7 \end{bmatrix} \end{array}$$

Clearly, R is dominant for the Coffee Party. Knowing this, the Tea Party should demand the repeal at first, but then negotiate (DN) to obtain the highest payoff (for them) of 0 seats gained by the Coffee Party. This agrees with part (b).

d) The conclusions of parts (b) and (c) are based on assuming both players know all the payoffs ('perfect information' implies this.) But the payoffs are in the future and perhaps difficult to predict. Suppose that the Tea Party thought that they would gain seats if the government shut down, for example they thought they would gain 5 seats, instead of losing 7 seats. Then the two players have different information - another basic assumption about game theory we can vary for sensitivity analysis. In this case, the Coffee Party has the correct information that they will gain 7 seats if the government shuts down, while the Tea Party erroneously thinks they will gain 5 seats in this case. How would that change the outcome of the game?

Solution: Since the Coffee Party has the correct information, they are using the correct payoff matrix from part (c) So they will choose their dominant strategy to resist the Tea Party demand to repeal health care. The Tea Party has the wrong information, so they are using a payoff matrix that looks the same as that in part (c), except $a_{25} = -5$. Since R is still dominant, the Tea Party (correctly) surmises that the Coffee Party will resist their demands. However, then they choose the shutdown instead of negotiating, thinking they will gain 5 seats. Or just note that $a_{25} = -5$ is a saddle point of the Tea Party's view of the matrix. If the Coffee Party really has the correct information, the outcome is there is a government shutdown, after which the Tea Party loses 7 seats.

Section 8.2 Ordinal Games

8.2.1 Dominance and Nash Equilibrium Points

No music references in the text of exercises in this subsection.

8.2.2 Prisoners' Dilemma and other Dilemmas

Music Reference in the text: The example used to illustrate the prisoners' dilemma pays homage to the new wave band the Talking Heads. Chris Frantz and Tina Weymouth are the drummer and bass player for the band, while the DA's name is an amalgamation of the names of their keyboardist Jerry Harrison and their vocalist David Byrne. The example alludes to their 1986 album True Stories, as well as to songs 'Burning Down the House' from the 1983 album Speaking in Tongues, and 'Puzzling Evidence' from the album True Stories. (Website: http://talking-heads.nl/)

8.2.3 Some Applications

Music References in the text: The example using grade inflation to illustrate the Prisoner's dilemma alludes to rock musician Alice Cooper (who had a 1972 hit with the song 'School's Out'), and also to folk-rock singer Graham Nash (who penned the 1969 song 'Teach Your Children' from Crosby, Stills, Nash, & Young.) (Websites: https://alicecooper.com/ and https://en.wikipedia.org/wiki/Crosby,_Stills,_Nash_&_Young)

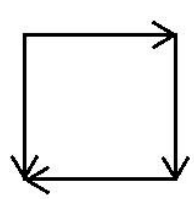
Solutions to exercises and more music references:

For Exercises 1-8, for each ordinal payoff matrix, draw the movement diagram, find all dominant strategies (if any), and all Nash equilibria (if any.) If possible, predict the outcome if the game is played as usual with the player choosing simultaneously and independently. State if the game is strictly determined.

1.

$$\begin{bmatrix} (1,2) & (2,3) \\ (3,4) & (4,1) \end{bmatrix}$$

Solution: The movement diagram:

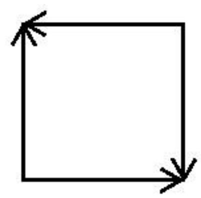


The second row is dominant, and there is a Nash equilibrium at $(3,4)$. It is the only Pareto efficient Nash equilibrium, so the game is strictly determined and the predicted outcome is $(3,4)$.

2.

$$\begin{bmatrix} (2,3) & (3,2) \\ (1,1) & (4,4) \end{bmatrix}$$

Solution: The movement diagram:

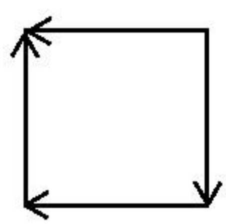


There are no dominant strategies for either player. There are two Nash equilibria, at $(2,3)$ and $(4,4)$. Only the one at $(4,4)$ is Pareto efficient, so the game is strictly determined. Each player tries for their favorite Nash equilibrium, so the result is they both play their second strategies and the outcome is $(4,4)$, the best possible outcome for the individual players as well as for the group.

3.

$$\begin{bmatrix} (2,3) & (3,2) \\ (1,4) & (4,1) \end{bmatrix}$$

Solution: The movement diagram:

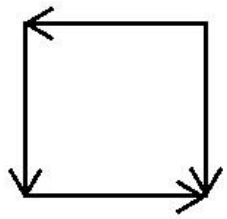


The first column dominates the second, and neither row dominates the other. The outcome at $(2,3)$ is a Pareto efficient Nash equilibrium, and the only one, so the game is strictly determined. The predicted outcome is $(2,3)$,

4.

$$\begin{bmatrix} (2,4) & (1,3) \\ (3,1) & (4,2) \end{bmatrix}$$

Solution: The movement diagram:

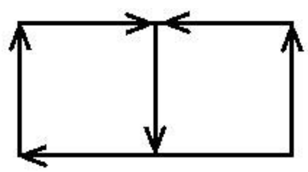


The second row is dominant, and neither column dominates the other. The $(4,2)$ payoff occurs at a Nash equilibrium, which is the only Pareto efficient one, so the game is strictly determined. The predicted outcome is $(4,2)$.

5.

$$\begin{bmatrix} (6,3) & (1,6) & (4,1) \\ (2,5) & (5,4) & (3,2) \end{bmatrix}$$

Solution: The movement diagram:

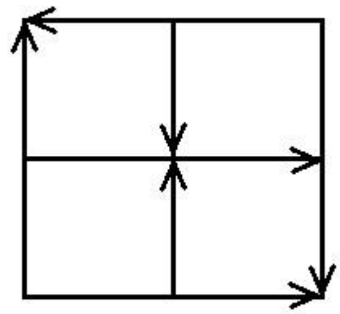


There are no dominant rows. The second column dominates the third one. However, there are no Nash equilibria. The game is not strictly determined. It is not clear what the outcome should be. One might guess that after crossing out the dominated third column, both players might choose their second strategy, resulting in the Pareto efficient outcome with payoff $(5,4)$. Both players seem to do fairly well with this outcome. However, it is not a stable outcome – the column player can improve his situation by unilaterally switching to the first column. But then, the row player can unilaterally switch away from $(2,5)$ to improve her payoff, etc., as we can see running around the left square of the movement diagram. It is not clear what would happen – especially under repeated play.

6.

$$\begin{bmatrix} (8,3) & (3,2) & (5,1) \\ (4,7) & (9,8) & (2,9) \\ (6,4) & (1,5) & (7,6) \end{bmatrix}$$

Solution: The movement diagram:

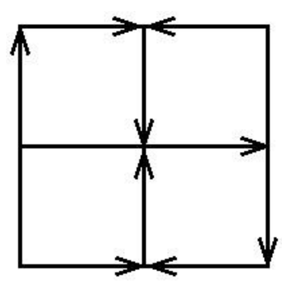


Clearly, there are no dominant strategies for either player. There are two Nash equilibria: one at $(8,3)$ and one at $(7,6)$. However, neither one is Pareto efficient, since $(9,8)$ is better for both players at either equilibrium point. It is not clear what the rational outcome is. One might guess that if each player tries for their preferred Nash equilibrium, then the row player chooses the first row, and the column player chooses the third column, in which case the outcome would be $(5,1)$ which is neither an equilibrium point nor is it Pareto efficient. While this might happen for a game played once, it would hardly last under repeated play.

7.

$$\begin{bmatrix} (8,3) & (3,7) & (5,5) \\ (4,6) & (9,2) & (2,9) \\ (6,1) & (1,8) & (7,4) \end{bmatrix}$$

Solution: The movement diagram:

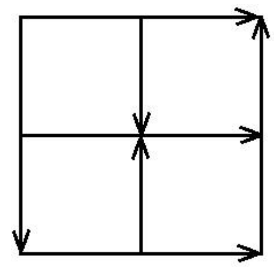


The third column dominates the first., but none of the rows dominates any other row. There are no Nash equilibria, so the game is not strictly determined. Again, it is not clear what the rational outcome would be, even after deleting the dominated column.

8.

$$\begin{bmatrix} (1,3) & (4,2) & (8,6) \\ (6,1) & (7,5) & (3,7) \\ (9,8) & (2,4) & (5,9) \end{bmatrix}$$

Solution: The movement diagram:



The third column dominates both of the other columns. No row dominates any other, but by higher order dominance, we arrive at the Nash equilibrium at the outcome with payoff $(8,6)$. Since this is the only equilibrium, it seems the most likely outcome of the game. However, note that since $(9,8)$ is better for both players, the Nash equilibrium is not Pareto efficient, and the game is not strictly determined.

9. The following story illustrates the *Battle of the Sexes dilemma*. James and Carly have plans to meet after work for a date. James wants to go to see his favorite football team, the Steamrollers, in their final game of the season, while Carly would prefer to go to the cinema to see the film 'Anticpation'. Of course, they both have a strong preference to be together rather than alone, and they left for work this morning without finalizing their plans. Carly's phone battery has died, so they cannot communicate before going to either event, both of which start at 6 PM – leaving just enough time to travel to each location from work. Model this as a 2×2 ordinal game, and show that the payoff matrix is indeed the Battle of the Sexes dilemma. If each person tries for their preferred Nash equilibrium, what is the predicted outcome of the game?

Music References: This exercise alludes to singer-songwriters James Taylor and Carly Simon. 'Steamroller Blues' is a song from Taylor's 1970 album Sweet Baby James, and 'Anticipation' is the title track from Simon's 1971 album. (Websites: https://www.jamestaylor.com/ and https://www.carlysimon.com/) (Note that in the text, the title of the film in this exercise is different than here – 'So Far Away' was the title of a song by another singer-songwriter Carole King, who wrote Taylor's first hit song 'You've Got a Friend'. The title will be corrected in the next edition.)

Solution: Each player must choose from two possible locations to go (football stadium or movie theater). For each player, think of the option of going to their own preferred activity as the 'defection' strategy and the going to the other person's preferred activity as the 'cooperation' strategy. Then the payoff matrix is:

$$\begin{array}{llcc} & & \multicolumn{2}{c}{\text{James}} \\ & & C\ (cinema) & D\ (football\ game) \\ \text{Carly} & \begin{array}{c} C\ (football\ game) \\ D\ (cinema) \end{array} & \multicolumn{2}{c}{\begin{bmatrix} (1,1) & (3,4) \\ (4,3) & (2,2) \end{bmatrix}} \end{array}$$

Indeed, each of them have a strong preference to be together, so the two outcomes where they both go to the same activity are their top two preferences. But James prefers the game to the movie, so if they both end up at the game, it would be James' first choice but Carly's second, explaining the $(3,4)$ payoff, and by symmetry, the $(4,3)$ payoff if they both end up at the cinema. The other two options, where they end up at different locations so miss each other, are their two least preferred outcomes. Clearly, it's even worse to miss each other AND end up at their least favorite activity, explaining the $(1,1)$ payoff if Carly goes to the football game and James goes to the cinema; that is, if they both cooperate. Almost as bad is going to your favorite activity but without your date, explaining the $(2,2)$ payoff if James goes to the game alone and Carly goes to the cinema alone; that is, if they both defect.

This is exactly the payoff matrix discussed in the text. It's a dilemma because, as noted in the text, if each player tries for their preferred Nash equilibrium, then each will defect (even though defection is not a dominant strategy.) The result is they end up at the $(2,2)$ payoff, which is neither a Nash equilibrium nor is it Pareto efficient.

10. The following story illustrates the *Follow the Leader dilemma*. Bruce and Clarence simultaneously pull up to stop signs across from each other on a busy intersection of Tenth Avenue. When a gap finally appears in the traffic, each driver has a choice to either drive directly into the gap, or else to concede the right of way to the other driver. If they both drive, they could crash, which would be the last preference of both drivers. If they both concede, then each driver is delayed and they may both lose the opportunity to drive into the gap before it vanishes, so that is the second last preference for both drivers (a "Tenth Avenue Freeze-Up".) However, if one driver goes and the other concedes, the one that goes gets his first preference, and there may still be time for the one who concedes to follow the leader into the gap before it vanishes. Model this situation as a 2×2 ordinal game and show that the payoff matrix is indeed the Follow the Leader dilemma. If each driver tries for their preferred Nash equilibrium, what is the predicted outcome?

Musical homage to: Bruce Springsteen, and his saxophone player from the E-Street Band, Clarence Clemons. Their 1975 album Born to Run contains a song called 'Tenth Avenue Freeze-Out' (Website: https://brucespringsteen.net/)

Solution: We consider the 'drive into the gap' strategy to be defection and the 'concede the right of way' strategy to be cooperation. Then the payoff matrix is:

$$\begin{array}{ll} & \text{Clarence} \\ & \begin{array}{cc} C\ (concede) & D\ (drive) \end{array} \\ \text{Bruce} \quad \begin{array}{r} C\ (concede) \\ D\ (drive) \end{array} & \begin{bmatrix} (2,2) & (3,4) \\ (4,3) & (1,1) \end{bmatrix} \end{array}$$

Indeed, each driver gets their most preferred outcome by driving first while the other driver waits (and obtains their second most preferred outcome), explaining the $(4,3)$ and $(3,4)$ payoffs. If they both drive at once, they crash, leading to the worst outcome $(1,1)$. If they both concede, they both might miss the gap, leading to their third preferred outcome, explaining the (2,2) payoff. The payoff matrix is exactly the one discussed in the text. It is a dilemma because if each player tries for their preferred Nash equilibrium, they will both defect (even though it is not a dominant strategy.) The result is the $(1,1)$ payoff and a possible crash.

11. Consider the following situation. Mr. Browne's car is in need of service, and his local garage down on Main Street has only two openings - a Friday appointment and a Saturday appointment, after which it is several weeks before they have another opening. Thus, Mr. Browne needs to make an appointment for one of those two days. Meanwhile, Mrs. Browne has an eye infection, and her eye doctor has openings also only on the same two days, Friday and Saturday. We suppose that each person will make their own appointment, but are not able to communicate with each other ahead of time about their choices, and we assume they make their calls to schedule the appointments simultaneously. They both need the car to travel to their appointments. The purpose of this exercise is to set up the situation as an ordinal game under various assumptions about the preferences of the players, and in each case we assume that each person is aware of the preferences of the other person.

Music references to: singer-songwriters Jackson Browne and Bob Seger. 'Doctor, My Eyes' was a song from Brone's eponymous 1972 debut album, and 'Down on Main Street' is a song from Seger's 1976 album Night Moves. (Websites: https://shop.jacksonbrowne.com/ and https://www.bobseger.com/)

a) Assume that both players' first preference is for the appointments to be on different days because it would be difficult (but not impossible) to work things out if they are the same day. Given that, each person has a minor preference for Friday over Saturday; Mr. Browne because the garage charges more for work done over the weekend, and Mrs. Browne because she is anxious to treat her infection as soon as possible. Write the payoff matrix as an ordinal game. If the matrix happens to be one of the four dilemmas we have studied, state which one. If the game is strictly determined, say what the predicted outcome is.

Solution: According to the stated preferences, the payoff matrix is

		Mr. Browne	
		$C\ (Saturday)$	$D\ (Friday)$
Mrs. Browne	$C\ (Saturday)$	$(1,1)$	$(3,4)$
	$D\ (Friday)$	$(4,3)$	$(2,2)$

This game is not strictly determined, but it is the same dilemma as the Battle of the Sexes.

b) Suppose that each person's highest priority is to have a Friday appointment (for the reasons given in part a), and the secondary preference for both players is to schedule the appointment on different days. Same directions as part a.

Solution: According to the preferences the payoff matrix is

		Mr. Browne	
		$C\ (Saturday)$	$D\ (Friday)$
Mrs. Browne	$C\ (Saturday)$	$(1,1)$	$(2,4)$
	$D\ (Friday)$	$(4,2)$	$(3,3)$

In this case, the game is not a dilemma – it is strictly determined. The D strategy is dominant for both players, and the intersection of those two strategies at the outcome with payoff $(3,3)$ is the unique Pareto efficient Nash equilibrium. So, the most likely outcome is that they both book Friday appointments.

c) Suppose that Mr. Browne most prefers a Saturday appointment so he will not have to leave work early on Friday, while Mrs. Browne most prefers a Friday appointment. For each of them, a secondary preference is to schedule the appointments on different days. Same directions as part a.

Solution: The payoff matrix is

		Mr. Browne	
		$C\ (Saturday)$	$D\ (Friday)$
Mrs. Browne	$C\ (Saturday)$	$(2,2)$	$(1,3)$
	$D\ (Friday)$	$(3,1)$	$(4,4)$

In this case, the game is not a dilemma – it is strictly determined. The D strategy is dominant for both players, and the intersection of those two strategies at the outcome with payoff $(4,4)$ is the unique Pareto efficient Nash equilibrium. It's also the best possible outcome for both players!

d) Suppose that Mr. Browne most prefers a Saturday appointment and Mrs. Browne most prefers a Friday appointment, but that for each of them, a secondary preference is to schedule the appointments on the same day so they can do all their errands at once. Same directions as part a.

Solution: The payoff matrix is

$$\begin{array}{llcc} & & \multicolumn{2}{c}{\text{Mr. Browne}} \\ & & C\ (Saturday) & D\ (Friday) \\ \text{Mrs.} & C\ (Saturday) & \left[(1,1) \right. & \left. (2,4) \right] \\ \text{Browne} & D\ (Friday) & \left[(4,2) \right. & \left. (3,3) \right] \end{array}$$

This case is just like part b. It's not a dilemma, but is strictly determined, with the Nash equilibrium at $(3,3)$ as the predicted outcome.

e) Suppose that for Mr. Browne, his main preference is for a Saturday appointment, and his secondary preference is to have appointments on different days. Mrs. Browne most prefers to have the appointment on the same day (so Mr. Browne can drive her and she can avoid driving with her compromised eye), and has a secondary preference for a Friday appointment since it is less time to wait to treat her infection. Same directions as part a.

Solution: The payoff matrix is

$$\begin{array}{llcc} & & \multicolumn{2}{c}{\text{Mr. Browne}} \\ & & C\ (Saturday) & D\ (Friday) \\ \text{Mrs.} & C\ (Saturday) & \left[(1,2) \right. & \left. (3,3) \right] \\ \text{Browne} & D\ (Friday) & \left[(4,1) \right. & \left. (2,4) \right] \end{array}$$

In this case, Mrs. Browne does not have a dominant strategy, but D is dominant for Mr. Brown. Knowing that, by higher order dominance Mrs. Browne will choose C so they can both go the same day. The outcome at $(3,3)$ is the unique Pareto efficient Nash equilibrium, so this game is also strictly determined.

f) Both players first preference is for the appointments to be on different days rather than the same day. Both players feel their secondary preference is for a Friday appointment if they go on different days, but Saturday if they go on the same day. Same directions as part a.

Solution: The payoff matrix is

$$\begin{array}{llcc} & & \multicolumn{2}{c}{\text{Mr. Browne}} \\ & & C\ (Saturday) & D\ (Friday) \\ \text{Mrs.} & C\ (Saturday) & \left[(2,2) \right. & \left. (3,4) \right] \\ \text{Browne} & D\ (Friday) & \left[(4,3) \right. & \left. (1,1) \right] \end{array}$$

This time, we have a dilemma – the payoff matrix is just like the Follow the Leader dilemma.

12. Angie and her sister Kiki are selecting colleges. They each have been accepted at a prestigious school in Boston, and at one in New York City. All they need to do is decide which school to attend, which, for the purposes of this exercise, we assume they do simultaneously and independently without talking with each other. After spending all that time together during their childhood operating their famous lemonade stand, they each feel a strong preference to spend some time apart and attend different schools. However, they each have a secondary preference to live in New York City over living in Boston. Model this as an ordinal game. Which of the dilemmas is the payoff matrix?

Solution: The payoff matrix is

		Kiki	
		$C\ (Boston)$	$D\ (New\ York)$
Angie	$C\ (Boston)$	$(1,1)$	$(3,4)$
	$D\ (New\ York)$	$(4,3)$	$(2,2)$

This is just like the Battle of the Sexes dilemma.

13. Show that, in Exercise 12, if the strength of the girls' preferences is reversed, so that their highest preference is to live in New York and their secondary preference is to attend different schools, then the game becomes strictly determined. What is the predicted outcome?

Solution: The payoff matrix is

		Kiki	
		$C\ (Boston)$	$D\ (New\ York)$
Angie	$C\ (Boston)$	$(1,1)$	$(2,4)$
	$D\ (New\ York)$	$(4,2)$	$(3,3)$

This time, the D strategy is mutually dominant, so the outcome with payoff $(3,3)$ is the unique Pareto efficient Nash equilibrium, so the game is strictly determined. The predicted outcome is that they both attend the school in New York.

14. Suppose that Angie and Kiki have been accepted at three schools: Harvard, Yale, and Princeton. Angie ranks them in exactly that order, while Kiki ranks them in the reverse order – Princeton, Yale, then Harvard. Suppose each girl's first priority is to go to a school they rank as high as possible, but they have a secondary priority to be located as close as they can be to each other (so attending the same school is ideal form this perspective.) For the purposes of deciding the preferences of the outcomes, note that Princeton and Harvard are quite far apart. Yale is closer to both, but somewhat closer to Harvard than it is to Princeton. Model this situation as a 3×3 ordinal game. Draw the movement diagram, find all Nash equilibria, and state if the game is strictly determined. What is the predicted outcome?

Solution: The payoff matrix is

$$\begin{array}{cc} & \text{Kiki} \\ & \begin{array}{ccc} H & Y & P \end{array} \\ \text{Angie} \begin{array}{c} H \\ Y \\ P \end{array} & \begin{bmatrix} (9,3) & (8,5) & (7,7) \\ (5,2) & (6,6) & (4,8) \\ (1,1) & (2,4) & (3,9) \end{bmatrix} \end{array}$$

This is a strictly determined game. Harvard dominates both of the other schools for Angie, and Princeton dominates both of the others for Kiki. The $(7,7)$ payoff at the intersection of the two dominant strategies is the unique Pareto efficient Nash equilibrium. Thus, Angie goes to Harvard and Kiki goes to Princeton.

15. Show that, in Exercise 14, if the strength of the girls preferences is reversed (but not their rankings of the schools), so that each girl's highest priority is to be as close as possible to her sister, while each girl's second priority is to attend a school they ranked as high as possible, then the game is not strictly determined; in fact, there are three Nash equilibria which are all Pareto efficient, but no two of them are equivalent and interchangeable. How would you resolve their dilemma?

Solution: The payoff matrix is

$$\begin{array}{cc} & \text{Kiki} \\ & \begin{array}{ccc} H & Y & P \end{array} \\ \text{Angie} \begin{array}{c} H \\ Y \\ P \end{array} & \begin{bmatrix} (9,7) & (6,6) & (2,2) \\ (5,5) & (8,8) & (4,4) \\ (1,1) & (3,3) & (7,9) \end{bmatrix} \end{array}$$

In this example, there are no dominant strategies for either player. At any outcome where the girls choose different schools, at least one of them can improve their payoff by a unilateral change to the school her sister is attending. Therefore, no Nash equilibrium cold be off the main diagonal. On the other hand, the three outcomes on the main diagonal are all Pareto efficient Nash equilibria. But they are not equivalent or interchangeable so the game is not strictly determined. Therefore, the question of how to resolve their dilemma is an open question. Trying for their favorite Nash equilibrium will land them at the outcome with payoff $(2,2)$, a pretty poor result. Perhaps they can negotiate a better outcome?

16. Make up a payoff matrix for a 2×2 ordinal game where all four outcomes are Pareto efficient, and the game is strictly determined, and also make up one which is not strictly determined.

Solution: Here is one solution. Consider the payoff matrices

$$A = \begin{bmatrix} (1,4) & (2,3) \\ (3,2) & (4,1) \end{bmatrix}$$

$$B = \begin{bmatrix} (3,2) & (1,4) \\ (2,3) & (4,1) \end{bmatrix}$$

Note that the payoffs $(1,4), (4,1), (2,3)$, and $(3,2)$ are all Pareto efficient outcomes. Indeed, switching from one of these to any of the others entails one payoff going up and the other going down, so at each of the four outcomes, it is impossible to improve on both payoffs. Thus, in both games above, every outcome is Pareto efficient.
The game with payoff matrix A is strictly determined. The first column dominates the second, and the second row dominates the first. Thus, the $(3,2)$ payoff is the only Nash equilibrium, and since it is Pareto efficient, the game is strictly determined, with predicted outcome $(3,2)$.
The game with payoff matrix B has no dominance for either player. Furthermore, there is no Nash equilibrium (refer to a movement diagram, for example), so the game is not strictly determined.

17. Johnny Gunn and Peter Rivers are spies working for the CIA. They have infiltrated a hostile foreign spy organization under cover, and are gathering inside information on the group. They have become aware of a plan to ro a bank in order to obtain funds for a secret arms deal. Agent Gunn and Agent Rivers are considering exposing the plot, which would mean exposing their cover and leaving the organization to return home. Each most wants to be credited for exposing the plot themselves, since they will advance in their jobs and get a nice raise. However, if one exposes the plot, the organization will be alerted to the existence of double agents, which will put the other agent in danger. Thus, the worst outcome for each is if the other one exposes the plot. However, if they both expose the plot, they each will have a blown cover and will no longer be able to operate as secret agents within the organization, thereby cutting off the flow of information to the CIA, so each regards the outcome of neither one exposing the plot as better than both exposing it. Model the situation as an ordinal game. IS the game strictly determined or not, and if not, which dilemma is it?

Music references: This exercise alludes to two songs which were used as theme songs for television shows. The first is the 'Peter Gunn Theme', composed by Henry Mancini for the Peter Gunn TV series which ran from 1958-1961. Many other artists have recorded covers of this song, including Duane Eddy, the Blue Brothers, Joe Jackson, The Tony Levin Band, and Emerson, Lake & Palmer among many others, including a vocal version by Sarah Vaughn. The other song is 'Secret Agent Man' recorded in 1966 by Johnny Rivers for the USA broadcast of the British TV series Danger Man (known as Secret Agent in the USA.) This song has also been covered by other artists, including Mel Torme, the Ventures, Blues Traveler, and heavy metal band Cirith Ungol. (Websites: https://en.wikipedia.org/wiki/Peter_Gunn_(song) and https://en.wikipedia.org/wiki/Secret_Agent_Man.)

Solution: The payoff matrix is

$$\begin{array}{ccc} & & \text{Agent Rivers} \\ & & \begin{array}{cc} C\ (keep\ plot\ secret) & D\ (Expose\ plot) \end{array} \\ \text{Agent Gunn} & \begin{array}{c} C\ (keep\ plot\ secret) \\ D\ (Expose\ plot) \end{array} & \begin{bmatrix} (3,3) & (1,4) \\ (4,1) & (2,2) \end{bmatrix} \end{array}$$

This matrix is exactly the Prisoners' dilemma, so the game is not strictly determined.

18. Consider the Cuban Missile Crisis discussed in the text. Suppose that we try to model this as an ordinal game which is larger than 2×2.

a) Suppose the USSR stays with just two strategies – either maintain the missiles, or Withdraw them. For the US, suppose there are three possible strategies, they are a Blockade, an Air Strike, or an Invasion, in order of increasing aggression. As in the text, assume each player most prefers to be aggressive while the other player is not aggressive, since it demonstrates power and resolve, and each side least prefers any outcome that would lead to nuclear war. From this perspective, it seems that if the USSR maintains the missiles, both an Airstrike and an Invasion are equally bad, since it is probable that either outcome will lead to war. For the purposes of setting up the payoff matrix, just assume than an invasion is worse if the USSR maintains the missiles, since it is more likely to lead to war than an airstrike. In fact, it is worse in any case, since the USSR might regard an attack on Cuba as an attack against them, and this could lead to war anyway, even if the missiles are withdrawn. Is the resulting game strictly determined or a dilemma?

Solution: The payoff matrix is:

$$\begin{array}{ccc} & & \text{USSR} \\ & & \begin{array}{cc} C\ (Withdraw) & D\ (Maintain) \end{array} \\ \text{USA} & \begin{array}{c} C\ (Blockade) \\ D\ (Air\ Strike \\ V\ (Invasion) \end{array} & \begin{bmatrix} (4,5) & (3,6) \\ (5,4) & (2,2) \\ (6,3) & (1,1) \end{bmatrix} \end{array}$$

Indeed, think of V as "very aggressive". The outcomes where the USA uses $D\ or\ V$ and the USSR uses D are the worst outcomes for both sides as they are most likely to lead to war, with invasion rated worse for the USSR because they will probably lose control of Cuba, and worse for the USA because it is even ore likely that the USSR would retaliate against an invasion than against an air strike. This explains the $(1,1)$ and $(2,2)$ payoffs. For the USA, all the remaining preferences are clear – they most prefer the three outcomes when the USSR uses C, with increasing preference the more aggressive their own action is, explaining the $4,5,6$ USA payoffs in the first column, an leaving the CD outcome to be the 3 payoff by elimination. For the USSR, it is clear that their most preferred outcomes are the ones when the USA uses C, with a preference for their more aggressive action D. It remains to decide whether the USSR prefers the USA to use D or V in the case they use C (withdraw the missiles.) But we stipulated that the more aggressive action is wore from the perspective of the USSR, so the USSR preferences are as they are shown in the matrix.

In this matrix, both the $(3,6)$ and $(6,3)$ outcomes are Pareto efficient Nash equilibria, but they are not equivalent or interchangeable, so the result is a dilemma. If both players pursue their preferred Nash equilibrium, the result is $(1,1)$, the worst possible outcome. This is just like the Chicken dilemma.

b) Suppose the USSR considers three options – Withdraw the Missiles, Maintain the Missiles, or Use the Missiles (a first strike against the United States, which would lead to War for sure as the US would undoubtedly retaliate with nuclear weapons.) In this model, assume the US options are just the two options to Invade or Promise not to invade. Same directions as part a.

Solution: The payoff matrix is

		USSR		
		$C\ (Withdraw)$	$D\ (Maintain)$	$V\ (Use)$
USA	$C\ (Promise\ not\ to\ invade)$	$(5,5)$	$(4,6)$	$(1,3)$
	$D\ (Invade\ Cuba)$	$(6,3)$	$(3,2)$	$(2,1)$

Again, the preferences for the USA are pretty clear. The best two outcomes are in the first column since the USSR removes the missiles. The worst two outcomes are in the third column since the USSR attacks the US with their missiles, leaving the middle column for the intermediate preferences. Within each column, the USA prefers to use the more aggressive strategy D, except in the middle column, since in that column D will lead to war, but C may not. For the USSR, the bottom row are their three least preferred outcomes, since the US invades Cuba, while the top row contains their three most preferred outcomes since the US promises not to invade. To order their preferences within each row, in the top row, they most prefer to maintain the missiles (preferring that over removing them), and preferring to remove them than use them (which would lead to war.) It's less clear what the preferences are in the second row, since any outcome in this row is likely to escalate to nuclear war. Nevertheless, we rank them higher (for the USSR) if the outcome is less likely to lead to war or the war is more limited in scope. This leads to the rankings shown.

Accepting these rankings, it is again the case that the $(4,6)$ and $(6,3)$ outcomes are inequivalent Pareto efficient Nash equilibria, so the game is a dilemma, not strictly determined. It is still similar to chicken, with the difference being that if both sides pursue their preferred Nash equilibrium, we end up at $(3,2)$ rather than the worst possible outcome. However, one could argue that once war broke out and the USSR maintained their missiles, they would be very likely to use them, so there is virtually no difference between the $(3,2)$ and $(2,1)$ outcomes.

c) Do you think that having a larger payoff matrix brings any insights to the conflict that were not already present in the 2×2 models?

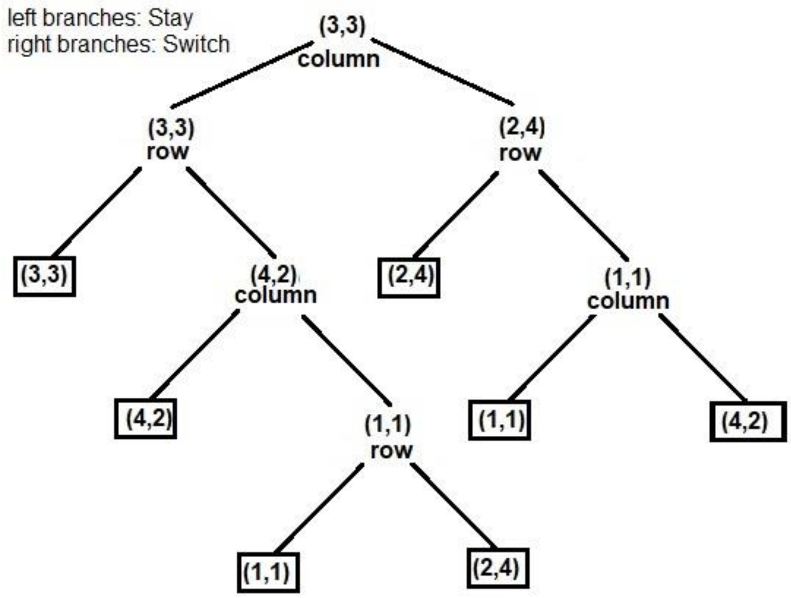
Solution: At least in these examples, in each case the result is a dilemma which is quite similar to Chicken anyway, so we'd have to say no, there weren't any new insights to analyzing the crisis that weren't already present in the 2×2 version. The 2×2 Chicken payoff matrix, while a major simplification of the actual crisis, does a remarkable job at reflecting the essence of the crisis.

8.3 Sequential Games: The Theory of Moves

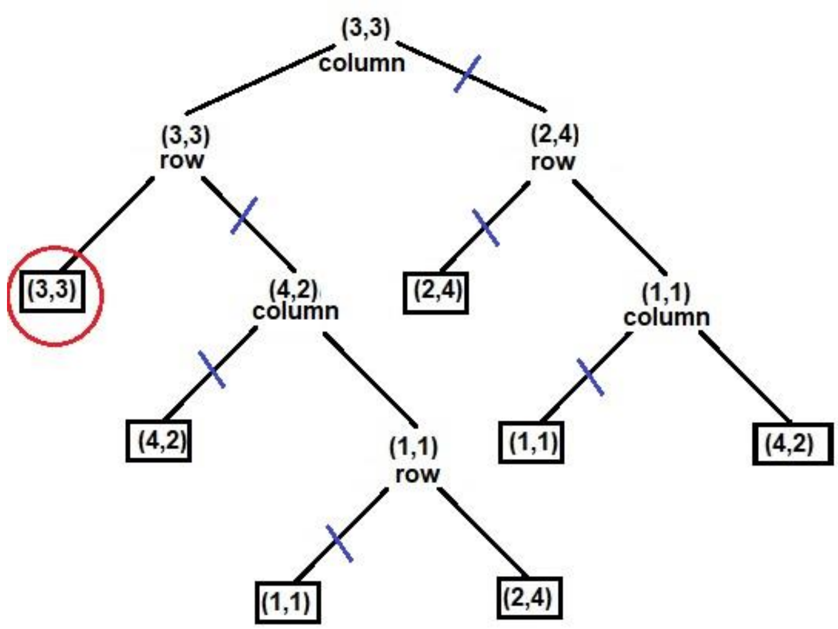
Solutions to exercises:

1. Show that in the Theory of Moves version of the Chicken dilemma, $(3,3)$ is a non-myopic equilibrium when the column player moves first.

Solution: The game tree is:



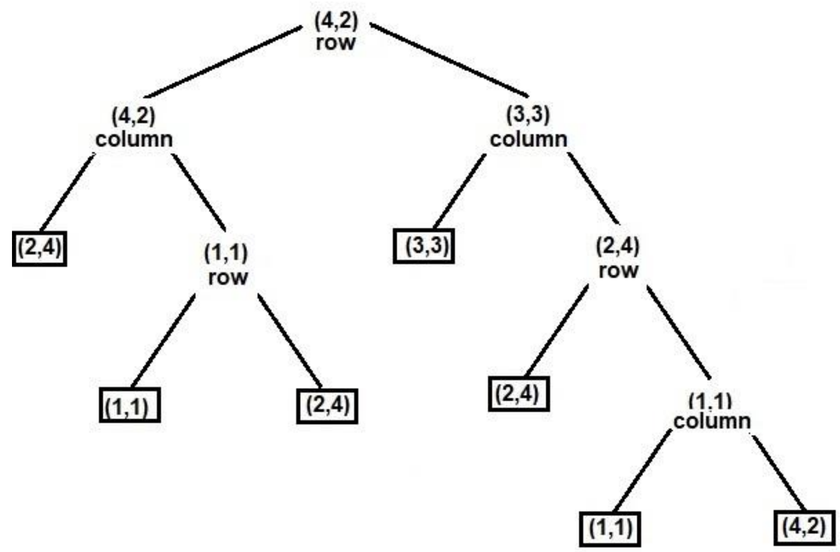
After applying reverse induction:



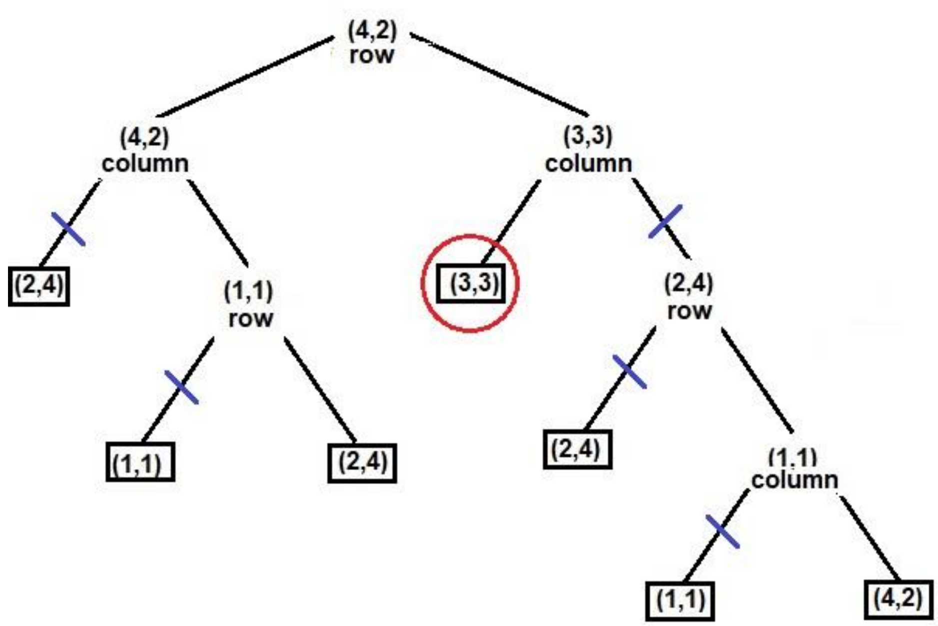
So, indeed, when $(3,3)$ is the starting position and the column player goes first, the final position (circled) is $(3,3)$, so this outcome is a non-myopic equilibrium when column moves first.

2. Draw the two tree diagrams in the Theory of Moves version of the Chicken dilemma when the starting position is $(4,2)$, and determine the final outcomes. Your answers should agree with the table in this section.

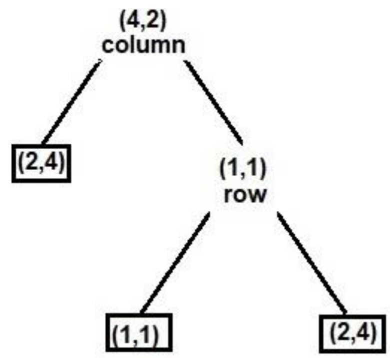
Solution: When the row player moves first:



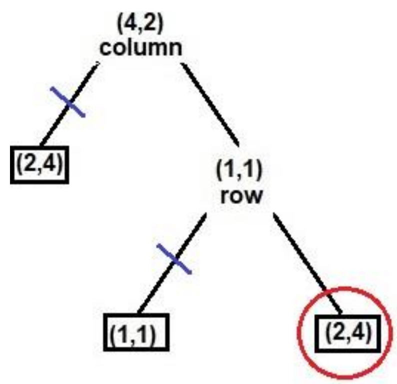
Applying reverse induction:



The ending outcome (circled) is $(3,3)$. The tree diagram when column goes first:



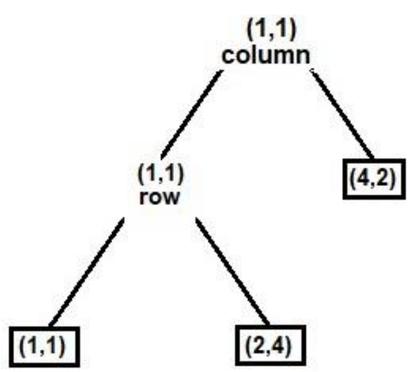
Applying reverse induction:



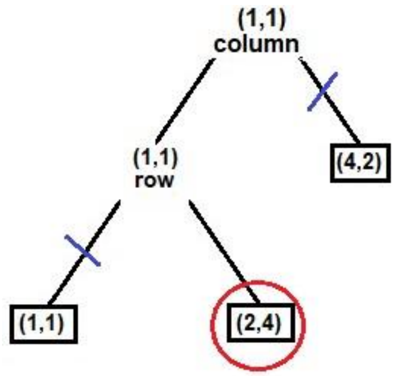
So, the ending position (circled) is $(2,4)$.

3. Draw the tree diagram for the Theory of Moves version of the Chicken dilemma when $(1,1)$ is the starting position and the column player moves first. Show that the final outcome is $(2,4)$.

Solution: The tree diagram:



Applying reverse induction:



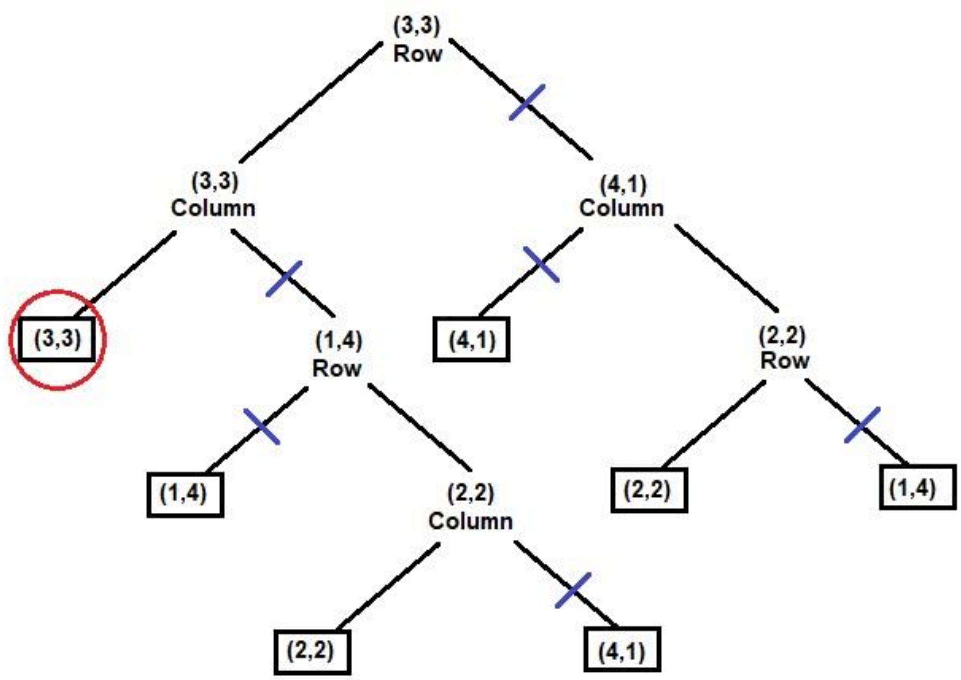
The final position (circled) is $(2,4)$.

4. Do a complete Theory of moves analysis of the prisoners' dilemma (the means drawing all 8 tree diagrams and making a table similar to the one we made for the chicken dilemma.) Show that there are two distinct non-myopic equilibria.

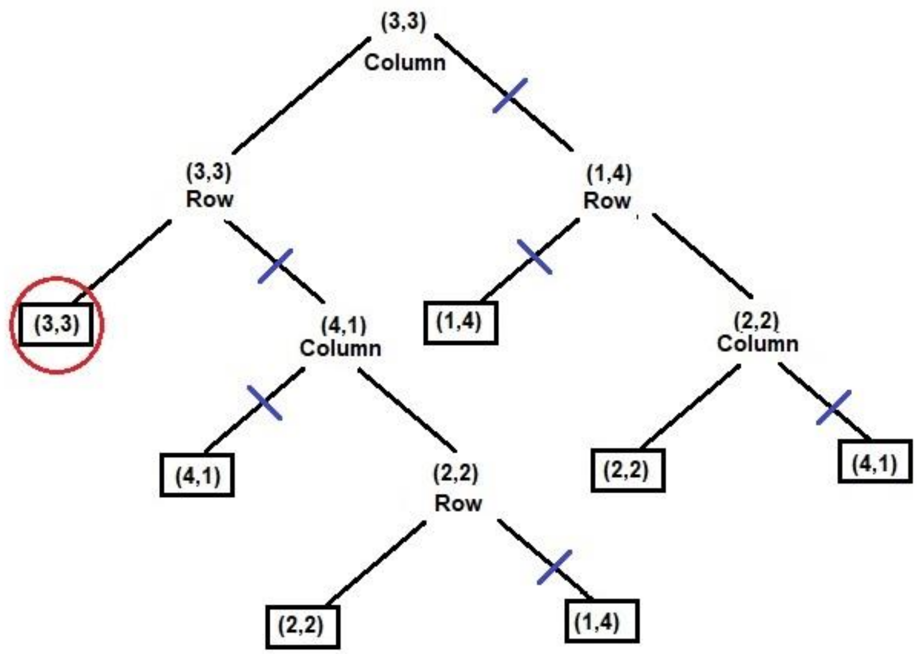
Solution: From this point on, we will only display the tree diagram after reverse induction has been applied, in order to save space.

Here are the tree diagrams:

When $(3,3)$ is the starting position:

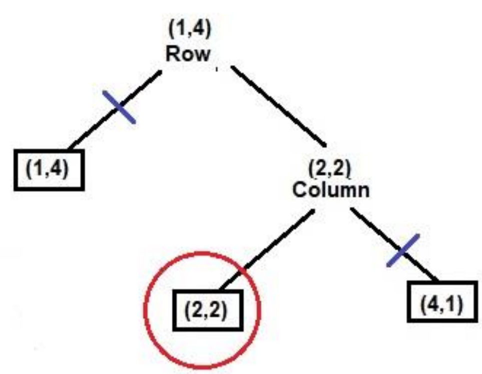


Starting position (3,3) and the row player moves first.



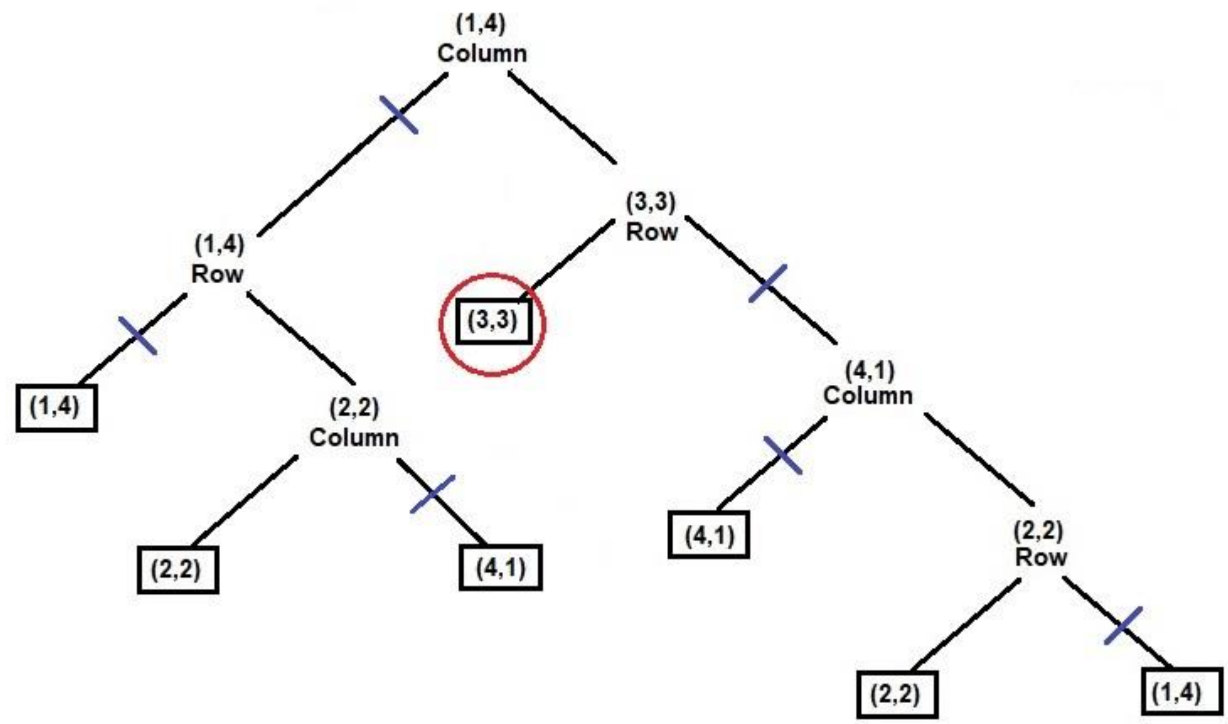
Starting position (3,3) and the column player moves first

In both cases the final position is $(3{,}3)$. Now suppose $(1{,}4)$ is the starting position:



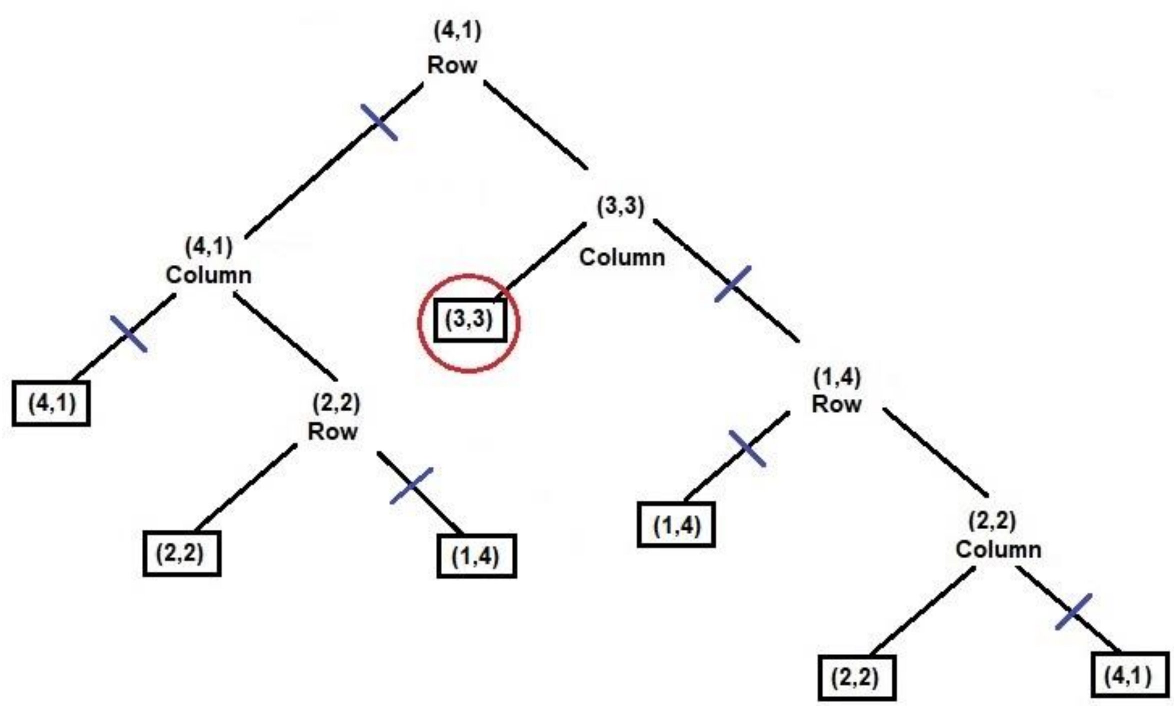
Starting position (1,4) and the row player moves first

The final position depends on who moves first. It is (2,2) if the row player goes first, but the next tree diagram shows it is $(3{,}3)$ if the column player goes first.

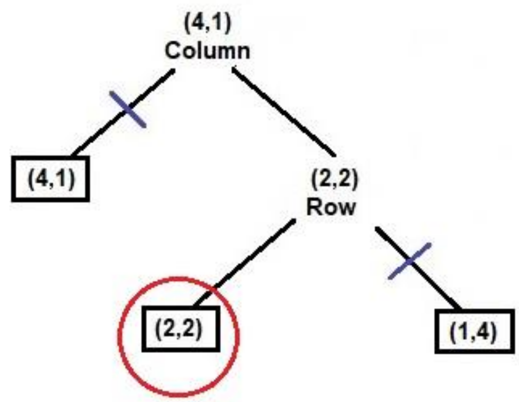


Starting position (1,4) and the column player moves first

Now suppose (4,1) is the starting position:

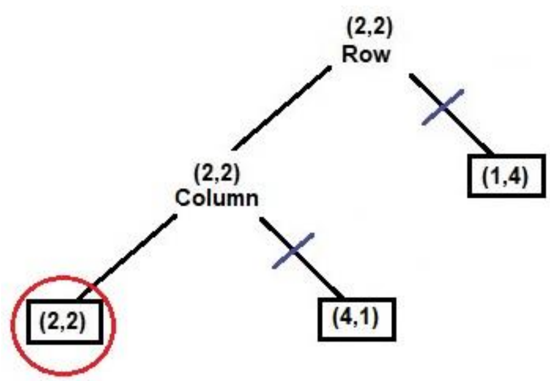


Starting position (4,1) and the row player moves first



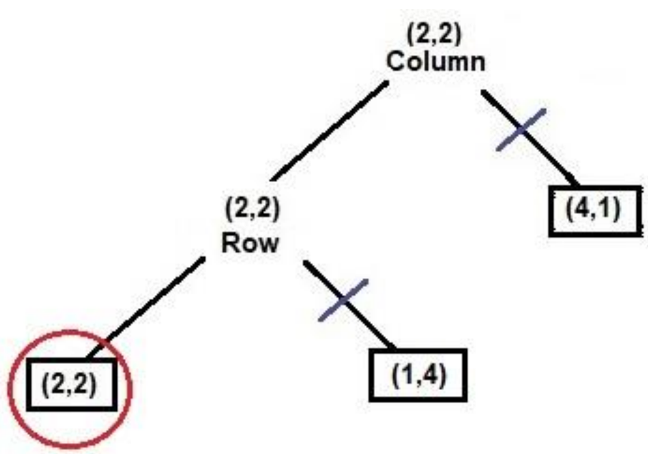
Starting position (4,1) and the column player moves first

Again, the final position depends on who moves first. Finally, if $(2,2)$ is the starting position:



Starting position is (2,2) and the row player moves first

In this last case the final position does not depend on who moves first – it is $(2,2)$ in both cases.



Starting position is (2,2) and the row player moves first

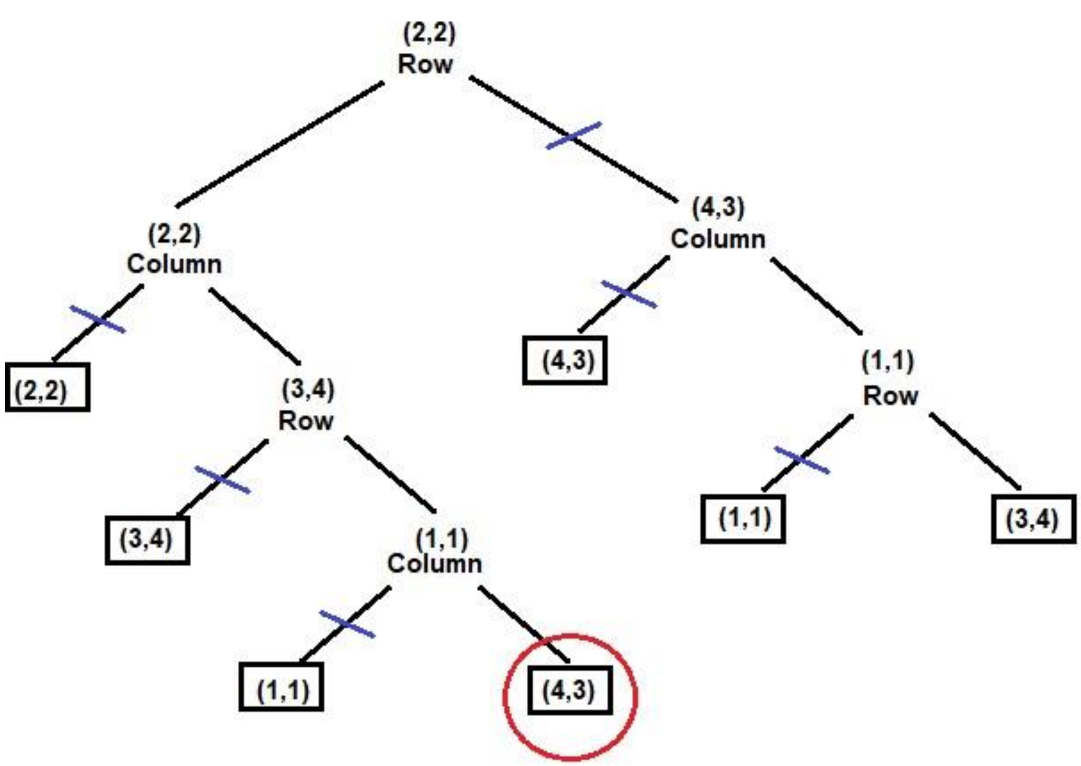
Tabulating the results:

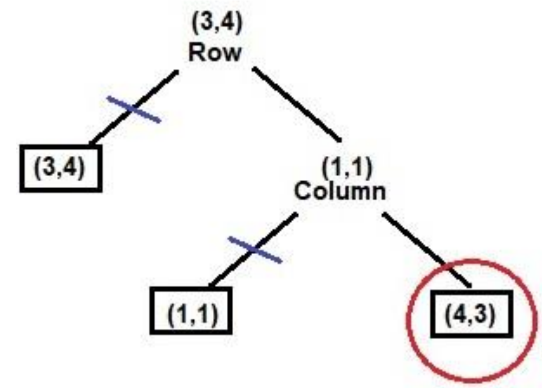
Starting position	Final Position if Row moves first	Final Position If Column moves first
(3,3)	(3,3)	(3,3)
(1,4)	(2,2)	(3,3)
(4,1)	(3,3)	(2,2)
(2,2)	(2,2)	(2,2)

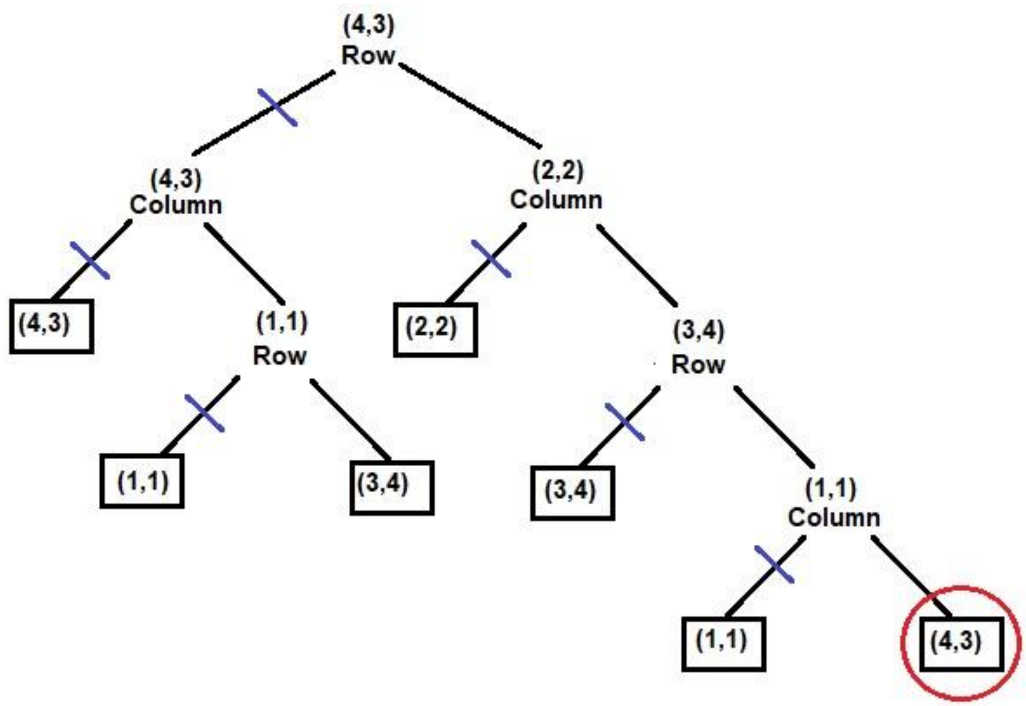
The table reveals that (2,2) and (3,3) are non-myopic equilibria.

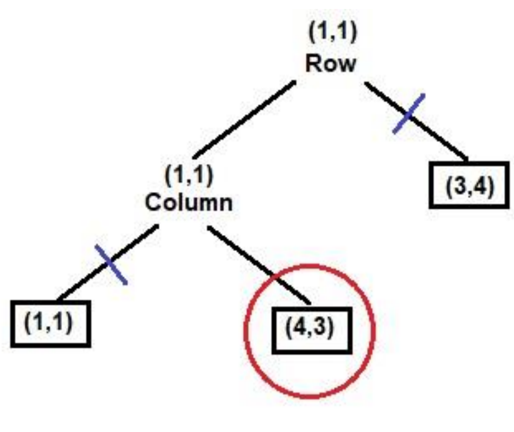
5. Do a complete Theory of Moves analysis of the Follow the Leader dilemma. Show that in this game, there are no non-myopic equilibria. However, show that whoever moves first gets their most preferred outcome, and whoever moves second gets their second most preferred outcome. (In other words, whoever moves first gets their preferred Nash equilibrium point!) Note that the simultaneous play version of this dilemma predicts (1,1) as most likely. Thinking of the story we used to illustrate this dilemma (see Exercise 10 in Section 8.2), and considering your experience of drivers in similar situations, which model is more realistic?

Solution: We present the four tree diagrams for when the row player makes the first move. The four diagrams when the column player makes the first move are left for the reader.




(3,4)
Row
(3,4)
(1,1)
Column
(1,1)
(4,3)


(4,3)
Row
(4,3)
Column
(2,2)
Column
(4,3)
(1,1)
Row
(2,2)
(3,4)
Row
(1,1)
(3,4)
(3,4)
(1,1)
Column
(1,1)
(4,3)


(1,1)
Row
(1,1)
Column
(3,4)
(1,1)
(4,3)

Note that the ending position is $(4,3)$ when the row player makes the first move, no matter what the starting position. Similarly, the ending position is $(3,4)$ whenever the column player moves first, for all starting positions. The table summarizes the situation:

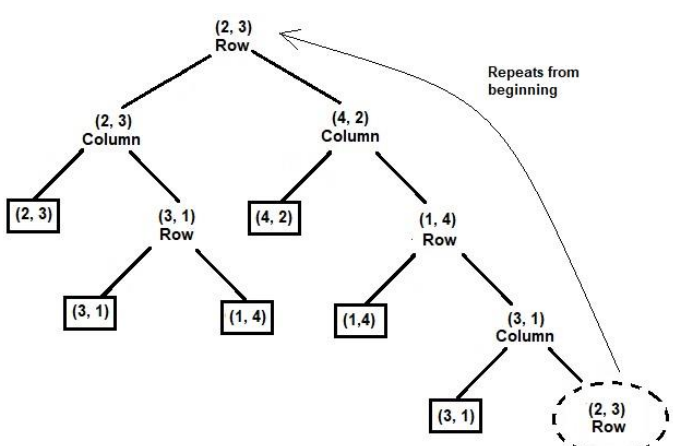
Starting position	Final Position if Row moves first	Final Position If Column moves first
$(2,2)$	$(4,3)$	$(3,4)$
$(3,4)$	$(4,3)$	$(3,4)$
$(4,3)$	$(4,3)$	$(3,4)$
$(1,1)$	$(4,3)$	$(3,4)$

In this game, there are no non-myopic equilibria, but each of the Nash equilibriums is also a non-myopic equilibrium when the player who prefers that outcome goes first. The upshot is that whoever moves first gets their most preferred outcome, whoever moves second gets their second most preferred outcome. The outcome of a collision $(1,1)$ shouldn't happen at all, unless one of the drivers makes an error. This is much more realistic than the simultaneous play version, which predicts $(1,1)$ every time!

6. It is possible that the tree diagram for a theory of moves version of a payoff matrix is not finite. For example, if after several branches you arrive at an outcome you have seen higher up in the tree, and with the same person's turn to move, the cycle will repeat forever. The following example, taken from Exercise 35 in Chapter 4 of Taylor and Pacelli (2008) illustrates this. Assume $(2,3)$ is the starting position and the row player moves first:

$$\text{Row}\quad \begin{matrix} \\ C \\ D \end{matrix} \quad \overset{\text{Column}}{\begin{matrix} C & D \\ \end{matrix}} \begin{bmatrix} (2,3) & (3,1) \\ (4,2) & (1,4) \end{bmatrix}$$

Solution: After several nodes, we arrive at $(2,3)$ at the row player's turn, which is the same as the starting position, so the tree repeats forever:

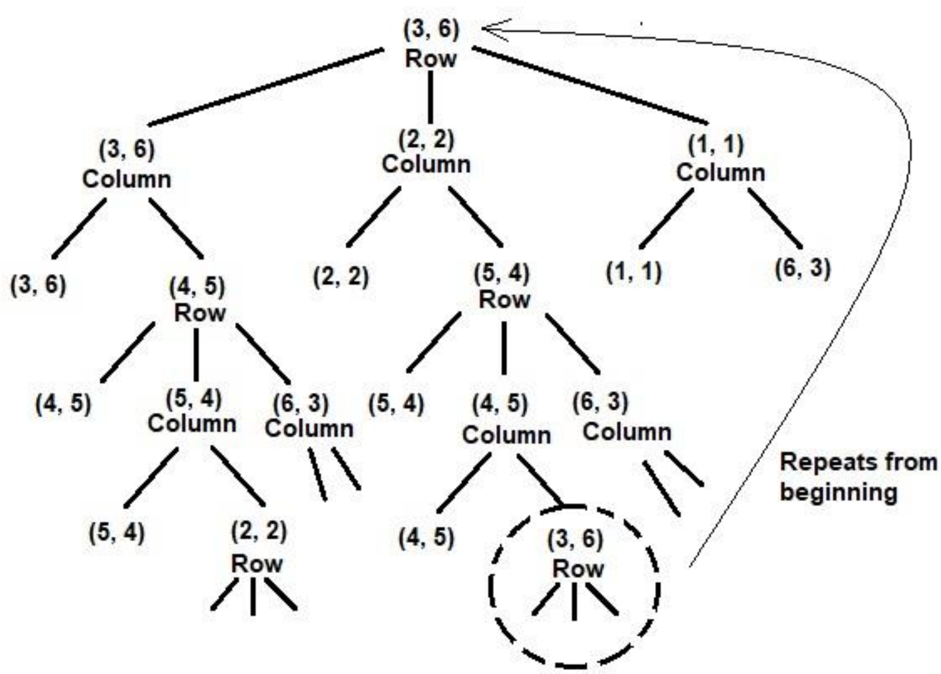


7. Verify that the game tree is infinite for the 3×2 Cuban Missile crisis using the usual ending rules for the theory of moves. Assume $(3,6)$ is the starting position, and either player moves first.

Solution: Recall the payoff matrix:

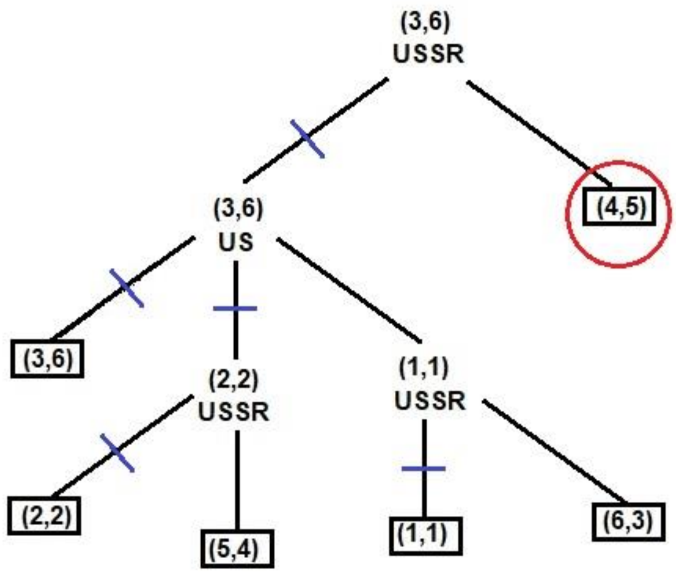
		USSR	
		$C\ (Withdraw)$	$D\ (Maintain)$
	$C\ (Blockade)$	$(4,5)$	$(3,6)$
USA	$D\ (Air\ Strike$	$(5,4)$	$(2,2)$
	$V\ (Invasion)$	$(6,3)$	$(1,1)$

This leads to the game tree:



8. a) Apply reverse induction to the game tree obtained from the Cuban Missile crisis (3×2 version, with the given thresholds of 4 for the US and 5 for the USSR) obtained in the text, and verify the final outcome is $(4,5)$, the 'mutual cooperation' outcome.

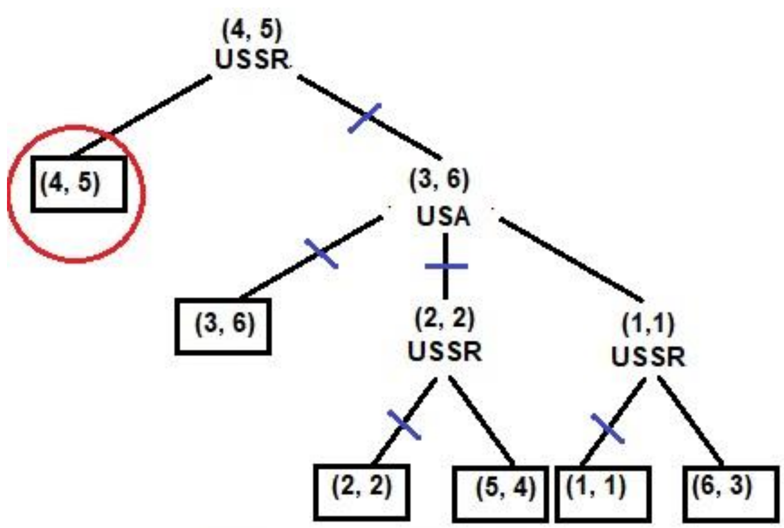
Solution:



The final position is circled – the US uses a blockade strategy and the USSR withdraws their missiles.

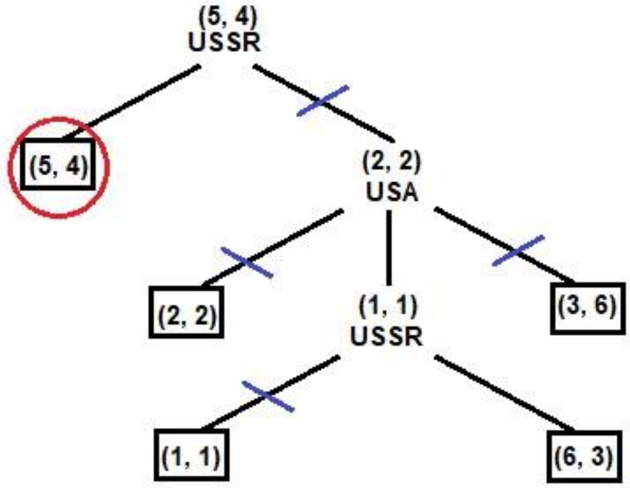
b) Complete the analysis for every possible starting position and for each player moving first. Determine if any outcomes are non-myopic equilibria.

Solution: There are twelve trees to draw. We present the six where the USSR moves first, leaving the remaining six for the reader. We summarize all twelve outcomes in the table below. The case when $(3,6)$ is the starting position is part (a). The other five are as follows:



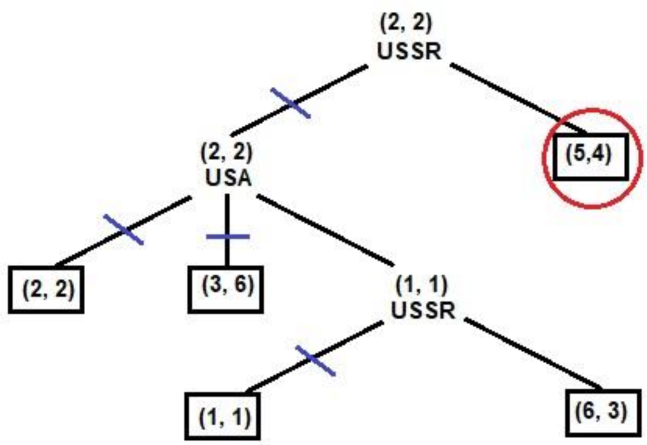
Starting position is (4, 5)

So, $(4,5)$ is a non-myopic equilibrium when the USSR (Column) goes first.

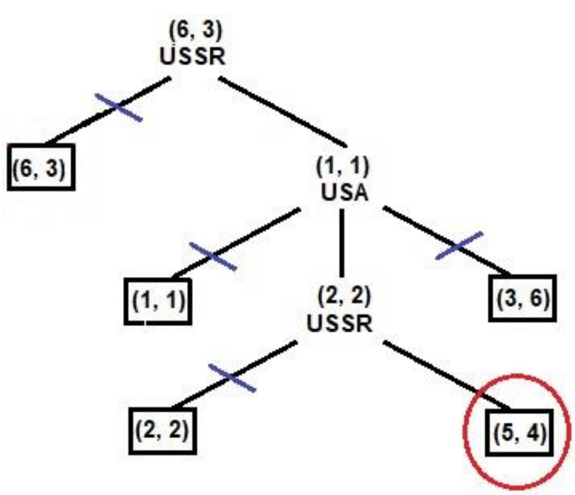


Starting position is (5, 4)

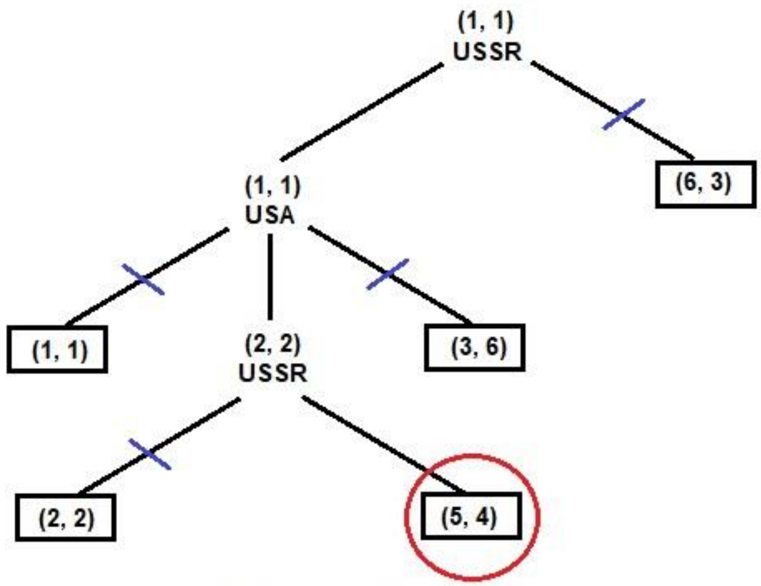
So, $(5, 4)$ is a non-myopic equilibrium when the USSR (Column) goes first.



Starting position is (2, 2)



Starting position is (6, 3)



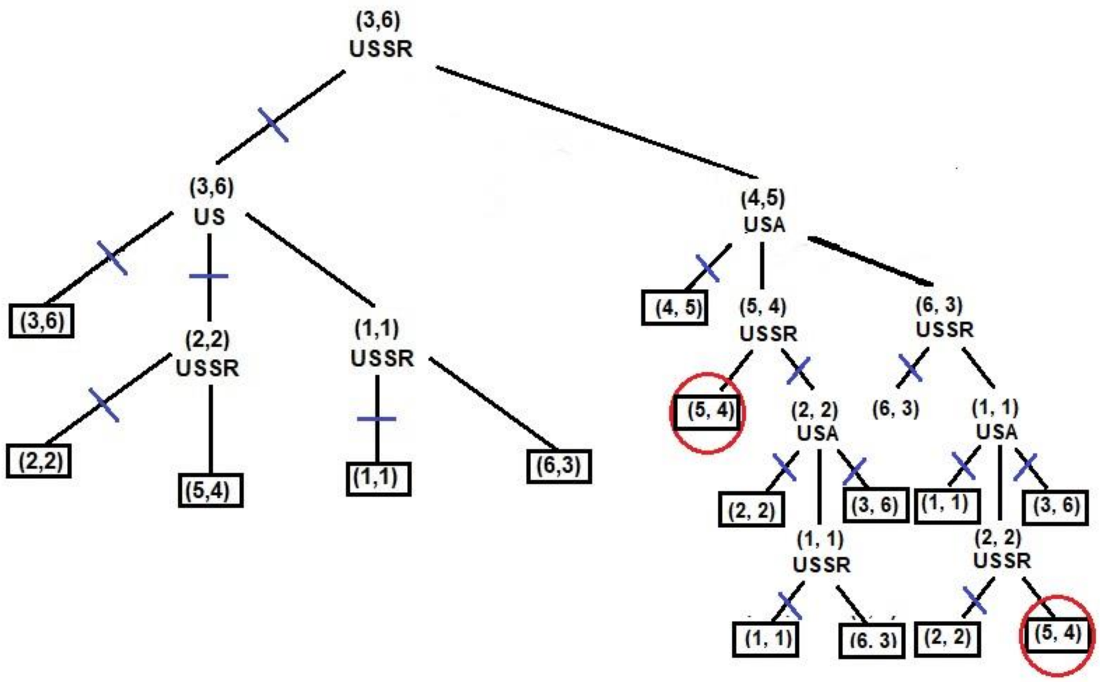
The table summarizes the situation:

Starting Position	Ending Position if the USSR moves first	Ending Position if the USA moves first
$(3,6)$	$(4,5)$	$(6,3)$
$(4,5)$	$(4,5)$	$(5,4)$
$(5,4)$	$(5,4)$	$(5,4)$
$(2,2)$	$(5,4)$	$(6,3)$
$(6,3)$	$(5,4)$	$(5,4)$
$(1,1)$	$(5,4)$	$(6,3)$

The reader should have noticed that in two of the tree diagrams when the USA moves first, there were actually two distinct ways to finish at the ending position. The only non-myopic equilibrium is at $(\mathbf{5},4)$**,** (although $(4,5)$ is a non-myopic equilibrium if the USSR moves first, as noted above.) Note also that those final positions with payoff $(6,3)$ happen only if the USSR truly think that nuclear war is the worst possible outcome, so that they would withdraw their missiles even after a US invasion or air strike. In reality, as noted in the text, the USSR may have felt into a retaliatory response if the US followed one of those more aggressive actions. Thus, the outcomes of $(6,3)$ and even $(5,4)$ may be less realistic a prediction in those cases, than the prediction of a $(4,5)$ outcome for those cases in the first two rows and first column of the table. Nevertheless, this example shows that the theory of moves can be applied to games with payoff matrices larger than 2×2 (modified by using threshold values for the terminal nodes.)

9. Give a theory of moves analysis of the 3×2 Cuban Missile crisis, assuming the threshold value for both players is 5. Assume $(3, 6)$ is the starting position and the USSR moves first.

Solution: By changing the threshold for the USA, the $(4{,}5)$ node is no longer a terminal mode when it is the USA's turn to move. Therefore, some new branches appear on the tree used in Exercise 8, emanating from $(4{,}5)$. There are two distinct paths on the tree that lead to it, but now the final position and predicted outcome is $(5{,}4)$ instead of $(4{,}5)$.



8.4 A Brief Introduction to n-Player Games

8.4.1 Dominance and Nash Equilibrium Points

No music references or exercises in this subsection.

8.4.2 Cooperative Games – Games in Characteristic Function Form

Solutions to exercises (and more music references):

1. Consider a coin game similar to the one in the text. Gain, Robert, Colleen, and Paige are sitting clockwise around a table. This time, Robert has a dime, Colleen has a half-dollar, and Paige has a dollar coin. Again, they simultaneously reveal whether their coin is heads up or tails up. The payoff rules are slightly different this time. If all the coins are heads up, no exchange is made. If one tail is up, that player wins both of the other players' coins. If two tails are up, each one wins half of the remaining coin. If three tails are up, they pass their coins counterclockwise to the next person.

a) Construct a table of payoffs and use it to draw the movement diagram of the game. Show that all three players have T as a dominant strategy, which leads to a unique Nash equilibrium. Thus, the game is strictly determined. What is the outcome?

Solution: We have (where, in "HTH" or other outcomes, the first letter is Robert's coin, the second is Colleen's, and the third is Paige's -the same order as the payoffs in the ordered triples):

Outcome	Payoffs to (Robert, Colleen, Paige)
$\boldsymbol{HHH}$	$(0,0,0)$
$\boldsymbol{HHT}$	$(-10,-50,60)$
$\boldsymbol{HTH}$	$(-10,110,-100)$
$\boldsymbol{THH}$	$(150,-50,-100)$
$\boldsymbol{TTH}$	$(50,50,-100)$
$\boldsymbol{THT}$	$(25,-50,25)$
$\boldsymbol{HTT}$	$(-10,5,5)$
$\boldsymbol{TTT}$	$(40,50,-90)$

Notice that this is a zero-sum game. In particular, every outcome is Pareto efficient. Thus, if we find a unique Nash equilibrium in pure strategies, the game will be strictly determined.

We can set up a three-dimensional payoff table as in the text, with the two 'pages' of the table given in the two tables below:

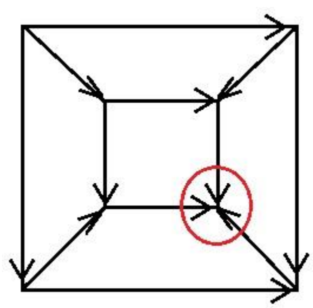
Paige selects H

$$\begin{array}{cc} & \text{Colleen} \\ & \begin{array}{cc} H & T \end{array} \\ \text{Robert} \begin{array}{c} H \\ T \end{array} & \begin{bmatrix} (0,0,0) & (-10,110,-100) \\ (150,-50,-100) & (50,50,-100) \end{bmatrix} \end{array}$$

Paige selects T

$$\begin{array}{cc} & \text{Colleen} \\ & \begin{array}{cc} H & T \end{array} \\ \text{Robert} \begin{array}{c} H \\ T \end{array} & \begin{bmatrix} (-10,-50,60) & (-10,5,5) \\ (25,-50,25) & (40,50,-90) \end{bmatrix} \end{array}$$

To construct the movement diagram, let the inner (back) square be when Paige selects T. We obtain:



The diagram reveals that T is dominant for each player – for Robert because all the vertical arrows point downwards, for Colleen because all the horizontal arrows point to the right, and for Paige because all the diagonal arrows point to the inner (back) face. This clearly implies a unique Nash equilibrium in pure strategies (circled in the diagram) at TTT, with payoff $(40,50,-90)$. This is the predicted outcome of this strictly determined game.

b) Find the security level for each player and each coalition, and thereby convert the game to characteristic function form.

Solution: If Colleen and Paige form a coalition against Robert, the game becomes:

$$\begin{array}{cc} & \text{Colleen and Paige} \\ & \begin{array}{cccc} HH & HT & TH & TT \end{array} \\ \text{Robert} \begin{array}{c} H \\ T \end{array} & \begin{bmatrix} 0 & -10 & -10 & -10 \\ 150 & 25 & 50 & 40 \end{bmatrix} \end{array}$$

In this game, T is still dominant for Robert, while HT is dominant for the team. Thus, there is a saddle point at $T, HT (= THT$ in our earlier notation), with payoff 25. Thus, $v(Robert) = 25$.

If Robert and Paige form a coalition against Colleen, the game becomes:

$$\text{Colleen} \quad \begin{matrix} & \text{Robert and Paige} \\ & \begin{matrix} HH & HT & TH & TT \end{matrix} \\ \begin{matrix} H \\ T \end{matrix} & \begin{bmatrix} 0 & -50 & -50 & -50 \\ 110 & 5 & 50 & 50 \end{bmatrix} \end{matrix}$$

In this game, T is still dominant for Colleen, and "HT" (in Robert, Paige order) is dominant for the team. Thus, the 5 at the intersection of these strategies is the unique saddle point of the matrix. (Be careful how to depict where the saddle is – it's in the "T, HT" position of the matrix above depicting Colleen against the team; but we are listing Colleen's payoff first since she is now the 'row' player of this matrix. So in the original ordered triple notation, it is the outcome denoted HTT in the first table in part a.) In particular, $v(Colleen) = 5$.
Finally, if Robert and Colleen form a coalition against Paige, the game becomes:

$$\text{Paige} \quad \begin{matrix} & \text{Robert and Colleen} \\ & \begin{matrix} HH & HT & TH & TT \end{matrix} \\ \begin{matrix} H \\ T \end{matrix} & \begin{bmatrix} 0 & -100 & -100 & -100 \\ 60 & 5 & 25 & -90 \end{bmatrix} \end{matrix}$$

In this game, T is still dominant for Paige, and TT dominates all other columns for the team. Thus, there is a saddle point at T, TT ($= TTT$ in the ordered triple notation), with payoff $v(Paige) = -90$,

We can now construct the entire characteristic function, in the tabular form:

Coalition	**value**
$\emptyset$	0
$\{Robert\}$	25
$\{Colleen\}$	5
$\{Paige\}$	-90
$\{Colleen, Paige\}$	-25
$\{Robert, Paige\}$	-5
$\{Robert, Colleen\}$	90
$\{Robert, Colleen, Paige\}$	0

c) Which coalition is likely to form? Show that the solution is the same outcome as predicted in part (a) using individual play.

Solution: Robert is guaranteed 25 when he plays alone. When he joins with Paige, the (joint) winnings drop to -5. When he joins with Colleen, the (joint) winnings go up to 90. Clearly, Colleen is Robert's preferred coalition partner.
Colleen is guaranteed 5 when she plays alone. When she joins with Paige, the winnings drop to -25. When she joins Robert, the winnings go up to 90. Clearly, Robert is also Colleen's preferred partner.
Paige is guaranteed -90 playing alone. When she joins Robert, go up to -5, whereas with Colleen, they go up to -25. Thus, she prefers Robert to Colleen. However, it doesn't matter, since neither Robert nor Colleen prefer Paige. The most likely coalition to form is Robert and Colleen (who prefer each other.)
When this happens, the predicted outcome is the TTT outcome with payoff $(40,50,-90)$, which is exactly what is predicted by individual play.

(So, this is an example of a game where there is no advantage gained by any player from cooperation with anyone else!)

d) In this example, we can answer the question of how the coalition in part (c) should split their winnings. Explain why there is only one reasonable way to split the winnings of this coalition. [**Hint**: Argue that if the proposed split was anything other than the obvious one, one or the other player in the coalition would refuse to join the coalition.

Solution: When Robert and Colleen join forces, they win $\$0.90$. One way to split it is $\$0.40$ for Robert and $\$0.50$ for Colleen, because that is what would happen if they played noncooperatively, as individuals. If Robert proposed a split to Colleen where she got anything less than $\$0.50$, she would refuse to join him. Similarly, if Colleen were to receive more than that, then Robert would receive less than the $\$0.40$ which he could get with individual play, and so Robert would refuse to join the coalition. Thus, the only reasonable way to split the winnings is $\$0.40$ for Robert and $\$0.50$ for Colleen.

(This happened because the game is strictly determined – the individual play solution affords each of Robert and Colleen higher payoff than their security levels. If Robert and Colleen refused to coalesce, and if one of them colluded with Paige instead, then the one player by themselves would drop to just their security level – like Paige in the case above when Robert and Colleen team up. Neither player would want that to happen, which is more incentive to stick with the split as in individual play.)

2. In the text, we did not consider what happens if the players use mixed strategies. Since the coin game in the text is not strictly determined, it seems reasonable to try mixed strategies. One difficulty is the lack of a quick formula to replace the formula $E = pAq$ we used for 2-player games. Another is a lack of a version of the minimax theorem. Nevertheless, let's see if we can remedy the former difficulty and try some examples.

a) As usual, let $p = [p_1 \quad p_2]$ denote the row player's (Robert's) mixed strategy, so p_1 is the probability of playing H, and p_2 the probability of playing T. Similarly, $q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$ is the Column player's (Colleen's) mixed strategy, and let $r = [r_1 \quad r_2]$ be Paige's mixed strategy. Suppose that $p = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$, $q = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$, and $r = \begin{bmatrix} \frac{1}{5} & \frac{4}{5} \end{bmatrix}$. By using a table of the eight possible outcomes, compute the expected payoff E_R for Robert. For example, argue that the probability of the HTH outcome is the product $\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(\frac{1}{5}\right) = \frac{1}{30}$ because the players choose their strategies independently. Do the same for each outcome, multiply by the appropriate values, and add. Repeat for E_C and E_P, the expected payoffs to Colleen and Paige.

Solution: If the players choose their strategies independently, then for each outcome we can find the probability of the intersection by multiplying the probabilities of the events as outlined in Chapter 6. Thus,

$$\begin{aligned} p(HTH) &= p(Robert\ chooses\ H, Colleen\ chooses\ T, and\ Paige\ chooses\ H\} \\ &= p(\{Robert\ chooses\ H\} \cap \{Colleen\ chooses\ T\} \cap Paige\ chooses\ H) \\ &= \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(\frac{1}{5}\right) = \frac{1}{30}. \end{aligned}$$

We then multiply the result by the value of that outcome (that is, the payoff) and add. The result is summarized in the table, where we keep track of the payoff (that is, of the value v_i), for each player simultaneously, as an ordered triple.

Outcome	p_i	v_i	$p_i v_i$
HHH	$\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{1}{5}=\frac{1}{15}$	$(0,0,0)$	$(0,0,0)$
HHT	$\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{4}{5}=\frac{4}{15}$	$(-5,-10,15)$	$\left(-\frac{20}{15},-\frac{40}{15},4\right)=\left(-\frac{4}{3},-\frac{8}{3},4\right)$
HTH	$\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{5}=\frac{1}{30}$	$(-5,30,-25)$	$\left(-\frac{5}{30},\frac{30}{30},-\frac{25}{30}\right)=\left(-\frac{1}{6},1,-\frac{5}{6}\right)$
THH	$\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{1}{5}=\frac{1}{15}$	$(35,-10,-25)$	$\left(\frac{35}{15},-\frac{10}{15},-\frac{25}{15}\right)=\left(\frac{7}{3},-\frac{2}{3},-\frac{5}{3}\right)$
TTH	$\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{5}=\frac{1}{30}$	$(5,-5,0)$	$\left(\frac{1}{6},-\frac{1}{6},0\right)$
THT	$\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{4}{5}=\frac{4}{15}$	$(20,0,-20)$	$\left(\frac{16}{3},0,-\frac{16}{3}\right)$
HTT	$\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{4}{5}=\frac{2}{15}$	$(0,15,-15)$	$(0,2,-2$
TTT	$\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{4}{5}=\frac{2}{15}$	$(20,-5,-15)$	$\left(\frac{8}{3},-\frac{2}{3},-2\right)$

Summing the vectors in the last column, we obtain

$$E=(E_R,E_C,E_P)=\left(9,-\frac{7}{6},-\frac{47}{6}\right)$$

b) Let $A_1=A_H$ be the payoff matrix, given that Paige chooses H (the 'front square' of the three-dimensional payoff table), and let $A_2=A_T$ be the payoff matrix, given that Paige chooses T (the 'back square' of the payoff table.) Show by direct comparison to the calculations in part (a), that E_R can be obtained by considering the payoffs to Robert in A_i $(i=1,2)$,and performing the following matrix product:

$$r\cdot\begin{bmatrix}pA_1q\\ pA_2q\end{bmatrix}=[r_1,r_1]\cdot\begin{bmatrix}pA_1q\\ pA_2q\end{bmatrix} \tag{8.2}$$

Show that we can also recover E_C $(resp.,E_P)$, by the same formula, but with the payoffs for Colleen (resp., Paige) for the entries in A_i.

Solution: Using Robert's payoffs:

$$A_1=\begin{bmatrix}0 & -5\\ 35 & 5\end{bmatrix}$$
$$A_2=\begin{bmatrix}-5 & 0\\ 20 & 20\end{bmatrix}$$

Observe that

$$pA_1q=\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}0 & -5\\ 35 & 5\end{bmatrix}\begin{bmatrix}\frac{2}{3}\\ \frac{1}{3}\end{bmatrix}$$
$$=\left(\frac{1}{2}\cdot 0+\frac{1}{2}\cdot 35\right)\cdot\frac{2}{3}+\left(\frac{1}{2}\cdot(-5)+\frac{1}{2}\cdot 5\right)\cdot\frac{1}{3}=\frac{35}{3}$$

Similarly,

$$pA_2q=\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}-5 & 0\\ 20 & 20\end{bmatrix}\begin{bmatrix}\frac{2}{3}\\ \frac{1}{3}\end{bmatrix}$$
$$=\left(\frac{1}{2}\cdot(-5)+\frac{1}{2}\cdot 20\right)\cdot\frac{2}{3}+\left(\frac{1}{2}\cdot 0+\frac{1}{2}\cdot 20\right)\cdot\frac{1}{3}=\frac{25}{3}$$

Therefore,

$$[r_1, r_1] \cdot \begin{bmatrix} pA_1q \\ pA_2q \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{4}{5} \end{bmatrix} \begin{bmatrix} \frac{35}{3} \\ \frac{25}{3} \end{bmatrix} =$$

$$= \frac{1}{5}\left(\left(\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 35\right)\cdot\frac{2}{3} + \left(\frac{1}{2}\cdot(-5) + \frac{1}{2}\cdot 5\right)\cdot\frac{1}{3}\right)$$

$$+\frac{4}{5}\left(\left(\frac{1}{2}\cdot(-5) + \frac{1}{2}\cdot 20\right)\cdot\frac{2}{3} + \left(\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 20\right)\cdot\frac{1}{3}\right) = 9$$

The important thing to notice is that the calculations are exactly the ones we did when we summed the $p_i v_i$ in the table above to obtain E_R in the first coordinate. The same holds for the second and third coordinates. Thus, using the payoffs for Colleen:

$$pA_1q = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 30 \\ -10 & -5 \end{bmatrix} \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} = \frac{5}{6}$$

$$pA_2q = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -10 & 15 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} = -\frac{5}{3}$$

Therefore,

$$E_C = r \cdot \begin{bmatrix} pA_1q \\ pA_2q \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{4}{5} \end{bmatrix} \begin{bmatrix} \frac{5}{6} \\ -\frac{5}{3} \end{bmatrix} = -\frac{7}{6}$$

Finally, using Paige's payoffs:

$$pA_1q = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & -25 \\ -25 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} = -\frac{25}{2}$$

$$pA_2q = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 15 & -15 \\ -20 & -15 \end{bmatrix} \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} = -\frac{20}{3}$$

Therefore,

$$E_P = r \cdot \begin{bmatrix} pA_1q \\ pA_2q \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ \overline{5} & \overline{5} \end{bmatrix} \begin{bmatrix} -\frac{25}{2} \\ -\frac{20}{3} \end{bmatrix} = -\frac{47}{6}$$

So, we obtain the same answers an in part (a).

c) Use the Eq. (8.2) to find the expected payoffs to the players if

$$p = [.3,.7], \qquad q = \begin{bmatrix} .4 \\ .6 \end{bmatrix}, \qquad and\ r = [.9,.1]$$

Solution: For Robert:

$$pA_1q = [.3 \quad .7] \begin{bmatrix} 0 & -5 \\ 35 & 5 \end{bmatrix} \begin{bmatrix} .4 \\ .6 \end{bmatrix} = 11.0$$

$$pA_2q = [.3 \quad .7] \begin{bmatrix} -5 & 0 \\ 20 & 20 \end{bmatrix} \begin{bmatrix} .4 \\ .6 \end{bmatrix} = 13.4$$

For Colleen:

$$pA_1q = [.3 \quad .7] \begin{bmatrix} 0 & 30 \\ -10 & -5 \end{bmatrix} \begin{bmatrix} .4 \\ .6 \end{bmatrix} = 0.5$$

$$pA_2q = [.3 \quad .7] \begin{bmatrix} -10 & 15 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} .4 \\ .6 \end{bmatrix} = -0.6$$

For Paige:

$$pA_1q = [.3 \quad .7] \begin{bmatrix} 0 & -25 \\ -25 & 0 \end{bmatrix} \begin{bmatrix} .4 \\ .6 \end{bmatrix} = -11.5$$

$$pA_2q = [.3 \quad .7] \begin{bmatrix} 15 & -15 \\ -20 & -15 \end{bmatrix} \begin{bmatrix} .4 \\ .6 \end{bmatrix} = -12.8$$

Thus, we can compute the expected payoff for all three players simultaneously:

$$(E_R, E_C, E_P) = r \cdot \begin{bmatrix} pA_1q & pA_1q & pA_1q \\ pA_2q & pA_2q & pA_2q \end{bmatrix},$$

where we use Robert's payoffs in the first column, Colleens in the second, and Paige's in the third:

$$(E_R, E_C, E_P) = [.9, .1] \cdot \begin{bmatrix} 11 & .5 & -11.5 \\ 13.4 & -.6 & -12.8 \end{bmatrix} = [11.24 \quad 0.39 \quad -11.63]$$

d) The equation (8.2) generalizes to 3-player games or arbitrary size $m \times n \times t$. In this case, since Paige has t strategies, the three-dimensional payoff table has t pages, and each page A_i is an $m \times n$ matrix. Thus, if

$$p = [p_1, p_2, \dots, p_m], \qquad q = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}, \qquad and\ r = [r_1, \dots, r_t],$$

then the expected payoff is $E = r \cdot \begin{bmatrix} pA_1q \\ \vdots \\ pA_lq \end{bmatrix}$. Suppose we have a $2 \times 4 \times 3$ game where

$$A_1 = \begin{bmatrix} (1,2,4) & (2,-3,5) & (-2,-3,7) & (5,0,-5) \\ (0,-2,6) & (3,3,4) & (-4,4,0) & (2,8,1) \end{bmatrix}$$

$$A_2 = \begin{bmatrix} (-1,0,4) & (3,2,1) & (0,10,0) & (2,-5,8) \\ (2,1,1) & (1,0,3) & (4,5,-4) & (3,4,0) \end{bmatrix}$$

$$A_3 = \begin{bmatrix} (0,3,-4) & (0,1,-4) & (2,1,2) & (-1,-2,-3) \\ (4,2,-1) & (2,6,-3) & (0,0,-2) & (-2,3,5) \end{bmatrix}$$

Find the expected payoffs for each player if the following mixed strategies are used:

$$p = \left[\frac{1}{3}, \frac{2}{3}\right], \qquad q = \begin{bmatrix} \frac{1}{10} \\ \frac{2}{10} \\ \frac{3}{10} \\ \frac{4}{10} \end{bmatrix}, \qquad and\ r = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \end{bmatrix}$$

(Observe that this game is variable-sum, but that does not affect the formula for E.)

Solution: Since Paige has three strategies, $t = 3$. Thus, A_1 is when Paige chooses her first strategy. Instead of computing each player's payoffs separately, we can do them all at once, using the vector entries in the A_i:

$$pA_1q = \begin{bmatrix}\frac{1}{3}, \frac{2}{3}\end{bmatrix} \begin{bmatrix} (1,2,4) & (2,-3,5) & (-2,-3,7) & (5,0,-5) \\ (0,-2,6) & (3,3,4) & (-4,4,0) & (2,8,1) \end{bmatrix} \begin{bmatrix} \frac{1}{10} \\ \frac{2}{10} \\ \frac{3}{10} \\ \frac{4}{10} \end{bmatrix}$$

$$= \begin{pmatrix} \frac{23}{30} & \frac{83}{30} & \frac{17}{10} \end{pmatrix}$$

When Paige chooses her second strategy we have

$$pA_2q = \begin{bmatrix}\frac{1}{3}, \frac{2}{3}\end{bmatrix} \begin{bmatrix} (-1,0,4) & (3,2,1) & (0,10,0) & (2,-5,8) \\ (2,1,1) & (1,0,3) & (4,5,-4) & (3,4,0) \end{bmatrix} \begin{bmatrix} \frac{1}{10} \\ \frac{2}{10} \\ \frac{3}{10} \\ \frac{4}{10} \end{bmatrix}$$

$$= \begin{pmatrix} \frac{23}{10} & \frac{13}{5} & \frac{14}{15} \end{pmatrix}$$

When Paige chooses her third strategy:

$$pA_3q = \begin{bmatrix}\frac{1}{3}, \frac{2}{3}\end{bmatrix} \begin{bmatrix} (0,3,-4) & (0,1,-4) & (2,1,2) & (-1,-2,-3) \\ (4,2,-1) & (2,6,-3) & (0,0,-2) & (-2,3,5) \end{bmatrix} \begin{bmatrix} \frac{1}{10} \\ \frac{2}{10} \\ \frac{3}{10} \\ \frac{4}{10} \end{bmatrix}$$

$$= \begin{pmatrix} \frac{119}{150} & \frac{58}{25} & \frac{61}{1575} \end{pmatrix} \approx (0.79333 \quad 2.32 \quad 0.81333)$$

3. Consider the following variation of the Anderson Household Chore game: The three players are the children Ian, Jon, and Laurie. There are only two chores to be done – the dishes D and the raking R. In this version, each child secretly writes down D or R on a piece of paper. When the papers are revealed, each child works on the chore they chose. If more than one child

elects the same chore, that is fine, each will get a partial payment. If a chore is not chosen by any of them, it is assigned to the three of them to split (which only happens if all three chose the other chore, so in this case all three children share both chores.)

Mr. Anderson has some asymmetry built into the payoffs so that if one or both of the boys choose the dishes, they are rewarded with extra pay, and if Laurie chooses the raking, she is paid more than the boys. However, if nobody chooses a chore, when it is assigned, the pay is less than for that chore than it would be if somebody chose it. The basic pay is \$6 for the dishes and \$12 for the raking. However, if Ian and John choose dishes, each earns an extra \$1. If only one of the boys chooses D, he earns the entire extra \$2. If Laurie chooses raking, she earns an extra \$3 if she does the job alone, and extra \$2 if she shares the job with one other person, and an extra \$1 if all three are raking. If the dishes have to be assigned because nobody chose the chore, the pay is \$1 per person (for a total of \$3 instead of \$6.) If the raking has to be assigned because nobody chose it, the pay is \$3 per person (for a total of \$9 instead of \$12.) Note that, unlike previous versions of the game, choosing no chore or both is NOT a possible strategy – their paper must say D or R only.

a) Model this as a $2 \times 2 \times 2$ game. Write down the payoff table (that is, the payoff matrices A_i for each choice that Laurie makes.) Also, write down the movement diagram. Determine all instances of dominance, if any, and any Nash equilibria. Is the game strictly determined?

Solution:

Laurie selects R

Jon

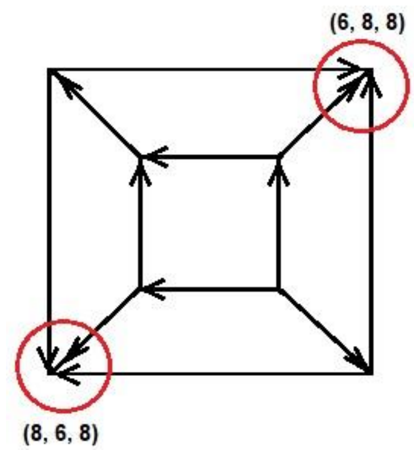
$$\text{Ian}\quad \begin{matrix} R \\ D \end{matrix} \overset{\begin{matrix} R & \quad D \end{matrix}}{\begin{bmatrix} (5,5,6) & (6,8,8) \\ (8,6,8) & (4,4,15) \end{bmatrix}}$$

Laurie selects D

Jon

$$\text{Ian}\quad \begin{matrix} R \\ D \end{matrix} \overset{\begin{matrix} R & \quad D \end{matrix}}{\begin{bmatrix} (6,6,6) & (12,5,3) \\ (5,12,3) & (6,6,5) \end{bmatrix}}$$

R is dominant for Laurie. No other player has a dominant strategy. The movement diagram (where Laurie choosing D is the inner square) is:



The two circle outcomes are Nash equilibria. They are Pareto efficient but inequivalent and not interchangeable, so this game is not strictly determined. Note that if each of the boys go for their preferred Nash equilibrium, the outcome is $(4,4,15)$, so that each is hurt and gets less than the payoff at either equilibrium point. Thus, ambiguity in what rational behavior means for two of the players (the boys in this example) can actually benefit the third player (Laurie, in this example.)

b) Convert the game to characteristic function form. Nate that the game is not constant-sum, so that for each coalition, to compute its security level, you must use the prudential strategy (use your own payoffs, ignore your opponent's payoffs, and play as if it were zero-sum.)

Solution: To find Ian's security level, form a coalition of Jon and Laurie to paly against Ian, but use Ian's payoffs:

Jon and Laurie

$$\text{Ian}\quad \begin{matrix} R \\ D \end{matrix} \overset{\begin{matrix} RR & RD & DR & DD \end{matrix}}{\begin{bmatrix} 5 & 6 & 6 & 12 \\ 8 & 5 & 4 & 6 \end{bmatrix}}$$

Pretending this is zero-sum, the third column dominates the second and fourth columns, so equalizing with the reduced matrix obtained form deleting the dominated columns:

$$[x, 1-x]\begin{bmatrix} 5 & 6 \\ 8 & 4 \end{bmatrix} = [8-3x \quad 2x+4]$$

$$8-3x = 2x+4$$

$$4 = 5x$$

$$x = \frac{4}{5}$$

$$v_I = 2\left(\frac{4}{5}\right) + 4 = \frac{28}{5} = 5.6$$

So, Ian's prudential strategy is $p = \begin{bmatrix} \frac{4}{5} & \frac{1}{5} \end{bmatrix}$.

Similarly, to find Jon's security level, use Jon's payoffs in a zero-sum game against a coalition of Ian and Laurie:

$$\text{Jon} \quad \begin{matrix} & \text{Ian and Laurie} \\ & \begin{matrix} RR & RD & DR & DD \end{matrix} \\ \begin{matrix} R \\ D \end{matrix} & \begin{bmatrix} 5 & 6 & 6 & 12 \\ 8 & 5 & 4 & 6 \end{bmatrix} \end{matrix}$$

This is the same payoff matrix as for Ian (which makes sense because the boys are treated symmetrically in this game.) Thus, Jon's prudential strategy is also $q = \begin{bmatrix} \frac{4}{5} \\ \frac{1}{5} \end{bmatrix}$ and his security level is also the same as Ian's: $V_J = 5.6$.

Finally, to find Laurie's security level, we use her payoffs in a zero-sum game against a coalition of Ian and Jon:

$$\text{Laurie} \quad \begin{matrix} & \text{Ian and Jon} \\ & \begin{matrix} RR & RD & DR & DD \end{matrix} \\ \begin{matrix} R \\ D \end{matrix} & \begin{bmatrix} 6 & 8 & 8 & 15 \\ 6 & 3 & 3 & 5 \end{bmatrix} \end{matrix}$$

Since Laurie has a dominant strategy, the matrix just reduces to the first row, and the row minimum is her security level (the saddle point of the matrix). Thus, Laurie's prudential strategy is a pure one – namely, her dominant one – always choose R, and her security level is $v_L = 6$.

We now have to compute the security levels of the coalitions of two or more. First, Jon and Laurie. For that, we make a zero-sum game where the team Jon and Laurie is the row player, against a coalition of the remaining players (just Ian in this case), and since we assume the payoffs are transferrable, we consider the payoff for the coalition to just be the sum of the payoffs for Jon and Laurie. We obtain:

$$\text{Jon and Laurie} \quad \begin{matrix} & \text{Ian} \\ & \begin{matrix} R & D \end{matrix} \\ \begin{matrix} RR \\ RD \\ DR \\ DD \end{matrix} & \begin{bmatrix} 11 & 14 \\ 12 & 15 \\ 16 & 19 \\ 8 & 11 \end{bmatrix} \end{matrix}$$

The 16 payoff is at a saddle point, so the security level for the coalition is $v_{JL} = 16$. Since the boys are treated symmetrically, the same is true of the coalition of Ian and Laurie: $v_{IL} = 16$. For the coalition consisting of Ian and Jon:

Laurie

$$\text{Ian and Jon} \quad \begin{matrix} RR \\ RD \\ DR \\ DD \end{matrix} \overset{\begin{matrix} R & D \end{matrix}}{\begin{bmatrix} 10 & 12 \\ 14 & 17 \\ 14 & 17 \\ 8 & 12 \end{bmatrix}}$$

The two outcomes with payoffs 14 are saddle points, so $v_{IJ} = 14$. Finally, for the grand coalition, again, since the payoffs are transferable, we consider the payoff for the coalition to be just the sum of its members payoffs. But clearly, the biggest total payoff sum occurs when both boys choose the dishes D and Laurie chooses raking R. At this outcome DDR, the payoffs are $(4,4,15)$, for a total payoff of 23 to the coalition. Since the three players can guarantee that payoff by coordinating their actions, that is the security level of the coalition: $v_{IJL} = 23$. In summary, the characteristic function is:

Coalition	**value**
$\emptyset$	0
$\{Ian\}$	5.6
$\{Jon\}$	5.6
$\{Laurie\}$	6
$\{Jon, Laurie\}$	16
$\{Ian, Laurie\}$	16
$\{Ian, Jon\}$	14
$\{Ian, Jon, Laurie\}$	23

Note that this game is superadditive (as all games which come from payoff matrices must be.)

c) Can you determine which coalition is likely to form?

Solution: If the coalition consists of two players, then both Ian and Jon prefer Laurie to each other, and, by symmetry, Laurie prefers the two boys equally. So, it's possible that one brother might team up with Laurie. However, it seems mutually beneficial for all in this game for the grand coalition to form, rather than a coalition of two.

4. Verify that the Divide the Dollar game is superadditive.

Solution: Denote the players by $A, B, and\ C$. We have:

$$v(\emptyset) = v(A) = v(B) = v(C) = 0$$
$$v(\{A,B\}) = v(\{A,C\}) = v(\{B,C\}) = v(\{A,B,C\}) = 1$$

Now let $X\ and\ Y$ be disjoint coalitions. We must compare $v(X \cup Y)$ to $v(X) + v(Y)$. If $X = \emptyset$ is empty, then $v(X \cup Y) = v(Y) = 0 + v(Y) = v(\emptyset) + v(Y) = v(X) + v(Y)$. Similarly, we have equality if $y = \emptyset$ is empty. So now assume both $X\ and\ Y$ are nonempty. If they are both singletons, then $X \cup Y$ is a doubleton (we stipulated they were disjoint, so the union can't be just one point). In this case:

$$v(X \cup Y) = 1 > 0 = 0 + 0 = v(x) + v(y).$$

If one of X or Y is a singleton and the other is a doubleton, then since they are disjoint, their union is all three $\{A, B, C\}$. Thus

$$v(X \cup Y) = 1 \geq 1 + 0 = v(X) + v(Y)$$

So, in all cases we have shown that

$$v(X \cup Y) \geq v(X) + v(Y),$$

which means the game is superadditive.

5. In the friendly town of Snowblind, there are three snow removal businesses operating. *Latimer's Snow Goose Plowing Service* plows driveways and small parking lots. *Harrison's Blow Away, Inc*. uses snow blowers to clear sidewalks and decks. *McGuire's Eaves of Congestion, Ltd.* Removes snow and ice from roofs. Because some customers want more than one service, and because they might be able to save by sharing equipment and overhead costs, the companies are exploring the possibility that some or all three of them may merge.
They've hired a consultant from *Midnight Sun and Hot Springs* marketing firm to study the situation. The consultant has provided the following data: In a typical winter month, with the companies working separately, the average profit for Snow Goose Plowing is \$12,000, the average profit for Blow Away is \$8,000, and the average profit for Eaves of Congestion is \$5,500. They estimate that the average monthly profit for Snow Goose and Blow Away would be \$22,000, the average for Blow Away and Eaves of Congestion working together would be \$15,000, and the average profit for Snow Goose and Eaves of Congestion working together would be \$18,000. Finally, they estimate that if all three companies join forces, the average monthly profit would be \$30,000.

Model this as a game in characteristic function form, and show that it is superadditive. Can you predict the most likely coalition to form?

Music references to: the band Camel, to former Beatle George Harrison, to singer-songwriter Barry McGuire, and to Led Zeppelin. Andrew Latimer is the guitarist for Camel, who in 1975 released a landmark progressive rock album entitled the Snow Goose. Harrison had a hit song 'Blow Away' from his 1979 eponymous album. Barry McGuire released the protest song 'Eve of Destruction' in 1965. Finally, the phrase 'Midnight Sun and Hot Springs' is taken from the lyrics (…'we come from the land of the ice and snow, from the midnight sun, where the hot springs flow') of the 1970 Led Zeppelin hit 'Immigrant Song'. (Websites: https://www.camelproductions.com/, https://www.georgeharrison.com/, https://en.wikipedia.org/wiki/Eve_of_Destruction, and https://lz50.ledzeppelin.com/.)

Solution: The players are the three companies:

$$N = \{Snow\ Goose, Blow\ Away, Eaves\ of\ Congestion\} = \{S, B, E\}$$

The characteristic function is:

Coalition	value
$\emptyset$	0
S	\$12,000
B	\$8,000
E	\$5,500
SB	22,000
BE	\$15,000
SE	\$18,000
SBE	\$30,000

To see it is superadditive, check all the possibilities of combining disjoint coalitions:

$$v(\emptyset \cup X) = v(X) = 0 + v(X) = v(\emptyset) + v(X)$$

Also

$$v(SB) = 22{,}000 > 12000 + 8000 = v(S) + v(B)$$
$$v(BE) = 15{,}000 > 8000 + 5500 = v(B) + v(E)$$
$$v(SE) = 18{,}000 > 12{,}000 + 5{,}500 = v(s) + v(E)$$
$$v(SBE) = 30{,}000 > 12000 + 15000 = v(S) + v(BE)$$
$$v(SBE) = 30{,}000 > 8000 + 18000 = v(B) + v(SE)$$
$$v(SBE) = 30{,}000 > 5500 + 22000 = v(E) + v(SB)$$

This shows the game is superadditive. Assuming the payoffs are transferable, we consider for each player the potential gain by joining a two-player coalition, and then by joining the grand coalition. For S, joining B gives an extra \$2,000 per month for the pair over working alone. Joining E gives an extra \$500 per month, and joining the grand coalition gives an extra \$4500 per month. We don't know how the extra money will be split (it being a fair division problem), but for simplicity let's assume it is split equally between the players. Thus, for S, joining B is

preferable to joining E (an extra \$1000 vs an extra \$250), while joining the grand coalition is (most preferred (an extra \$1500 per month for each player.)
For B, joining S gives an extra \$1000 (each), while joining E gives an extra \$750 (each), while joining the grand coalition again gives an extra \$1500 (each). So B prefers S to E, but again, joining the grand coalition is best.
Finally, for E, Joining S gives an extra \$250 (each), joining B gives an extra \$750 (each), and joining the grand coalition again yields and extra \$1500 per month (each).
So, assuming the extra money from merging is equally split, the most likely coalition to form is the grand coalition – that is, all three companies merge. Of course, if the extra profit from merging is not equally split, this could affect which coalition is most likely to form.

6. Consider the following game in characteristic function form: $\{a, b, c, d\}$ and characteristic function given by:

Value function $v(S)$ for $S \subseteq N$
$\boldsymbol{v(\emptyset) = 0}$
$\boldsymbol{v(a) = v(b) = 1,\ \ v(c) = 2,\ \ v(d) = 3}$
$\boldsymbol{v(\{a, b\}) = 2,\ \ \ v(\{a, c\}) = 3,\ \ v(\{a, d\}) = 4}$
$\boldsymbol{v(\{b, c\}) = 3,\ \ v(\{b, d\}) = 5,\ \ v(\{c, d\}) = 6}$
$\boldsymbol{v(\{a, b, c\}) = 4,\ \ v(\{a, b, d\}) = 6,\ \ v(\{a, c, d\}) = 6,\ \ v(\{b, c, d\}) = 6}$
$\boldsymbol{v(\{a, b, c, d\}) = v(N) = 8}$

Determine whether or not the game is superadditive.

Solution: The game is not superadditive because:

$$v(\{b, c, d\}) = 6 < 7 = 5 + 2 = v(\{b, d\}) + v(\{c\})$$

8.5 Legislative Voting and Political Power

8.5.1 Legislative Voting Systems

Music Reference in the Text: Example 8.16 is a musical nod to the band Big Brother and the Holding Company. Between 1966 and 1968, the vocalist for the band was legendary singer Janis Joplin. Sam Andrew and James Gurley were guitarists who were two of the founding members of the band, along with guitarist Peter Albin. (Website: https://www.bbhc.com/)

No exercises this subsection.

8.5.2 Political Power

Music Reference in the Text: Example 8.21 (King Heart, Ltd.) is a musical nod to the band to the English progressive rock group Van der Graaf Generator. Members of the band are vocalist and multi-instrumentalist Peter Hammill, keyboard player Hugh Banton, drummer Guy Evans, and (for most of their career) saxophone player David Jackson, among others. In 1971 they released an album entitled Pawn Hearts. (Website: https://www.vandergraafgenerator.co.uk/)

1. In the *United Nations Security Council* there are 15 countries that are voters in the system. Five of the countries are called *permanent members*. They are China, England, France, Russia, and the United States. The remaining 10 nonpermanent members rotate periodically among the other member countries of the UN. If the Security Council wants to pass a proposal, it requires the support of at least 9 of the 15 countries on the council. However, any one of the five permanent members has veto power - a vote of 'no' from one of them will kill the proposal. (We ignore the possibility of abstentions.) With this description, the UN Security Council is not described as a weighted system. Show that it is equivalent to a weighted system. [**Hint**: Give each nonpermanent member a weight of 1. The five permanent members seem to have equal power, so give them all the same weight, call it x. Now just make x and the quota q large enough, so that no coalition is winning unless all five permanent members are in the coalition. You can do this by trial and error, or you can try to be more efficient and set up some inequalities to solve…]

Solution: The idea is to make x and q large enough so that no coalition without all the permanent members can be winning. An example of a guessed solution is this: Set $x = 20$, and set the quota at $q = 104$. Clearly, without all five of the permanent members, a coalition cannot exceed the quota, and even with all five, the coalition still needs 4 more countries to join (so no winning coalitions have fewer than 9 countries.) Thus, to be winning in this weighted system, the coalitions which win are those with 9 or more countries, except those missing a permanent member, which is the same as in the system as described. We can get away with smaller values of x and q. For example, $x = 10$ and $q = 54$ also works. Assuming x

is an integer, to see the smallest value of x that can work you can consider inequalities coming from the winning and losing coalitions. For example, $5x + 4 \geq q$ must hold because a minimal winning coalition has weight $5x + 4$. On the other hand, four permanent members together with all ten of the other countries is insufficient to be winning (a sort of maximal losing coalition) because of the veto power, so we must also have $4x + 10 < q$. Combining these inequalities leads to $4x + 10 < 5x + 4$, or $x > 6$. If you try $x = 7$, it works if you set $q = 5x + 4 = 39$.

2. Consider the following weighted voting system: *Velour Secret* is a fashion clothing company, and all production decisions are made by a committee of department managers: Lou has 8 votes, Christa has 8 votes, John has 7 votes, Maureen has 6 votes, Sterling has 5 votes, and Doug has 3 votes. To pass a production idea and start production, a quota of 20 votes is required.

Music reference: to the New York based rock band the Velvet Underground (velour is a type of velvet and 'underground' sometimes means secret or little-known.) Members of the band included vocalist/guitarist Lou Reed, singer/model/actress Nico (real name: Christa Paffgen), John Cale (bass, keyboards, viola, backing vocals), Maureen 'Moe' Tucker (drums), Sterling Morrison (guitar, bass, backing vocals), Doug Yule (vocals, bass, guitar, keyboards), and others. (Website: https://www.velvetundergroundmusic.com/)

a) Consider the following two coalitions: $X = \{Lou, John, Sterling\}$ and $Y = \{Lou, Christa, Maureen, Doug\}$. Show that both coalitions are winning.

Solution: observe that, if $w(A)$ denotes the weight of the coalition A (the sum of the weights of the voters in A), then

$$w(X) = 8 + 7 + 5 = 20 = q$$
$$w(Y) = 8 + 8 + 6 + 3 = 25 > q$$

This shows both coalitions are winning.

b) Consider what happens when the two coalitions swap John for Christa. Do both coalitions remain winning, just one, or neither?

Solution: After swapping John for Christa, we obtain modified coalitions $X' = \{Lou, Christa, Sterling\}$, and $Y' = \{Lou, John, Maureen, Doug\}$. We have

$$w(X') = 8 + 8 + 5 = 21 > q$$
$$w(Y') = 8 + 7 + 6 + 3 = 24 > q$$

So, they both remain winning coalitions after the swap.

c) Same question as (b) if the two coalitions swap Sterling for Doug.

Solution: The new coalitions are $X' = \{Lou, Joh, Doug\}$ and $Y' = \{Lou, Christa, Maureen, Sterling\}$. We have

$$w(X') = 8 + 7 + 3 = 18 < q$$
$$w(Y') = 8 + 8 + 6 + 5 = 27 > q$$

So, one of the coalitions (X), becomes losing, while the other (Y) remains winning.

d) Is it possible for these coalitions to swap two voters which will render both coalitions losing? Explain.

Solution: No, it is not possible. The sum of the weights of the original two coalitions is $w(X) + w(Y) = 20 + 25 = 45$. When making a one for one swap between two voters in X and Y, none of the voters is lost or gained. It follows that after the swap, we still have $w(X') + w(Y') = 45$. Therefore, in order for them both to become losing, they each would total less than 20, so together they could not total the required 45. It follows that at least one of them must remain heavier than 20 (and so remain winning.) Perhaps a more succinct way of putting it is this – if the average weight of two coalitions is greater than or equal to q, then at least one of them must be greater than or equal to q. Note that swapping a voter does not affect the average.

3. The considerations of Exercise 2 lead to the following definition (explored in Taylor and Pacelli (2008) and in Hodge and Klima (2005)): A voting system is said to be ***swap-robust*** if whenever you have two winning coalitions X and Y, and a one for one trade (a "swap") is made between X and Y, then at least one of the resulting coalitions must remain winning. (The voters that are traded must not belong to the intersection of the two coalitions.) Prove that every weighted system is swap-robust. [**Hint**: Let q be the quota and let X and Y be two winning coalitions so that $w(X) \geq q$ and $w(Y) \geq q$, where $w(X)$ means the sum of the weights of the voters in X. Now select $x \in X$ and $y \in Y$ to trade, and consider what happens to the weights of the coalitions in two cases. First, what if the weights of x and y are the same, and second, what is the weights of x and y are different?]

Solution: Let q be the quota, and so for winning coalitions we have $w(X) \geq q$ and $w(Y) \geq q$. Select voters x and y to trade. If $w(x) = w(y)$, then swapping these voters has no effect on the weights of X and Y, so they both remain winning coalitions after the swap. If $w(x) \neq w(y)$, then one of these voters has a larger weight than the other. Without any loss of generality, assume that x is the voter with the higher weight. When x is swapped for y, the total weight of X goes down while the total weight of Y goes up. It is possible that the weight of X might drop below the quota, but since the weight of Y increased, it still exceeds q, so at least Y is still winning. They cannot both be rendered losing by a swap.

4. In Exercise 3, you showed a weighted system must be swap-robust. It follows that if a system is not swap robust, it cannot be weighted. Show that the US Federal system is not swap-robust, and therefore, it is not weighted, as suggested in the text. This shows that voting systems exist that are not weighted.

Solution: Let X be a coalition consisting of the President, the Vice President, exactly 50 senators, including both senators from Virginia, and exactly 218 representatives, not including any representative from California. Let Y be another coalition with the President, the Vice President, exactly 50 senators (but none from Virginia), and exactly 218 representatives, including all of California's representatives. Note that both coalitions are winning. Now swap a senator from Virginia in X with a representative from California in Y. The coalition X' is now losing because it has only 49 senators (although it does have 219 representatives.) Similarly, Y' is losing because it has only 217 representatives (although it does have 51 senators.) Thus, because both winning coalitions were rendered losing by a swap, the system is not swap-robust, and therefore it is not weighted, either, by Exercise 3.

5. The condition of swap-robustness does not completely characterize weighted systems. It's true that every weighted system is swap robust (Exercise 3), but there are also non-weighted systems which are swap robust as well. For example, given that the following system is not equivalent to a weighted system, show that, nevertheless, it is swap robust.

The *Windy Association* is an Investment Firm, which has a board of directors consisting of six people. Each potential investment is voted on. A particular investment is made if it receives at least three of the six possible votes, subject to the restriction that the people voting for the investment must have a total of at least 25 years experience with the firm between them. The3 number of years of experience of each board member is given by the following table:

Board member	Years experience
Mary	18
Terry	12
Jules	8
Brian	5
Jim	4
Ted	3

Show that this system is swap-robust. [**Hint**: There are two conditions for a coalition to be winning. It needs a minimum of 3 voters, and a minimum of 25 years experience. If there are two winning coalitions and a swap is performed, consider separately how the swap affects each of the two conditions.]

Music reference to: The Pop-Rock band the Association, who had a big hit in 1967 with the song 'Windy', and a hit in 1968 with 'Six Man Band'. Their line-up changed over the years, but in 1965 the original line-up included Terry Kirkman, Jules Gary Alexander, Brian Cole, and Ted Bluechel, Jr., among others. They were soon joined by Jim Yester (later of the Lovin' Spoonful.) 'Mary' is a reference to the song 'Along Comes Mary', the 1966 breakout hit for the band. (Website: https://www.theassociationwebsite.com/)

Solution: Let X and Y denote two winning coalitions, and suppose a one for one swap is made between $x \in X$ and $y \in Y$. To be winning means at least three voters plus a total of at least 25 years experience. A one for one swap does not affect the number of voters in the coalitions, so they each will have at least three voters after the swap. Since no two voters have the same number of years experience, any swap entails the total number of years experience increasing for one coalition and decreasing for the other (much like the argument for the weights in Exercise 3.) The one that goes down might be rendered losing if the total number of years experience drops below 25, but the other one will remain winning since the number of years experience went up after the swap. Thus, any swap leaves at least one winning coalition, so the system is swap-robust. (We'll see below that this system is not weighted, even though it is swap robust.)

6. A condition that a voting system may satisfy that is stronger than swap robustness is called ***trade robustness***. Its definition is similar to swap robustness, except that the trade of voters can be an arbitrary trade between any number of winning coalitions, instead of a one for one swap between just two winning coalitions. To be precise, a voting system is **trade robust** if whenever you have two or more winning coalitions, and these coalitions perform an arbitrary trade of voters, then at least one of the resulting coalitions must still be winning.

Show that the Windy Association board of directors system from Exercise 5 is not trade-robust. [**Hint**: Try to make up two winning coalitions X and Y, such that a two for one trade between them renders them both losing.]

Solution: Consider the following coalitions: $X = \{Mary, Jim, Ted\}$ and $Y = \{Terry, Jules, Brian\}$. Both are winning with exactly 25 years experience. Consider what happens if X trades Jim and Ted for Jules. This leads to $X' = \{Mary, Jules\}$ and $Y' = \{Terry, Jim, Ted, Brian\}$. Then X' has a total of $18 + 8 = 26$ years experience. However, X' is losing because it only has two voters. Y' has four voters, but it is losing because the total years experience is $12 + 4 + 3 + 5 = 24$, less than the required 25 years. Since we found a trade that renders both coalitions losing, it cannot be trade-robust.

(Note that this system has disjoint winning coalitions, namely X and Y, so it fails to be superadditive.)

Remark 8.28 In 1992, Alan Taylor and William Zwicker proved that the condition of trade-robustness completely characterizes a weighted system. That is, a voting system is weighted if and only if it is trade-robust. The Windy Association voting system in Exercise 5 is not weighted because in Exercise 6, you showed it is not trade-robust, even though it is swap-robust as you saw in Exercise 5. This characterization theorem for weighted systems is discussed in Taylor and Pacelli (2008), and a full proof appears in Taylor and Zwicker (1999) and Taylor and Zwicker (1992).

7. Suppose the six New England states form a voting system for regional economic development. Passage on any initiative requires at least three of the six votes, subject to the requirement that the states supporting the initiative must constitute at least 50% of the New England population, and also subject to the condition that either Maine or Connecticut supports the measure. Assume the population percentages are given by the table:

State	Population Percentage
Connecticut	20%
Massachusetts	26%
Maine	19%
Rhode Island	15%
New Hampshire	10%
Vermont	10%

Show that this system is not swap-robust, and therefore, not a weighted system.

Solution: Consider the coalitions $X = \{Maine, Massachusetts, Vermont\}$ and $Y = \{Connecticut, Rhode\ Island, New\ Hampshire, Vermont\}$. Each is a winning coalition because: X has 3 members, a total of 55% of the New England population and has the support of Maine, while Y has 4 members, 55% of the population, and has the support of Connecticut. Make the following swap: Massachusetts for Connecticut. This results in
$X' = \{Maine, Connecticut, Vermont\}$ and
$Y' = \{Massachusetts, Rhose\ Island, New\ Hampshire, Vermont\}$

Now X' has a total of 49% of the population, so is a losing coalition. Also Y' fails to have the support of either Maine or Connecticut, so is a losing coalition also, even though it has 4 members and more than enough population. Thus, we found a swap which rendered both coalitions losing, so the system is not swap robust. Therefore, it also fails to be trade-robust or weighted.

8. Consider the weighted voting system $[8; 4,3,3,1,1]$. List all the winning coalitions (there are eleven of them.) Which ones are minimal?

Solution: In descending order of the weights, let's name the voters A, B, C, D, E. The winning coalitions (sorted by total weight) are:

Total weight	Coalitions
8	$ABD, ABE, ACD, ACE, BCDE$
9	$ABDE, ACDE$
10	ABC
11	$ABCD, ABCE$
12	$ABCDE$

The minimal ones are those which would become losing if any voter defected from it. That includes (obviously) all the coalitions of weight $8 = q$, since any defection would make the weight go below 8. But it also includes ABC, since, if any voter defected, the weight would drop to 6 or 7. None of the other winning coalitions are minimal, since D or E could defect without making the coalition go below the quota. Thus, the minimal ones are $ABD, ABE, ACD, ACE, BCDE$, and ABC.

9. a) Explain why David is a dummy voter in the King Heart, Ltd. Example from the text (Example 8.21).

Solution: We noted in the text that this system is the weighted system $[10; 8,6,4,1]$. Note that David is the only voter with an odd number (1) as weight. That means the total weight of a coalition is odd if and only if David belongs to it. However, to be winning, the coalition must weigh 10 or more, but any winning coalition with David has odd weight, so must actually be 11 or more. In that case, David can defect and the coalition is still winning since it weighs at least $10 = q$. It follows that any winning coalition with David cannot be a minimal winning coalition. By definition, a dummy voter is a voter not part of any minimal winning coalition, so this shows David is a dummy voter.

b) Verify the list of winning coalitions for this example given in the text.

Solution: Again, the system is $[10; 8,6,4,1]$ with P (Peter) having 8 votes, H (Hugh) having 6 votes, G (Guy) having 4 votes and D (David) having 1 vote. We list the winning coalitions (according to total weight):

Total weight	**coalition**
10	HG
11	HGD
12	PG
13	PGD
14	PH
15	PHD
18	PHG
19	$PHGD$

There are no coalitions with total weight 16 or 17. This list agrees with what is in the text, except there they are organized according to the number of voters rather than according to total weight.

10. Consider the weighted voting system $[9; 4,3,3,1]$. Explain why the voter with 1 vote is a dummy voter. Do any of the voters have veto power? Is any voter a dictator?

Solution: There are no dictators, since every singleton set is a losing coalition (showing each voter can be part of a losing coalition.) Since $3 + 3 + 1 = 7 < 9$, any winning coalition must contain the voter A with 4 votes. In particular, A has veto power since he belongs to every winning coalition. Similarly, $4 + 3 + 1 = 8 < 9$, so B must be part of any winning coalition with 3, votes, and by symmetry do does C with 3 votes. Thus, A, B, C all have veto power. Since ABC is a winning coalition, this show that the only winning coalitions are ABC and $ABCD$. In particular the grand coalition is the only winning coalition containing D. But D can defect from this coalition, leaving the other winning coalition ABC, so the grand coalition is not minimal. We have shown that D is not a member of any minimal winning coalition, and therefore, D is a dummy voter.

11. Suppose we lower the quota in the weighted system in Exercise 10 from 9 to 8.

a) Compute the Shapley-Shubik Index for each voter.

Solution: We are considering the system $[8; 4,3,3,1]$. The system is small enough to list all 24 orderings of the voters. The table below summarizes for each ordering, who is the pivotal voter.

ordering	Pivotal voter	ordering	Pivotal voter	ordering	Pivotal voter	ordering	Pivotal voter
$ABCD$	C	$BACD$	C	$CABD$	B	$DABC$	B
$ABDC$	D	$BADC$	D	$CADB$	D	$DACB$	C
$ABCD$	B	$BCAD$	A	$CBAD$	A	$DBAC$	A
$ACBD$	D	$BCDA$	A	$CBDA$	A	$DBCA$	A
$ADBC$	B	$BDAC$	A	$CDAB$	A	$DCAB$	A
$ADCB$	C	$BDCA$	A	$CDBA$	A	$DCBA$	A

Therefore, we have:

$$SSI(A) = \frac{12}{24} = .5$$
$$SSI(B) = SSI(C) = \frac{4}{24} = \frac{1}{6} \approx .1667$$
$$SSI(D) = \frac{4}{24} = \frac{1}{6} \approx .16777$$

b) Same question if we lower the quota to 7 instead of 8.

Solution: We are considering the system $[7; 4,3,3,1]$. The analogous table now becomes:

ordering	Pivotal voter	ordering	Pivotal voter	ordering	Pivotal voter	ordering	Pivotal voter
$ABCD$	B	$BACD$	A	$CABD$	A	$DABC$	B
$ABDC$	B	$BADC$	A	$CADB$	A	$DACB$	C
$ABCD$	C	$BCAD$	A	$CBAD$	A	$DBAC$	A
$ACBD$	C	$BCDA$	D	$CBDA$	D	$DBCA$	A
$ADBC$	B	$BDAC$	A	$CDAB$	A	$DCAB$	A
$ADCB$	C	$BDCA$	C	$CDBA$	B	$DCBA$	A

The results are:

$$SSI(A) = \frac{12}{24} = .5$$
$$SSI(B) = SSI(C) = \frac{5}{24} \approx .20833$$
$$SSI(D) = \frac{2}{24} = \frac{1}{12} \approx .0833$$

12. Compute the Banzhaf Index for the voters of the system in Exercise 8.

Solution: We must count the total Banzhaf power for each voter, that is, the total number of critical defections they have. From Exercise 8, we have a complete list of winning coalitions. The following table lists, for each winning coalition, which voters have a critical defection form it:

Winning coalition	Critical defections
ABD	A, B, D
ABE	A, B, E
ACD	A, C, D
ACE	A, C, E
$BCDE$	B, C, D, E
$ABDE$	A, B
$ACDE$	A, C
ABC	A, B, C
$ABCD$	A, B, C
$ABCE$	A, B, C
$ABCDE$	none

Thus, we have:

$$TBP(A) = 9$$
$$TBP(B) = TBP(C) = 7$$
$$TBP(D) = TBP(E) = 3$$

The total Banzhaf power for all voters is $9 + 7 + 7 + 3 + 3 = 29$, so

$$BI(A) = \frac{9}{29} \approx .3103$$
$$BI(B) = BI(C) = \frac{7}{29} \approx .2424$$
$$BI(D) = BI(E) = \frac{3}{29} \approx .1034$$

Remark: See Taylor and Pacelli (2008) for an algorithm for computing the Banzhaf power index which is very easy!

13. Compute the Banzhaf Power Index for the voters in the Windy Association board of directors in Exercise 5. (You will first need to list all the winning coalitions.)

Solution: The 28 winning coalitions, along with their critical defections, are as listed in the table:

Winning coalition	Critical defections	Winning coalition	Critical defections
Mary, Terry, Jules	Mary, Terry, Jules	Mary, Jules, Brian, Jim	Mary
Mary, Terry, Brian	Mary, Terry, Brian	Mary, Jules, Brian, Ted	Mary
Mary, Terry, Jim	Mary, Terry, Jim	Mary, Jules, Jim, Ted	Mary
Mary, Terry, Ted	Mary, Terry, Ted	Mary, Brian, Jim, Ted	Mary
Mary, Jules, Brian	Mary, Jules, Brian	Terry, Jules, Brian, Jim	Terry, Jules, Brian
Mary, Jules, Jim	Mary, Jules, Jim	Terry, Jules, Brian, Ted	Terry, Jules, Brian
Mary, Jules, Ted	Mary, Jules, Ted	Terry, Jules, Jim, Ted	Terry, Jules, Jim, Ted
Mary, Brian, Jim	Mary, Brian, Jim	Mary, Terry, Jules, Brian. Jim	None
Mary, Brian, Ted	Mary, Brian, Ted	Mary, Terry, Jules, Brian, Ted	None
Mary, Jim, Ted	Mary, Jim, Ted	Mary, Terry, Jules, Jim, Ted	None
Terry, Jules, Brian	Terry, Jules, Brian	Mary, Terry, Brian, Jim, Ted	Mary
Mary, Terry, Jules, Brian	None	Mary, Jules, Brian, Jim, Ted	Mary
Mary, Terry, Jules, Jim	Mary	Terry, Jules, Brian, Jim, Ted	Terry, Jules
Mary, Terry, Jules, Ted	Mary	Mary, Terry, Jules, Brian, Jim, Ted	None

The total Banzhaf power for each voter is then:

$$TBP(Mary) = 18$$
$$TBP(Terry) = 9$$
$$TBP(Jules) = 9$$
$$TBP(Brian) = 7$$
$$TBP(Jim) = 5$$
$$TBP(Ted) = 5$$

The total is 53, so

$$BI(Mary) = \frac{18}{53} \approx .3397$$

$$BI(Terry) = BI(Jules) = \frac{9}{53} \approx .1698$$
$$BI(Jim) = BI(Ted) = \frac{5}{53} \approx .09434$$

14. Compute the Banzhaf Index for the voters in the voting system of New England states in Exercise 7. (First list the winning coalitions.) This exercise shows that the Banzhaf index makes sense for non-weighted systems and is calculated the same way.

Solution: Denote each state by its first letter, except Ma for Massachusetts. The 29 winning coalitions, together with their critical defections, are listed in the table:

Winning coalition	Critical defections	Winning coalition	Critical defections	Winning coalition	Critical defections
C, Ma, M	C, Ma, M	C, Ma, M, V	Ma	Ma, M, N, V	Ma, M
C, Ma, R	C, Ma, R	C, Ma, R, N	C, Ma	M, R, N, V	M, R, N, V
C, Ma, N	C, Ma, N	C, Ma, R, V	C, Ma	C, Ma, M, R, N	None
C, Ma, V	C, Ma, V	C, Ma, N, V	C, Ma	C, Ma, M, R, V	None
C, M, R	C, M, R	C, M, R, N	C, M, R	C, Ma, M, N, V	None
Ma, M, R	Ma, M, R	C, M, R, V	C, M, R	C, Ma, R, N, V	C
Ma, M, N	Ma, M, N	C, M, N, V	C, M, N, V	C, M, R, N, V	None
Ma, M, V	Ma, M, V	C, R, N, V	C, R, N, V	Ma, M, R, N, V	M
C, Ma, M, R	none	Ma, M, R, N	Ma, M	C, Ma, M, R, N, V	None
C, Ma, M, N	Ma	Ma, M, R, V	Ma, M		

The total Banzhaf power for each voter is:

$$TBP(C) = 13$$
$$TBP(Ma) = 15$$
$$TBP(M) = 13$$
$$TBP(R) = 7$$
$$TBP(N) = TBP(V) = 5$$

The total is 58, so

$$BP(Ma) = \frac{15}{58} \approx .2586$$
$$BP(C) = BP(M) = \frac{13}{58} \approx .2241$$
$$BP(N) = BP(V) = \frac{5}{58} \approx .0862$$

15. A voting system can have more than one voter with veto power (see Exercise 1.) Can it have more than one dictator? Explain.

Solution: No, it cannot. Suppose A and B are dictators with $A \neq B$. Then $\{A\}$ is a winning coalition (because A is a dictator) which does not contain B, contradicting the fact that B is a dictator. Similarly, $\{B\}$ is a winning coalition to which A does not belong, so A cannot be a dictator. So, we must have $A = B$.

For more exercises on the Shapley-Shubik power index, see the exercises after Section 10.2.

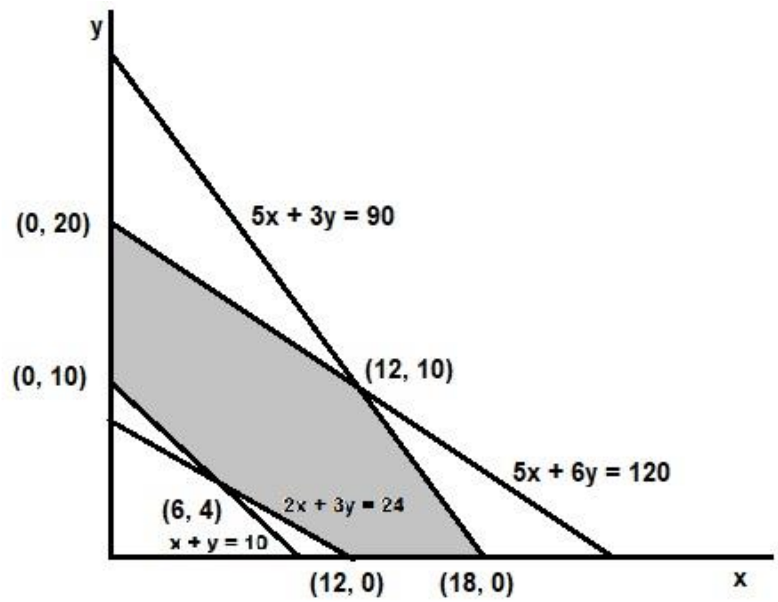
Chapter 9. More Linear Programming

Section 9.1 Phase I Pivoting

Solutions to exercises:

1. Solve the problem in Example 9.1 by graphing in the decision space. Note that the origin is not feasible.

Solution: The feasible set:



Evaluate the objective function at the corners:

(x, y)	$P = 2x + 3y$
(0,10)	30
(0,20)	60
(12,10)	54
(18,0)	36
(12,0)	24
(6,4)	24

Thus, the optimal solution is:

$$x = 0$$
$$y = 20$$

$$P = 60 \text{ (maximized)}$$
$$S_1 = 0$$
$$S_2 = 30$$
$$S_3 = 10$$
$$S_4 = 36$$

2. Finish doing the pivots in Example 9.1 and verify that the final tableau is as claimed in the text, and verify that the answer is the same as the one you obtained in Exercise 1. On your graph in Exercise 1, follow and mark the corners we covered in the pivots for this example so you can see which corners were visited on the way to the feasible set, and once there, to the optimal point.

Solution: In the text we pivoted as far as this tableau (As usual we shade in the next pivot row and column):

	x	y	S_1	S_2	S_3	S_4	$Capacity$
S_1	0	0	1	0	3	1	66
S_2	0	0	0	1	9	−2	48
x	1	0	0	0	−3	1	6
y	0	1	0	0	2	−1	4
P	0	0	0	0	0	−1	24

Pivoting on the shaded row and column leads to:

	x	y	S_1	S_2	S_3	S_4	$Capacity$
S_1	−1	0	1	0	6	0	60
S_2	2	0	0	1	3	0	60
S_4	1	0	0	0	−3	1	6
y	1	1	0	0	−1	0	10
P	1	0	0	0	−3	0	30

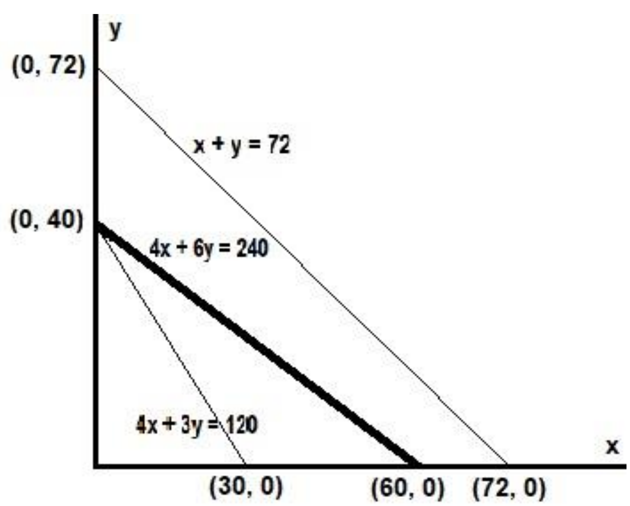
Performing the final pivot:

	x	y	S_1	S_2	S_3	S_4	$Capacity$
S_3	$-\frac{1}{6}$	0	$\frac{1}{6}$	0	1	0	10
S_2	$\frac{5}{2}$	0	$-\frac{1}{2}$	1	0	0	30
S_4	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	1	36
y	$\frac{5}{6}$	1	$\frac{1}{6}$	0	0	0	20
P	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0	60

This final tableau agrees with what was displayed in the text (and the optimal data agrees with Exercise 1.)

3. a) Solve the problem in Example 0.2 by graphing in the decision space and verify that the answer is the same as the one obtained in the text.

Solution: The feasible set is just the bold line segment (because of the exact equality constraint):



Evaluating the objective function at the two corners:

(x, y)	$P = 2x + 5y$
$(\mathbf{0}, \mathbf{40})$	200
$(\mathbf{60}, \mathbf{0})$	120

Thus, the optimal solution is:

$$x = 0$$
$$y = 40$$
$$P = 200 \text{ (maximized)}$$
$$S_1 = 32$$
$$A_2 = 0$$
$$S_3 = 0$$

This agrees with the answer in the text. Also, the fact that the problem is degenerate is visible in the graph – three constraints are concurrent at the optimal point $(0,40)$.

b) As suggested in the text, continue the pivoting after the second tableau to drive out the artificial variable.

Solution: The pivots in the text led to this tableau:

	x	y	S_1	S_3	A_2	A_3	$Capacity$
S_1	$\frac{2}{6}$	0	1	$-\frac{1}{6}$	0	0	32
y	$\frac{4}{6}$	1	0	$\frac{1}{6}$	0	0	40
$-A_3$	-2	0	0	$\frac{3}{6}$	1	-1	0
P	$\frac{8}{6}$	0	0	$\frac{5}{6}$	0	0	200
z	-2	0	0	$\frac{9}{6}$	1	0	0

The suggestion was to pivot on the third row, first column, even though this tableau represents an optimal value of z (and hence a feasible point in the original problem.) Performing this pivot leads to:

	x	y	S_1	S_3	A_2	A_3	$Capacity$
S_1	0	0	1	$-\frac{1}{12}$	$\frac{1}{6}$	$-\frac{1}{6}$	32
y	0	1	0	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	40
x	1	0	0	$-\frac{1}{4}$	$-\frac{1}{2}$	$\frac{1}{2}$	0
P	0	0	0	$\frac{7}{6}$	$\frac{2}{3}$	$-\frac{1}{3}$	200
z	0	0	0	1	0	1	0

This tableau still represents an optimal value of z since $z = 0$. However, now the artificial variables have been completely driven out of the solution. Furthermore, as claimed in the text, this actually represents the same geometric corner point – albeit with a different set of basic variables (possible because the problem is degenerate.) Indeed, reading off the data from the tableau:

$x = 0$ (now a basic variable with value 0)
$y = 40$
$P = 200$ (maximized, because the tableau is optimal
For P as well as for z)
$S_1 = 32$
$S_3 = 0$

And, of course, $A_2 = A_3 = 0$ because $z = 0$.

c) As suggested in the text, in the third tableau, after we deleted the artificial variable columns, finish the problem by pivoting in the third row, fourth column, instead of the choice we made in the text. Verify that the solution point is still the same, but with a different basic variable with value 0.

Solution: The pivoting in the text led to:

	x	y	S_1	S_3	$Capacity$
S_1	$\frac{2}{6}$	0	1	$-\frac{1}{6}$	32
y	$\frac{4}{6}$	1	0	$\frac{1}{6}$	40
$-A_3$	-2	0	0	$\frac{3}{6}$	0
P	$\frac{8}{6}$	0	0	$\frac{5}{6}$	200

The claim in the text is that we could pivot on the $\frac{3}{6}$ in the fourth column to get rid of the artificial variable from the basis, instead of pivoting on the -2 in the first column as we did in the text. Indeed, performing the pivot leads to:

	x	y	S_1	S_3	$Capacity$
S_1	-1	0	1	0	32
y	$\frac{4}{3}$	1	0	0	40
S_3	-4	0	0	1	0
P	$\frac{14}{3}$	0	0	0	200

Clearly, this tableau represents the same point $(0,40)$ as we obtained in the text, except that S_3 is a basic variable with value 0 (whereas with the pivot displayed in the text, S_3 ended up being non-basic.)

4. Solve the following problem:

$$\text{Maximize } P = 5x + 4y$$
$$\text{Subject to:}$$
$$6x + 4y \leq 200$$
$$x + 2y \leq 96$$
$$x + y = 50$$
$$x \geq 0,\ \ y \geq 0$$

Solution: According to our algorithm, we subtract an artificial variable A_3 from the exact equality constraint, and we first must maximize $z = -A_3$:

	x	y	S_1	S_2	A_3	$Capacity$
S_1	6	4	1	0	0	240
S_2	1	2	0	1	0	96
$-A_3$	-1	-1	0	0	1	-50
P	-5	-4	0	0	0	0
z	-1	-1	0	0	0	-50

Pivoting, we obtain:

	x	y	S_1	S_2	A_3	$Capacity$
x	1	$\frac{4}{6}$	$\frac{1}{6}$	0	0	40
S_2	0	$\frac{8}{6}$	$-\frac{1}{6}$	1	0	56
$-A_3$	0	$-\frac{2}{6}$	$\frac{1}{6}$	0	1	-10
P	0	$-\frac{4}{6}$	$\frac{5}{6}$	0	0	200
z	0	$-\frac{2}{6}$	$\frac{1}{6}$	0	0	-10

	x	y	S_1	S_2	A_3	$Capacity$
x	1	0	$\frac{3}{6}$	0	2	20
S_2	0	0	$\frac{3}{6}$	1	4	16
y	0	1	$-\frac{3}{6}$	0	-3	30
P	0	0	$\frac{3}{6}$	0	-2	220
z	0	0	0	0	-1	0

Having reached a feasible point, we can now delete the z row and the artificial variable column, and finish via phase II pivoting:

	x	y	S_1	S_2	$Capacity$
x	1	0	$\frac{3}{6}$	0	20
S_2	0	0	$\frac{3}{6}$	1	16
y	0	1	$-\frac{3}{6}$	0	30
P	0	0	$\frac{3}{6}$	0	220

However, this tableau is already optimal, so no further pivoting is required. The solution is:

$$x = 20$$
$$y = 30$$
$$P = 220 \text{ (maximized)}$$
$$S_1 = 0, \quad M_1 = \frac{1}{2}$$
$$S_2 = 16, \quad M_2 = 0$$
$$A_3 = 0$$

5. Solve the problem:

$$\text{Maximize } P = x + 4y$$
$$\text{Subject to:}$$
$$4x + 7y \leq 140$$
$$5x + 22y = 228$$
$$3x + 2y \geq 36$$
$$x \geq 0, \quad y \geq 0$$

Solution: The exact equality constraint and the greater-than type inequality are modified to include artificial variables:

$$-5x - 22y - A_2 = -228$$
$$-3x - 2y + S_3 - A_3 = -36$$

Thus,

$$z = -A_2 - A_3$$
$$= -228 + 5x + 22y + (-36 + 3x + 2y - S_3)$$
$$= 8x + 24y - S_3 - 264$$

The initial tableau is

	x	y	S_1	S_3	A_2	A_3	$Capacity$
S_1	4	7	1	0	0	0	140
$-A_2$	-5	-22	0	0	-1	0	-228
$-A_3$	-3	-2	0	1	0	-1	-36
P	-1	-4	0	0	0	0	0
z	-8	-24	0	1	0	0	-264

	x	y	S_1	S_3	A_2	A_3	$Capacity$
S_1	$\frac{53}{22}$	0	1	0	$-\frac{7}{22}$	0	$\frac{1484}{22}$
y	$\frac{5}{22}$	1	0	0	$\frac{1}{22}$	0	$\frac{228}{22}$
$-A_3$	$-\frac{56}{22}$	0	0	1	$\frac{2}{22}$	-1	$-\frac{336}{22}$
P	$-\frac{2}{22}$	0	0	0	$\frac{4}{22}$	0	$\frac{912}{22}$
z	$-\frac{56}{22}$	0	0	1	$\frac{24}{22}$	0	$-\frac{336}{22}$

	x	y	S_1	S_3	A_2	A_3	$Capacity$
S_1	0	0	1	$\frac{53}{56}$	$-\frac{13}{56}$	$\frac{53}{56}$	53
y	0	1	0	$\frac{5}{56}$	$\frac{3}{56}$	$-\frac{5}{56}$	9
x	1	0	0	$-\frac{22}{56}$	$-\frac{2}{56}$	$\frac{22}{56}$	6
P	0	0	0	$\frac{2}{56}$	$\frac{10}{56}$	$\frac{2}{56}$	42
z	0	0	0	0	1	1	0

Phase I is now complete since $z = 0$. We delete the z row and the columns headed with artificial variables and continue with Phase II:

	x	y	S_1	S_3	$Capacity$
S_1	0	0	1	$\frac{53}{56}$	53
y	0	1	0	$\frac{5}{56}$	9
x	1	0	0	$-\frac{22}{56}$	6
P	0	0	0	$\frac{2}{56}$	42

But this tableau represents an optimal point and no further pivoting is required. The solution is:

$$x = 6$$
$$y = 9$$
$$P = 42 \text{ (maximized)}$$
$$S_1 = 53, \quad M_1 = 0$$
$$S_3 = 0, \;\; M_3 = \frac{2}{56} = \frac{1}{28}$$

And, of course, $A_2 = A_3 = 0$.

6. In Section 3.4, you solved the following diet mix problem via a graphical method. Now solve it by the simplex method: *Kerry's Kennel* is mixing two commercial brands of dog food for its canine guests. A bag of *Dog's Life Canine Cuisine* contains 3 lbs. fat, 2 lbs. carbohydrates, 5 lbs. of protein, and 3 oz. vitamin C. A giant size bag of *Way of Life Healthy Mix* contains 1 lb. fat, 5 lbs. carbohydrates, 10 lbs. protein, and 7 oz. vitamin C. The requirements for a week's supply of food for the kennel are that there should be at most 21 lbs. fat, at most 40 lbs. carbohydrates, and at least 21 oz. vitamin C. How many bags of each type should be mixed in order to design a diet that maximizes protein?

Solution: Recall the setup of the problem: Let x be the number of bags of Dog's Life Canine Cuisine, and y the number of bags of Way of Life Healthy Mix. Then

Maximize Protein $P = 5x + 10y$

Subject to:

$3x + y \leq 21$ lbs. fat

$2x + 5y \leq 40$ lbs. carbohydrates

$3x + 7y \geq 21$ oz. vitamin C

$x \geq 0, \;\; y \geq 0$

We add appropriate artificial and slack variables to the third constraint to convert it to:

$$-3x - 7y + S_3 - A_3 = -21$$

Thus,

$$z = -A_3 = 3x + 7y - S_3 - 21$$

The initial tableau is

	x	y	S_1	S_2	S_3	A_3	$Capacity$
S_1	3	1	1	0	0	0	21
S_2	2	5	0	1	0	0	40
$-A_3$	-3	-7	0	0	1	-1	-21
P	-5	-10	0	0	0	0	0
z	-3	-7	0	0	1	0	-21

	x	y	S_1	S_2	S_3	A_3	$Capacity$
S_1	$\frac{18}{7}$	0	1	0	$\frac{1}{7}$	$-\frac{1}{7}$	18
S_2	$-\frac{1}{7}$	0	0	1	$\frac{5}{7}$	$-\frac{5}{7}$	25
y	$\frac{3}{7}$	1	0	0	$-\frac{1}{7}$	$\frac{1}{7}$	3
P	$-\frac{5}{7}$	0	0	0	$-\frac{10}{7}$	$\frac{10}{7}$	30
z	0	0	0	0	1	0	0

At this point, since $z = 0$, this tableau represents a feasible point and phase I is over. Deleting the z row and column headed by artificial variables, we are left with:

	x	y	S_1	S_2	S_3	$Capacity$
S_1	$\frac{18}{7}$	0	1	0	$\frac{1}{7}$	18
S_2	$-\frac{1}{7}$	0	0	1	$\frac{5}{7}$	25
y	$\frac{3}{7}$	1	0	0	$-\frac{1}{7}$	3
P	$-\frac{5}{7}$	0	0	0	$-\frac{10}{7}$	30

We continue with Phase II:

	x	y	S_1	S_2	S_3	$Capacity$
S_1	$\frac{13}{5}$	0	1	$-\frac{1}{5}$	0	13
S_3	$-\frac{1}{5}$	0	0	$\frac{7}{5}$	1	35
y	$\frac{2}{5}$	1	0	$\frac{1}{5}$	0	8
P	-1	0	0	2	0	80

	x	y	S_1	S_2	S_3	$Capacity$
x	1	0	$\frac{5}{13}$	$-\frac{1}{13}$	0	5
S_3	0	0	$\frac{1}{13}$	$\frac{18}{13}$	1	36
y	0	1	$-\frac{2}{13}$	$\frac{3}{13}$	0	6
P	0	0	$\frac{5}{13}$	$\frac{25}{13}$	0	85

We have reached the optimal tableau. The solution is:

$x = 5$ bags of Dog's Life Canine Cuisine

$y = 6$ bags of Way of Life Healthy Mix

$P = 85$ lbs. Protein (Maximized)

$S_1 = 0$ lbs. Fat $\quad M_1 = \frac{5}{13}$ lbs. protein/lb. fat

$S_2 = 0$ lbs. Carbohydrate, $\quad M_2 = \frac{25}{13}$ lbs. protein/lb. carbohydrate

$S_3 = 36$ oz. surplus Vitamin C, $\quad M_3 = 0$

This agrees with our previous solution.

7. Solve the following linear programming problem from Section 3.4 via the simplex algorithm:
The *Jefferson Plastic Fantastic Assembly Corporation* manufactures gadgets and widgets for airplanes and starships. Each case of gadgets uses 2 kg. steel and 5 kg. plastic. Each case of widgets uses 2 kg. steel and 3 kg. plastic. The profit for a case of gadgets is $360 and the profit for a case of widgets is $200. Suppose they have 80 kg. steel available and 150 kg. plastic available on a daily basis, and can sell everything they manufacture. How many cases of each should they manufacture if they are obligated to produce at least 10 cases of widgets per day? The objective is to maximize daily profit.

Solution: Recall the setup of the problem: Let x be the number of cases of gadgets produced per day and y the number of widgets produced per day. Then

Maximize Profit $P = 360x + 200y$
Subject to:
$2x + 2y \leq 80$ kg. steel
$5x + 3y \leq 150$ kg. plastic
$y \geq 10$ (contractual obligation)
$x \geq 0, \ \ y \geq 0$

We now modify the third constraint:

$$-y + S_3 - A_3 = -10,$$

so that

$$z = -A_3 = y - S_3 - 10$$

Phase I:

Initial tableau:

	x	y	S_1	S_2	S_3	$-A_3$	$Capacity$
S_1	2	2	1	0	0	0	80
S_2	5	3	0	1	0	0	150
$-A_3$	0	-1	0	0	1	-1	-10
P	-360	-200	0	0	0	0	0
z	0	-1	0	0	1	0	-10

	x	y	S_1	S_2	S_3	$-A_3$	$Capacity$
S_1	2	0	1	0	2	−2	60
S_2	5	0	0	1	3	−3	120
y	0	1	0	0	−1	1	10
P	−360	0	0	0	−200	200	2000
z	0	0	0	0	0	1	0

This tableau represents a feasible point because $z = 0$. We continues with Phase II, after deleting the z row and the column headed by an artificial variable:

	x	y	S_1	S_2	S_3	$Capacity$
S_1	2	0	1	0	2	60
S_2	5	0	0	1	3	120
y	0	1	0	0	−1	10
P	−360	0	0	0	−200	2000

	x	y	S_1	S_2	S_3	$Capacity$
S_1	0	0	1	$-\frac{2}{5}$	$\frac{4}{5}$	12
x	1	0	0	$\frac{1}{5}$	$\frac{3}{5}$	24
y	0	1	0	0	−1	10
P	0	0	0	72	16	10640

This tableau represents an optimal point, so the solution is:

$x = 24$ cases Gadgets
$y = 10$ cases Widgets
$P = \$10{,}640$ (Maximized)
$S_1 = 12$ kg. leftover steel, $M_1 = 0$
$S_2 = 0$ kg. leftover plastic, $M_2 = \$72$/kg. plastic
$S_3 = 0$ surplus cases widgets, $M_3 = \$16$/case

This agrees with our graphical solution.

8. Solve the following linear programming problem from Section 3.4 via the simplex algorithm: Mr. Cooder, a farmer in the Purple Valley, has at most 400 acres to devote to two crops: rye and barley. Each acre of rye yields \$100 profit per week, while each acre of barley yields \$80 profit per week. Due to local demand, Mr. Cooder must plant at least 100 acres of barley. The federal government provides a subsidy to grow these crops in the form of tax credits. They credit Mr. Cooder 4 units for each acre of rye and 2 units for each acre of barley. The exact value of a 'unit' of tax credit is not important. Mr. Cooder has decided that he needs at least 600 units of tax credits in order to be able to afford his loan payments on a new harvester. How many acres of each crop should he plant in order to maximize his profit?

Solution: Recall the setup: Let x be the number of acres of rye and y the number of acres of barley. Then

$$\text{Maximize weekly profit } P = 100x + 80y$$
$$\text{Subject to:}$$
$$x + y \leq 400 \ (acres\ land)$$
$$y \geq 100 \ (acres\ barley\ demand)$$
$$4x + 2y \geq 600 \ (unit\ tax\ credit)$$
$$x \geq 0, \qquad y \geq 0$$

The modified greater than type constraints are:

$$-y + S_2 - A_2 = -100$$
$$-4x - 2y + S_3 - A_3 = -600$$

This leads to:

$$z = -A_2 - A_3 = 4x + 3y - S_2 - S_3 - 700$$

Phase I:

	x	y	S_1	S_2	S_3	A_2	A_3	$Capacity$
S_1	1	1	1	0	0	0	0	400
$-A_2$	0	-1	0	1	0	-1	0	-100
$-A_3$	-4	-2	0	0	1	0	-1	-600
P	-100	-80	0	0	0	0	0	0
z	-4	-3	0	1	1	0	0	-700

	x	y	S_1	S_2	S_3	A_2	A_3	$Capacity$
S_1	0	$\frac{1}{2}$	1	0	$\frac{1}{4}$	0	$-\frac{1}{4}$	250
$-A_2$	0	-1	0	1	0	-1	0	-100
x	1	$\frac{1}{2}$	0	0	$-\frac{1}{4}$	0	$\frac{1}{4}$	150
P	0	-30	0	0	-25	0	25	15000
z	0	-1	0	1	0	0	1	-100

	x	y	S_1	S_2	S_3	A_2	A_3	$Capacity$
S_1	0	0	1	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{2}$	$-\frac{1}{4}$	200
y	0	1	0	-1	0	1	0	100
x	1	0	0	$\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{2}$	$\frac{1}{4}$	100
P	0	0	0	-30	-25	30	25	18000
z	0	0	0	0	0	1	1	0

This tableau represents a feasible point, so we delete the z row and the columns headed by artificial variables, and continue with Phase II:

	x	y	S_1	S_2	S_3	$Capacity$
S_1	0	0	1	$\frac{1}{2}$	$\frac{1}{4}$	200
y	0	1	0	-1	0	100
x	1	0	0	$\frac{1}{2}$	$-\frac{1}{4}$	100
P	0	0	0	-30	-25	18000

	x	y	S_1	S_2	S_3	$Capacity$
S_1	-1	0	1	0	$\frac{1}{2}$	100
y	2	1	0	0	$-\frac{1}{2}$	300
S_2	2	0	0	1	$-\frac{1}{2}$	200
P	60	0	0	0	-40	24000

	x	y	S_1	S_2	S_3	$Capacity$
S_3	-2	0	2	0	1	200
y	1	1	1	0	0	400
S_2	1	0	1	1	0	300
P	-20	0	80	0	0	32000

	x	y	S_1	S_2	S_3	$Capacity$
S_3	0	0	4	2	1	800
y	0	1	0	-1	0	100
x	1	0	1	1	0	300
P	0	0	100	20	0	38000

The optimal solution is:

$x = 300$ acres rye
$y = 100$ acres barley
$p = \$38{,}000$ (Maximized)
$S_1 = 0$ leftover acres land, $M_1 = \$100/\text{acre}$
$S_2 = 0$ surplus acres barley, $M_2 = \$20/\text{acre barley}$
$S_3 = 800$ units tax credits, $M_3 = 0$

This agrees with our previous solution.

9. Solve the following linear programming problem from Section 3.1: Arnold and Penny Layne have up to \$60,000 to invest for a one-year period. There are three investment options they are considering. Certificates of Deposit, which have an expected annual return of 3%, Municipal bonds (the proceeds of which go to the cleaning and upkeep of local busses, police cars, and fire engines), which have an expected annual return of 5%, and Stocks on women's apparel, with an expected annual return of 11%. They would like to maximize their expected return for the year, but are adhering to the following guidelines suggested by their financial advisor:

- The amount invested in Municipal Bonds should be at least \$10,000.
- Because of stock volatility, the amount invested in Stocks should be at most \$10,000 more than the amount invested in Bonds.
- Because of the reliability of Certificates of Deposit, at least $\frac{1}{3}$ of the total investment should be in Certificates of Deposit.

Solution: Recall the setup: Let x be the amount invest in Certificates of Deposit, y the amount invested in Municipal Bonds, and z the amount invested in Stocks of women's apparel.

$$\text{Maximize Interest} \quad R = .03x + .05y + .11z$$

Subject to:

$x + y + z \leq 60$ (amount invested)
$y \geq 10000$ (municipal bonds)

$$-y + z \leq 10000 \text{ (stock volatility)}$$
$$2x - y - z \geq 0 \text{ (reliability of CD's)}$$
$$x \geq 0, \quad y \geq 0, \quad z \geq 0$$

Now rewrite the fourth constraint as:

$$-2x + y + z \leq 0$$

By doing so, we do not need an artificial variable for that constraint (possible because the right side of the inequality is 0.) Then, the second constraint is modified to say:

$$-y + S_2 - A_2 = -10000$$

Which leads to (using a capital Z to distinguish from z, the amount invested in Stocks of women's apparel.)

$$Z = -A_2 = y - S_2 = 10000$$

Phase I:

	x	y	z	S_1	S_2	S_3	S_4	A_2	$Capacity$
S_1	1	1	1	1	0	0	0	0	60000
$-A_2$	0	-1	0	0	1	0	0	-1	-10000
S_3	0	-1	1	0	0	1	0	0	10000
S_4	-2	1	1	0	0	0	1	0	0
R	$-.03$	$-.05$	$-.11$	0	0	0	0	0	0
Z	0	-1	0	0	1	0	0	0	-10000

The 0 in the capacity column indicates that this problem is degenerate. Nevertheless, after two pivots, we arrive at the following tableau:

	x	y	z	S_1	S_2	S_3	S_4	A_2	$Capacity$
S_1	1	0	$\frac{3}{2}$	1	$\frac{3}{2}$	0	$\frac{1}{2}$	$-\frac{3}{2}$	45000
x	0	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	5000
S_3	0	0	1	0	-1	1	0	1	20000
y	0	1	0	0	-1	0	0	1	10000
R	0	0	$-.125$	0	$-.065$	0	$-.015$	$.065$	650
Z	0	0	0	0	0	0	0	1	0

This completes phase I. Deleing the appropriate rows and columns, we continue with Phase II:

	x	y	z	S_1	S_2	S_3	S_4	$Capacity$
S_1	1	0	$\frac{3}{2}$	1	$\frac{3}{2}$	0	$\frac{1}{2}$	45000
x	0	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$	5000
S_3	0	0	1	0	-1	1	0	20000
y	0	1	0	0	-1	0	0	10000
R	0	0	$-.125$	0	$-.065$	0	$-.015$	650

After two more pivots, we obtain:

	x	y	z	S_1	S_2	S_3	S_4	$Capacity$
S_2	0	0	0	$\frac{1}{3}$	1	$-\frac{1}{2}$	$\frac{1}{6}$	5000
x	1	0	0	$\frac{1}{3}$	0	0	$-\frac{1}{3}$	20000
z	0	0	1	$\frac{1}{3}$	0	$\frac{1}{2}$	$\frac{1}{6}$	25000
y	0	1	0	$\frac{1}{3}$	0	$-\frac{1}{2}$	$\frac{1}{6}$	15000
R	0	0	0	$.04667$	0	$.055$	$.008333$	3850

The optimal solution is:

$x = \$20{,}000$ invested in CDs
$y = \$15{,}000$ invested in municipal bonds
$z = \$25{,}000$ invested in women's apparel stocks
$R = \$3{,}850$ Interest earned (Maximized)
$S_1 = 0$ leftover funds to invest, $M_1 = \$.04667$ interest/$ invested
$S_2 = \$5{,}000$ surplus invested in municipal bonds, $M_2 = 0$
$S_3 = 0,\ M_3 = \$.055$
$S_4 = 0,\quad M_4 = \$.00833$

10. a) Solve the following problem:

Maximize $P = 2x + 3y + 5z$
Subject to:
$x + y + 2z \leq 60$

$$2x + y + z \geq 12$$
$$x + 2y + z \geq 10$$
$$7x + 8y + 5z \geq 54$$
$$x \geq 0, \quad y \geq 0, \quad z \geq 0$$

Solution: Adding the appropriate artificial variables, the three modified constraints become:

$$-2x - y - z + S_2 - A_2 = -12$$
$$-x - 2y - z + S_3 - A_3 = -10$$
$$-7x - 8y - 5z + S_4 - A_4 = -54$$

This leads to (again using a capital Z to avoid confusion with the decision variable z):

$$Z = -A_2 - A_3 - A_4 = 10x + 11y + 7z - S_2 - S_3 - S_4 - 76$$

Phase I:

	x	y	z	S_1	S_2	S_3	S_4	A_2	A_3	A_4	$Capacity$
S_1	1	1	2	1	0	0	0	0	0	0	60
$-A_2$	-2	-1	-1	0	1	0	0	-1	0	0	-12
$-A_3$	-1	-2	-1	0	0	1	0	0	-1	0	-10
$-A_4$	-7	-8	-5	0	0	0	1	0	0	-1	-54
P	-2	-3	-5	0	0	0	0	0	0	0	0
Z	-10	-11	-7	0	1	1	1	0	0	0	76

After two pivots, we arrive at the following tableau:

	x	y	z	S_1	S_2	S_3	S_4	A_2	A_3	A_4	$Capacity$
S_1	0	0	$\frac{4}{3}$	1	$\frac{1}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$	$-\frac{1}{3}$	0	$\frac{158}{3}$
x	1	0	$\frac{1}{3}$	0	$-\frac{2}{3}$	$\frac{1}{3}$	0	$\frac{2}{3}$	$-\frac{1}{3}$	0	$\frac{14}{3}$
y	0	1	$\frac{1}{3}$	0	$\frac{1}{3}$	$-\frac{2}{3}$	0	$-\frac{1}{3}$	$\frac{2}{3}$	0	$\frac{8}{3}$
$-A_4$	0	0	0	0	-2	-3	1	2	3	-1	0
P	0	0	$-\frac{10}{3}$	0	$-\frac{1}{3}$	$-\frac{4}{3}$	0	$\frac{1}{3}$	$\frac{4}{3}$	0	$\frac{52}{3}$
Z	0	0	0	0	-2	-3	1	3	4	0	0

Although there are still negative numbers in the last row, this tableau represents a feasible point because $Z = 0$. In order to continue with Phase II, we must first drive the artificial variable out of solution:

	x	y	z	S_1	S_2	S_3	S_4	$Capacity$
S_1	0	0	$\frac{4}{3}$	1	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{158}{3}$
x	1	0	$\frac{1}{3}$	0	$-\frac{2}{3}$	$\frac{1}{3}$	0	$\frac{14}{3}$
y	0	1	$\frac{1}{3}$	0	$\frac{1}{3}$	$-\frac{2}{3}$	0	$\frac{8}{3}$
$-A_4$	0	0	0	0	-2	-3	1	0
P	0	0	$-\frac{10}{3}$	0	$-\frac{1}{3}$	$-\frac{4}{3}$	0	$\frac{52}{3}$

After five pivots (the first of which changes nothing in the table except the row label $-\boldsymbol{A_4}$ is replaced by the column label $\boldsymbol{S_4}$), we arrive at the final tableau:

	x	y	z	S_1	S_2	S_3	S_4	$Capacity$
S_2	-1	0	1	1	1	0	0	48
S_3	1	0	3	2	0	1	0	110
y	1	1	2	1	0	0	0	60
S_4	1	0	11	8	0	0	1	426
P	1	0	1	3	0	0	0	180

Thus, the optimal solution is:

$$x = 0$$
$$y = 60$$
$$z = 0$$
$$P = 180 \text{ (Maximized)}$$
$$S_1 = 0, \quad M_1 = 3$$
$$S_2 = 48, \quad M_2 = 0$$
$$S_3 = 110, \quad M_3 = 0$$
$$S_4 = 426, \quad M_4 = 0$$

b) Show that the fourth constraint is a linear combination of the second and third, and is therefore redundant. (Therefore, an alternate approach to solving the problem is to delete the fourth constraint altogether, before starting any pivoting.)

Solution: Indeed, two times the second constraint plus three times the third is exactly the fourth.

11. Solve the brainteaser in Exercise 21 of Section 3.1 using the simplex algorithm.

Solution: Recall the setup: Let x be the hundreds digit, y the tens digit, and z the units' digit. Then $N = 100x + 10y + z$ is what is to be maximized. The number $M = 100z + 10y + x$. Notice that $N - M = 99x - 99z$, and $99x - 99z \leq 693$ is equivalent to $x - z \leq 7$. Finally note that digits must be at most 9, while x cannot be 0 since we stated that N is a three-digit number. Thus:

$$Maximize\ N = 100x + 10y + z$$

Subject to:

$$\begin{gathered} x + y + z \leq 12 \\ x - y - z \leq 4 \\ x - z \leq 7 \\ x \leq 9 \\ y \leq 9 \\ z \leq 9 \\ x \geq 1 \\ y \geq 0, \quad z \geq 0 \end{gathered}$$

(**Remark**: Also, x, y, z must be integers.) Ignoring the integer constraint for the moment, we set up Phase I by adding an artificial variable just to the last constraint (the greater than type inequality):

$$-x + S_7 - A_7 = -1$$

This leads to:

$$Z = -A_7 = x - S_7 - 1$$

	x	y	z	S_1	S_2	S_3	S_4	S_5	S_6	S_7	A_7	***Capacity***
S_1	1	1	1	1	0	0	0	0	0	0	0	12
S_2	1	−1	−1	0	1	0	0	0	0	0	0	4
S_3	1	0	−1	0	0	1	0	0	0	0	0	7
S_4	1	0	0	0	0	0	1	0	0	0	0	9
S_5	0	1	0	0	0	0	0	1	0	0	0	9
S_6	0	0	1	0	0	0	0	0	1	0	0	9
$-A_7$	−1	0	0	0	0	0	0	0	0	1	−1	−1
N	−100	−10	−1	0	0	0	0	0	0	0	0	0
Z	−1	0	0	0	0	0	0	0	0	−1	0	−1

After one pivot, we arrive at a tableau representing a feasible point:

	x	y	z	S_1	S_2	S_3	S_4	S_5	S_6	S_7	A_7	$Capacity$
S_1	0	1	1	1	0	0	0	0	0	1	−1	11
S_2	0	−1	−1	0	1	0	0	0	0	1	−1	3
S_3	0	0	−1	0	0	1	0	0	0	1	−1	6
S_4	0	0	0	0	0	0	1	0	0	1	−1	8
S_5	0	1	0	0	0	0	0	1	0	0	0	9
S_6	0	0	1	0	0	0	0	0	1	0	0	9
x	1	0	0	0	0	0	0	0	0	−1	1	1
N	0	−10	−1	0	0	0	0	0	0	−100	100	100
Z	0	0	0	0	0	0	0	0	0	−2	1	0

We delete the Z row and the A_7 column, and continue with phase II:

	x	y	z	S_1	S_2	S_3	S_4	S_5	S_6	S_7	$Capacity$
S_1	0	1	1	1	0	0	0	0	0	1	11
S_2	0	−1	−1	0	1	0	0	0	0	1	3
S_3	0	0	−1	0	0	1	0	0	0	1	6
S_4	0	0	0	0	0	0	1	0	0	1	8
S_5	0	1	0	0	0	0	0	1	0	0	9
S_6	0	0	1	0	0	0	0	0	1	0	9
x	1	0	0	0	0	0	0	0	0	−1	1
N	0	−10	−1	0	0	0	0	0	0	−100	100

After three more pivots, we arrive at the final tableau:

	x	y	z	S_1	S_2	S_3	S_4	S_5	S_6	S_7	$Capacity$
z	0	0	1	$\frac{1}{2}$	$\frac{1}{2}$	-1	0	0	0	0	1
S_7	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	1	7
y	0	1	0	0	-1	1	0	0	0	0	3
S_4	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	1	0	0	0	1
S_5	0	0	0	0	1	-1	0	1	0	0	6
S_6	0	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	1	0	0	1	0	8
x	1	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	0	8
N	0	0	0	$\frac{101}{2}$	$\frac{81}{2}$	9	0	0	0	0	831

Thus, the solution to the brainteaser is $N = 831$.

12. Solve Problem 4 above using Algorithm 9.5 (dual pivoting.)

Solution: The initial tableau is like that in Exercise 4, except without the z row and without the A_3 column:

	x	y	S_1	S_2	$Capacity$
S_1	6	4	1	0	240
S_2	1	2	0	1	96
$-A_3$	-1	-1	0	0	-50
P	-5	-4	0	0	0

We first select the row with the artificial variable (to satisfy the exact equality constraints first.) The ratios of the entries in the objective row to the (nonzero) entries in the pivot row (shaded) are 5, 4, and we select the maximum ratio 5. Thus, the pivot column is the x column.

	x	y	S_1	S_2	$Capacity$
S_1	0	-2	1	0	-60
S_2	0	1	0	1	46
x	1	1	0	0	50
P	0	1	0	0	250

Continuing with this algorithm, the next tableau is:

	x	y	S_1	S_2	$Capacity$
y	0	1	$-\frac{1}{2}$	0	30
S_2	0	0	$\frac{1}{2}$	1	16
x	1	0	$\frac{1}{2}$	0	20
P	0	0	$\frac{1}{2}$	0	220

This is now a feasible point, since every entry in the Capacity column is nonnegative. We continue with phase II. However, all the entries in the Objective row are also nonnegative so this point is optimal. The solution agrees with what we found in Exercise 4.

13. Solve Problem 6 above using Algorithm 9.5 (dual pivoting.)

Solution: Phase I (via Algorithm 9.5):

	x	y	S_1	S_2	S_3	$Capacity$
S_1	3	1	1	0	0	21
S_2	2	5	0	1	0	40
$-A_3$	-3	-7	0	0	1	-21
P	-5	-10	0	0	0	0

	x	y	S_1	S_2	S_3	$Capacity$
S_1	0	-6	1	0	1	0
S_2	0	$\frac{1}{3}$	0	1	$\frac{2}{3}$	26
x	1	$\frac{7}{3}$	0	0	$-\frac{1}{3}$	7
P	0	$\frac{5}{3}$	0	0	$-\frac{5}{3}$	35

This tableau represents a feasible point, so we are done with Phase I, and continue with phase II:

	x	y	S_1	S_2	S_3	$Capacity$
S_3	0	-6	1	0	1	0
S_2	0	$\frac{13}{3}$	$-\frac{2}{3}$	1	0	26
x	1	$\frac{1}{3}$	$\frac{1}{3}$	0	0	7
P	0	$-\frac{25}{3}$	$\frac{5}{3}$	0	0	35

	x	y	S_1	S_2	S_3	$Capacity$
S_3	0	0	$\frac{1}{13}$	$\frac{18}{13}$	1	36
y	0	1	$-\frac{2}{13}$	$\frac{3}{13}$	0	6
x	1	0	$\frac{5}{13}$	$-\frac{1}{13}$	0	5
P	0	0	$\frac{5}{13}$	$\frac{25}{13}$	0	85

This is the final tableau, and the solution agrees with what we found in Exercise 6.

14. Solve Problem 7 above using Algorithm 9.5 (dual pivoting.)

Solution: Phase I:

	x	y	S_1	S_2	S_3	$Capacity$
S_1	2	2	1	0	0	80
S_2	5	3	0	1	0	150
$-A_3$	0	-1	0	0	1	-10
P	-360	-200	0	0	0	0

	x	y	S_1	S_2	S_3	$Capacity$
S_1	2	0	1	0	2	60
S_2	5	0	0	1	3	120
y	-360	1	0	0	-1	10
P	-360	0	0	0	-200	2000

This represents a feasible point so phase I is complete. We continue with phase II:

	x	y	S_1	S_2	S_3	$Capacity$
S_1	1	0	1	$-\frac{2}{5}$	$\frac{4}{5}$	12
x	0	0	0	$\frac{1}{5}$	$\frac{3}{5}$	24
y	0	1	0	0	-1	10
P	0	0	0	72	16	10640

This is the optimal tableau, and the solution agrees with what we found in Exercise 7.

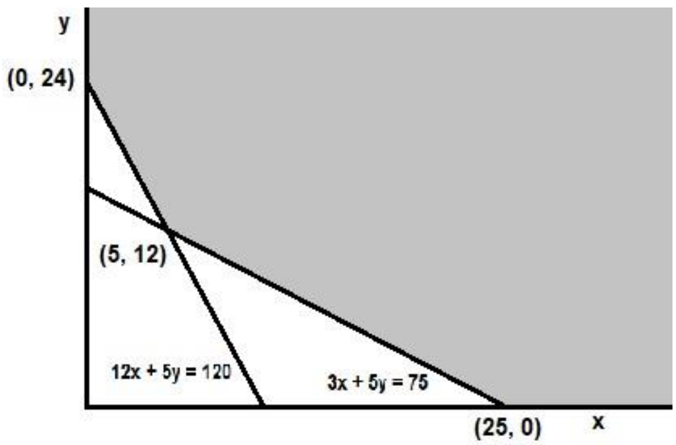
Section 9.2 Alternate Approach to Minimization Problems

Music Reference in the text: The "Sunday Noir" Medical Labs of Example 9.7 alludes to the band Black Sabbath. They released songs with titles 'Paranoid' (1970), 'Iron Man' (1970), 'Sweet Leaf' (1971) and 'Black Moon' (1987). (Website: https://www.blacksabbath.com/)

Solutions to exercises:

1. a) Solve Example 9.6 above by graphing in the decision space, and verify your answer agrees with what we got before.

Solution: The feasible set:



Evaluating the objective function at the corners:

(x, y)	$C = 12x + 8y$
$(0, 24)$	192
$(5, 12)$	156
$(25, 0)$	300

The solution is:

$$x = 5$$
$$y = 12$$
$$C = 156 \text{ (Minimized)}$$
$$S_1 = 0$$
$$S_2 = 0$$

This agrees with the solution in the text.

b) Since the example 9.1 is in standard form, you can also solve it by using duality. Do so, and verify that you obtain the same answer as we did above.

Solution: The dual problem is:

$$\text{Maximize } P = 120u + 75v$$
$$\text{Subject to:}$$
$$12u + 3v \leq 12$$
$$5u + 5v \leq 8$$
$$u \geq 0, \quad v \geq 0$$

The simplex method (phase II) applied to the dual problem:

	u	v	T_1	T_2	$Capacity$
T_1	12	3	1	0	12
T_2	5	5	0	1	8
P	-120	-75	0	0	0

	u	v	T_1	T_2	$Capacity$
u	1	$\frac{3}{12}$	$\frac{1}{12}$	0	1
T_2	0	$\frac{45}{12}$	$-\frac{5}{12}$	1	3
P	0	-45	10	0	120

	u	v	T_1	T_2	$Capacity$
u	1	0	$\frac{5}{45}$	$-\frac{3}{45}$	$\frac{36}{45}$
v	0	1	$-\frac{5}{45}$	$\frac{12}{45}$	$\frac{36}{45}$
P	0	0	5	12	156

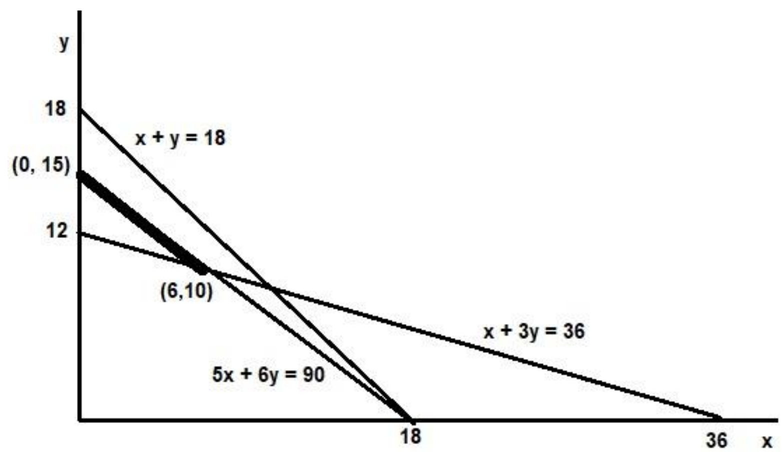
So, the solution to the primal problem is:

$$x = 5$$
$$y = 12$$
$$C = 156 \text{ (Minimized)}$$
$$S_1 = 0, \quad M_1 = u = \frac{36}{45} = \frac{4}{5}$$
$$S_2 = 0, \quad M_2 = v = \frac{36}{45} = \frac{4}{5}$$

This agrees with the solution we got in part (a), as well as with the marginal values we obtained in the text.

2. a) Solve Example 9.7 above by graphing in the decision space, and verify the answer we obtained above.

Solution: The feasible set is the line segment in bold:



Evaluating the objective function at the corners:

(x, y)	$C = 3x + 5y$
$(0, 15)$	75
$(6, 10)$	68

Thus, the solution is:

$$x = 6$$
$$y = 10$$
$$C = 68 \text{ (Minimized)}$$
$$S_1 = 2$$
$$A_2 = 0$$
$$S_3 = 0$$

This agrees with the solution in the text.

b) Complete the pivoting in Example 9.2 above.

Solution:

	x	y	S_1	S_3	A_2	A_3	$Capacity$
S_1	1	1	1	0	0	0	18
$-A_2$	−5	−6	0	0	−1	0	−90
$-A_3$	−1	−3	0	1	0	−1	−36
$-C$	3	5	0	0	0	0	0
z	−6	−9	0	1	0	0	−126

	x	y	S_1	S_3	A_2	A_3	$Capacity$
S_1	$\frac{2}{3}$	0	1	$\frac{1}{3}$	0	$-\frac{1}{3}$	6
$-A_2$	-3	0	0	-2	-1	2	-18
y	$\frac{1}{3}$	1	0	$-\frac{1}{3}$	0	$\frac{1}{3}$	12
$-C$	$\frac{4}{3}$	0	0	$\frac{5}{3}$	0	$-\frac{5}{3}$	-60
z	-3	0	0	-2	0	3	-18

	x	y	S_1	S_3	A_2	A_3	$Capacity$
S_1	0	0	1	$-\frac{1}{9}$	$-\frac{2}{9}$	$\frac{1}{9}$	2
x	1	0	0	$\frac{2}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	6
y	0	1	0	$-\frac{5}{9}$	$-\frac{1}{9}$	$\frac{5}{9}$	10
$-C$	0	0	0	$\frac{7}{9}$	$-\frac{4}{9}$	$-\frac{7}{9}$	-68
z	0	0	0	0	1	1	0

Phase I is complete as $z = 0$. We delete the appropriate row and columns for phase II:

	x	y	S_1	S_3	$Capacity$
S_1	0	0	1	$-\frac{1}{9}$	2
x	1	0	0	$\frac{2}{3}$	6
y	0	1	0	$-\frac{5}{9}$	10
$-C$	0	0	0	$\frac{7}{9}$	-68

However, this tableau is already optimal, so no further pivoting is required. The solution is:

$$x = 6$$
$$y = 10$$
$$C = 68 \text{ (Minimized)}$$
$$S_1 = 2, \quad M_1 = 0$$
$$A_2 = 0, \quad M_2 = -\tfrac{4}{9}$$
$$S_3 = 0, \quad M_3 = \frac{7}{9}$$

This agrees with our previous solutions and with the solution in the text.

Remark: The value of M_2 is negative (visible on the table before we deleted the A_2 column) because it is an exact equality constraint. Marginal values of such constraints are unrestricted in sign.

3. Solve Exercise 4 from Section 3.4, reproduced below, using the method of this section:

$$\text{Minimize } C = 20x + 12y$$
$$\text{Subject to}$$
$$5x + 2y \geq 30$$
$$5x + 7y \geq 70$$
$$x \geq 0, \quad y \geq 0$$

Solution: So, we must maximize $-C = -20x - 12y$, and the constraints are adjusted with artificial variables to read:

$$-5x - 2y + S_1 - A_1 = -30$$
$$-5x - 7y + S_2 - A_2 = -70$$

This leads to:

$$z = -A_1 - A_2 = 10x + 9y - S_1 - S_2 - 100$$

Phase I:

	x	y	S_1	S_2	A_1	A_2	$Capacity$
$-A_1$	-5	-2	1	0	-1	0	-30
$-A_2$	-5	-7	0	1	0	-1	-70
$-C$	20	12	0	0	0	0	0
z	-10	-9	1	1	0	0	-100

	x	y	S_1	S_2	A_1	A_2	$Capacity$
x	1	$\frac{2}{5}$	$-\frac{1}{5}$	0	$\frac{1}{5}$	0	6
$-A_2$	0	-5	-1	1	1	-1	-40
$-C$	0	4	4	0	-4	0	-120
z	0	-5	-1	1	2	0	-40

	x	y	S_1	S_2	A_1	A_2	$Capacity$
x	1	0	$-\frac{7}{25}$	$\frac{2}{25}$	$\frac{7}{25}$	$-\frac{2}{25}$	$\frac{14}{5}$
y	0	1	$\frac{1}{5}$	$-\frac{1}{5}$	$-\frac{1}{5}$	$\frac{1}{5}$	8
$-C$	0	0	$\frac{16}{5}$	$\frac{4}{5}$	$-\frac{16}{5}$	$-\frac{4}{5}$	-152
z	0	0	0	0	1	1	0

Since $z = 0$, phase I is complete. Continuing with phase II:

	x	y	S_1	S_2	$Capacity$
x	1	0	$-\frac{7}{25}$	$\frac{2}{25}$	$\frac{14}{5}$
y	0	1	$\frac{1}{5}$	$-\frac{1}{5}$	8
$-C$	0	0	$\frac{16}{5}$	$\frac{4}{5}$	-152

But this represents an optimal point, so no further pivoting is required. The solution is:

$$x = \frac{14}{5} = 2.8$$
$$y = 8$$
$$C = 152 \text{ (Minimized)}$$
$$S_1 = 0, \quad M_1 = \frac{16}{5} = 3.2$$
$$S_2 = 0, \quad M_2 = \frac{4}{5} = .8$$

This agrees with the solution we found in Section 3.4.

4. Solve Exercise 8 from Section 3.4, reproduced below, using the method of this section: *Toys in the Attic, Inc*. operates two workshops to build toys for needy children. Mr. Tyler's shop can produce 36 *Angel* dolls, 16 *Kings and Queens* board games, and 16 *Back in the Saddle* rocking horses each day it operates. Mr. Perry's shop can produce 10 *Angel* dolls, 10 *Kings and Queens* board games, and 20 *Back in the Saddle* rocking horses each day it operates. It costs $144 to operate Mr. Tyler's shop for one day and $166 to operate Mr. Perry's shop for one day. Suppose the company receives an order from *Kids Dream On* Charity Foundation for at least 720 *Angel* dolls, at least 520 *Kings and Queens* board games,

and at least 640 *Back in the Saddle* rocking horses. How many days should they operate each shop in order to fill the order at least possible cost?

Solution: Recall the setup: : Let x be the number of days to operate Mr. Tyler's shop and y the number of days to operate Mr. Perry's shop.

$$\text{Minimize Cost } C = 144x + 166y$$

Subject to:

$$36x + 10y \geq 720 \text{ Angel Dolls}$$
$$16x + 10y \geq 520 \quad \text{Kings and Queens Board Games}$$
$$16x + 20y \geq 640 \quad \text{Back in the Saddle Rocking Horses}$$
$$x \geq 0 \qquad y \geq 0$$

So, we must Maximize $-C = -144x - 166y$. The constraints are modified as follows:

$$-36x - 10y + S_1 - A_1 = -720$$
$$-16x - 10y + S_2 - A_2 = -520$$
$$-16x - 20y + S_3 - A_3 = -640$$

This leads to:

$$z = -A_1 - A_2 - A_3 = 68x + 40y - S_1 - S_2 - S_3 = -1880$$

Phase i:

	x	y	S_1	S_2	S_3	A_1	A_2	A_3	$Capacity$
$-A_1$	-36	-10	1	0	0	-1	0	0	-720
$-A_2$	-16	-10	0	1	0	0	-1	0	-520
$-A_3$	-16	-20	0	0	1	0	0	-1	-640
$-C$	144	166	0	0	0	0	0	0	0
z	-68	-40	1	1	1	0	0	0	-1880

	x	y	S_1	S_2	S_3	A_1	A_2	A_3	$Capacity$
x	1	$\frac{10}{36}$	$-\frac{1}{36}$	0	0	$\frac{1}{36}$	0	0	20
$-A_2$	0	$-\frac{200}{36}$	$-\frac{16}{36}$	1	0	$\frac{16}{36}$	-1	0	-200
$-A_3$	0	$\frac{560}{36}$	$-\frac{16}{36}$	0	1	$\frac{16}{36}$	0	-1	-320
$-C$	0	126	4	0	0	-4	0	0	-2880
z	0	$-\frac{760}{36}$	$-\frac{32}{36}$	1	1	$\frac{68}{36}$	0	0	-520

	x	y	S_1	S_2	S_3	A_1	A_2	A_3	$Capacity$
x	1	0	$-\frac{1}{28}$	0	$\frac{1}{56}$	$\frac{1}{28}$	0	$-\frac{1}{56}$	$\frac{100}{7}$
$-A_2$	0	0	$-\frac{2}{7}$	1	$-\frac{5}{14}$	$\frac{2}{7}$	-1	$\frac{5}{14}$	$-\frac{600}{7}$
y	0	1	$\frac{1}{35}$	0	$-\frac{9}{140}$	$-\frac{1}{35}$	0	$\frac{9}{140}$	$\frac{144}{7}$
$-C$	0	0	$\frac{2}{5}$	0	$\frac{81}{10}$	$-\frac{2}{5}$	0	$-\frac{81}{10}$	-5472
z	0	0	$-\frac{2}{7}$	1	$-\frac{5}{14}$	$\frac{9}{7}$	0	$\frac{19}{14}$	$-\frac{600}{7}$

	x	y	S_1	S_2	S_3	A_1	A_2	A_3	$Capacity$
x	1	0	$-\frac{1}{20}$	$\frac{1}{20}$	0	$\frac{1}{20}$	$-\frac{1}{20}$	0	10
S_3	0	0	$\frac{4}{5}$	$-\frac{14}{5}$	1	$-\frac{4}{5}$	$\frac{14}{5}$	-1	240
y	0	1	$\frac{2}{25}$	$-\frac{9}{50}$	0	$-\frac{2}{25}$	$\frac{9}{50}$	0	36
$-C$	0	0	$-\frac{152}{25}$	$\frac{567}{25}$	0	$\frac{152}{25}$	$-\frac{567}{25}$	0	-7416
z	0	0	0	0	0	1	1	1	0

This completes phase I. Continuing with phase II:

	x	y	S_1	S_2	S_3	$Capacity$
x	1	0	$-\frac{1}{20}$	$\frac{1}{20}$	0	10
S_3	0	0	$\frac{4}{5}$	$-\frac{14}{5}$	1	240
y	0	1	$\frac{2}{25}$	$-\frac{9}{50}$	0	36
$-C$	0	0	$-\frac{152}{25}$	$\frac{567}{25}$	0	-7416

	x	y	S_1	S_2	S_3	$Capacity$
x	1	0	0	$-\frac{1}{8}$	$\frac{1}{16}$	25
S_1	0	0	1	$\frac{7}{2}$	$\frac{5}{4}$	300
y	0	1	0	$\frac{1}{10}$	$-\frac{1}{10}$	12
$-C$	0	0	0	$\frac{7}{5}$	$\frac{38}{5}$	-5592

This completes phase II. The solution is:

$x = 25$ days operating Mr. Tyler's shop
$y = 12$ days operating Mr. Perry's shop
$C = \$5{,}592$ (Minimized)
$S_1 = 300$ surplus Angel Dolls, $M_1 = 0$
$S_2 = 0$ surplus Kings and Queens board games, $M_2 = \frac{7}{5} = \$1.40$/game
$S_3 = 0$ surplus Back in the Saddle rocking horses, $M_3 = \frac{38}{5} = \$7.60$/rocking horse

This agrees with our previous solutions.

5. Solve the following exercise from Section 3.4 using the method of this section: The *Poseidon's Wake* petroleum company operates two refineries. The *Cadence Refinery* can produce 40 units of low grade oil, 10 units medium grade oil, and 10 units high grade oil in a single day. (Each unit is 1000 barrels.) The *Cascade Refinery* can produce 10 units low grade oil, 10 units medium grade oil, and 30 units high grade oil in a single day. They receive an order from the Mars Triangle Oil Retailers for at least 80 units low grade oil, at least 50 units medium grade oil, and at least 90 units high grade oil. If it costs Poseidon's Wake \$1,800 to operate the Cadence refinery for a day, and \$2,000 to operate the Cascade Refinery for a day, how many days should they operate each refinery to fill the order at least cost?

Solution: Recall the setup: Let x be the number of days to operate the Cadence refinery, and y the number of days to operate the Cascade refinery. Then

Minimize Cost $C = 1800x + 2000y$
Subject to:
$$40x + 10y \geq 80\ (units\ low\ grade\ oil)$$
$$10x + 10y \geq 50\ (units\ medium\ grade\ oil)$$
$$10x + 30y \geq 90\ (units\ high\ grade\ oil)$$
$$x \geq 0, \qquad y \geq 0$$

We must maximize $-C = -1800x - 2000y$, and modify the constraints as follows:

$$-40x - 10y + S_1 - A_1 = -80$$
$$-10x - 10y + S_2 - A_2 = -50$$
$$-10x - 30y + S_3 - A_3 = -90$$

This leads to:

$$z = -A_1 - A_2 - A_3 = 60x + 50y - S_1 - S_2 - S_3 - 220$$

Phase I:

	x	y	S_1	S_2	S_3	A_1	A_2	A_3	$Capacity$
$-A_1$	-40	-10	1	0	0	-1	0	0	-80
$-A_2$	-10	-10	0	1	0	0	-1	0	-50
$-A_3$	-10	-30	0	0	1	0	0	-1	-90
$-C$	1800	2000	0	0	0	0	0	0	0
z	-60	-50	1	1	1	0	0	0	-220

	x	y	S_1	S_2	S_3	A_1	A_2	A_3	$Capacity$
x	1	$\frac{1}{4}$	$-\frac{1}{40}$	0	0	$\frac{1}{40}$	0	0	2
$-A_2$	0	$-\frac{15}{2}$	$-\frac{1}{4}$	1	0	$\frac{1}{4}$	-1	0	-30
$-A_3$	0	$-\frac{55}{2}$	$-\frac{1}{4}$	0	1	$\frac{1}{4}$	0	-1	-70
$-C$	0	1550	45	0	0	-45	0	0	-3600
z	0	-35	$-\frac{1}{2}$	1	1	$\frac{3}{2}$	0	0	-100

	x	y	S_1	S_2	S_3	A_1	A_2	A_3	$Capacity$
x	1	0	$-\frac{3}{110}$	0	$\frac{1}{110}$	$\frac{3}{110}$	0	$-\frac{1}{110}$	$\frac{15}{11}$
$-A_2$	0	0	$-\frac{2}{11}$	1	$-\frac{3}{11}$	$\frac{2}{11}$	-1	$\frac{3}{11}$	$-\frac{120}{11}$
y	0	1	$\frac{1}{110}$	0	$-\frac{2}{55}$	$-\frac{1}{110}$	0	$\frac{2}{55}$	$\frac{28}{11}$
$-C$	0	0	$\frac{340}{11}$	0	$\frac{620}{11}$	$-\frac{340}{11}$	0	$-\frac{620}{11}$	$-\frac{83000}{11}$
z	0	0	$-\frac{2}{11}$	1	$-\frac{3}{11}$	$\frac{13}{11}$	0	$\frac{14}{11}$	$-\frac{120}{11}$

	x	y	S_1	S_2	S_3	A_1	A_2	A_3	*Capacity*
x	1	0	$-\frac{1}{30}$	$\frac{1}{30}$	0	$\frac{1}{30}$	$-\frac{1}{30}$	0	1
S_3	0	0	$\frac{2}{3}$	$-\frac{11}{3}$	1	$-\frac{2}{3}$	$\frac{11}{3}$	-1	40
y	0	1	$\frac{1}{30}$	$-\frac{2}{15}$	0	$-\frac{1}{30}$	$\frac{2}{15}$	0	4
$-C$	0	0	$-\frac{20}{3}$	$\frac{620}{3}$	0	$-\frac{20}{3}$	$-\frac{620}{3}$	0	-9800
z	0	0	0	0	0	1	1	1	0

Since $z = 0$, phase I is complete. Continuing with phase II:

	x	y	S_1	S_2	S_3	*Capacity*
x	1	0	$-\frac{1}{30}$	$\frac{1}{30}$	0	1
S_3	0	0	$\frac{2}{3}$	$-\frac{11}{3}$	1	40
y	0	1	$\frac{1}{30}$	$-\frac{2}{15}$	0	4
$-C$	0	0	$-\frac{20}{3}$	$\frac{620}{3}$	0	-9800

	x	y	S_1	S_2	S_3	*Capacity*
x	1	0	0	$-\frac{3}{20}$	$\frac{1}{20}$	3
S_1	0	0	1	$-\frac{11}{2}$	$\frac{3}{2}$	60
y	0	1	0	$\frac{1}{20}$	$-\frac{1}{20}$	2
$-C$	0	0	0	170	10	-9400

That completes phase II. The optimal solution is:

$x = 3$ days operating Cadence refinery
$y = 2$ days operating Cascade refinery
$C = \$9,400$ (Minimized)

$$S_1 = 60 \text{ units surplus low grade oil,} \quad M_1 = 0$$
$$S_2 = 0 \text{ units surplus medium grade oil,} \quad M_2 = \$170/\text{unit}$$
$$S_3 = 0 \text{ units surplus high grade oil,} \quad M_3 = \$10/\text{unit}$$

This agrees with our previous solutions.

6. Solve the following problem from Section 3.4 using the method of this section:

Minimize $z = 15x + 51y$
Subject to:
$$x + y \leq 100$$
$$3x + 10y \geq 300$$
$$18x + 10y \geq 900$$
$$-6x + 50y \geq 240$$
$$x \geq 0, \quad y \geq 0$$

Solution: Let us use C for the objective function (instead of z) to avoid confusion. Then we must maximize $-C = -15x - 51y$. We adjust the last three constraints as follows:

$$-3x - 10y + S_2 - A_2 = -300$$
$$-18x - 10y + S_3 - A_3 = -900$$
$$6x - 50y + S_4 - A_4 = -240$$

This leads to

$$z = -A_2 - A_3 - A_4 = 15x + 70y - S_2 - S_3 - S_4 - 1440$$

Phase I

	x	y	S_1	S_2	S_3	S_4	A_2	A_3	A_4	***Capacity***
S_1	1	1	1	0	0	0	0	0	0	100
$-A_2$	−3	−10	0	1	0	0	−1	0	0	−300
$-A_3$	−18	−10	0	0	1	0	0	−1	0	−900
$-A_4$	6	−50	0	0	0	1	0	0	−1	−240
$-C$	15	51	0	0	0	0	0	0	0	0
z	−15	−70	0	1	1	1	0	0	0	−1440

	x	y	S_1	S_2	S_3	S_4	A_2	A_3	A_4	***Capacity***
S_1	$\frac{28}{25}$	0	1	0	0	$\frac{1}{50}$	0	0	$-\frac{1}{50}$	$\frac{476}{5}$
$-A_2$	$-\frac{21}{5}$	0	0	1	0	$-\frac{1}{5}$	-1	0	$\frac{1}{5}$	-252
$-A_3$	$-\frac{96}{5}$	0	0	0	1	$-\frac{1}{5}$	0	-1	$\frac{1}{5}$	-852
y	$-\frac{3}{25}$	1	0	0	0	$-\frac{1}{50}$	0	0	$\frac{1}{50}$	$\frac{24}{5}$
$-C$	$\frac{528}{25}$	0	0	0	0	$\frac{51}{20}$	0	0	$-\frac{51}{50}$	$-\frac{1224}{5}$
z	$-\frac{117}{5}$	0	0	1	1	$-\frac{2}{5}$	0	0	$\frac{7}{5}$	-1104

	x	y	S_1	S_2	S_3	S_4	A_2	A_3	A_4	***Capacity***
S_1	0	0	1	0	$\frac{7}{120}$	$\frac{1}{120}$	0	$-\frac{7}{120}$	$-\frac{1}{120}$	$\frac{91}{2}$
$-A_2$	0	0	0	1	$-\frac{7}{32}$	$-\frac{5}{32}$	1	$\frac{7}{32}$	$\frac{5}{32}$	$-\frac{525}{8}$
x	1	0	0	0	$-\frac{5}{96}$	$\frac{1}{96}$	0	$\frac{5}{96}$	$-\frac{1}{96}$	$\frac{355}{8}$
y	0	1	0	0	$-\frac{1}{160}$	$-\frac{3}{160}$	0	$\frac{1}{160}$	$\frac{3}{160}$	$\frac{81}{8}$
$-C$	0	0	0	0	$\frac{11}{10}$	$\frac{4}{5}$	0	$-\frac{11}{10}$	$-\frac{4}{5}$	-1182
z	0	0	0	1	$-\frac{7}{32}$	$-\frac{5}{32}$	0	$\frac{39}{32}$	$\frac{37}{32}$	$-\frac{525}{8}$

	x	y	S_1	S_2	S_3	S_4	A_2	A_3	A_4	***Capacity***
S_1	0	0	1	$\frac{4}{15}$	0	$-\frac{1}{30}$	$-\frac{4}{15}$	0	$\frac{1}{30}$	28
S_3	0	0	0	$-\frac{32}{7}$	1	$\frac{5}{7}$	$\frac{32}{7}$	-1	$-\frac{5}{7}$	300
x	1	0	0	$-\frac{5}{21}$	0	$\frac{1}{21}$	$\frac{5}{21}$	0	$-\frac{1}{21}$	60
y	0	1	0	$-\frac{1}{35}$	0	$-\frac{1}{70}$	$\frac{1}{35}$	0	$\frac{1}{70}$	12
$-C$	0	0	0	$\frac{176}{35}$	0	$\frac{1}{70}$	$-\frac{176}{35}$	0	$-\frac{1}{70}$	-1512
z	0	0	0	0	0	0	1	1	1	0

Since $z = 0$, this tableau represents a feasible point so phase I is complete. We continue with phase II:

	x	y	S_1	S_2	S_3	S_4	$Capacity$
S_1	0	0	1	$\frac{4}{15}$	0	$-\frac{1}{30}$	28
S_3	0	0	0	$-\frac{32}{7}$	1	$\frac{5}{7}$	300
x	1	0	0	$-\frac{5}{21}$	0	$\frac{1}{21}$	60
y	0	1	0	$-\frac{1}{35}$	0	$-\frac{1}{70}$	12
$-C$	0	0	0	$\frac{176}{35}$	0	$\frac{1}{70}$	-1512

But this is an optimal tableau, so no further pivoting is required. The solution is:

$$x = 60$$
$$y = 12$$
$$C = (the\ original\ z) = 1512 \text{ (Minimized)}$$
$$S_1 = 28, \quad M_1 = 0$$
$$S_1 = 0, \quad M_1 = \frac{176}{35} \approx 5.0286$$
$$S_3 = 300, \quad M_3 = 0$$
$$S_4 = 0, \quad M_4 = \frac{1}{70} \approx 0.014286$$

This agrees with the previous solution.

7. Solve the following problem from Section 3.4 using the method of this section: As a result of a federal discrimination lawsuit, the town of Yankee, New York is required to build a low-income housing project. The outcome of the lawsuit specifies that Yankee should build enough units to be able to house at least 44 adults and at least 72 children. They must also meet a separate requirement to be able to house at least 120 people altogether. They have available up to 54,000 square feet on which to build. Each townhouse requires 1,800 square feet and can house 6 people (2 adults and 4 children.) Each apartment requires 1,500 square feet and can house 4 people (2 adults and 2 children.) Each townhouse costs $100,000 to build and each apartment costs $80,000 to build. How many of each type of housing unit should they build in order to minimize the total cost?

Solution: Recall the setup: Let x be the number of townhouses and y the number of apartments. Then the town must

$$\text{Minimize Cost } C = 100{,}000x + 80{,}000y$$

Subject to:

$$2x + 2y \geq 44 \text{ (adults housed)}$$
$$4x + 2y \geq 72 \text{ (children housed}$$
$$6x + 4y \geq 120 \text{ (people housed)}$$
$$1800x + 1500y \leq 54{,}000 \text{ (square feet of land)}$$
$$x \geq 0, \qquad y \geq 0$$

Also, for a realistic model, x and y must be integers. We will ignore the integer constraint. We must maximize $-C = -100000x - 80000y$. We modify the first three constraints:

$$-2x - 2y + S_1 - A_1 = -44$$
$$-4x - 2y + S_2 - A_2 = -72$$
$$-6x - 4y + S_3 - A_3 = -120$$

This leads to:

$$z = -A_1 - A_2 - A_3 = 12x + 8y - S_1 - S_2 - S_3 - 236$$

Phase I:

	x	y	S_1	S_2	S_3	S_4	A_1	A_2	A_3	$Capacity$
$-A_1$	-2	-2	1	0	0	0	-1	0	0	-44
$-A_2$	-4	-2	0	1	0	0	0	-1	0	-72
$-A_3$	-6	-4	0	0	1	0	0	0	-1	-120
S_4	1800	1500	0	0	0	1	0	0	0	54000
$-C$	100000	80000	0	0	0	0	0	0	0	0
z	-12	-8	1	1	1	0	0	0	0	-236

	x	y	S_1	S_2	S_3	S_4	A_1	A_2	A_3	$Capacity$
$-A_1$	0	-1	1	$-\frac{1}{2}$	0	0	-1	$\frac{1}{2}$	0	-8
x	1	$\frac{1}{2}$	0	$-\frac{1}{4}$	0	0	0	$\frac{1}{4}$	0	18
$-A_3$	0	-1	0	$-\frac{3}{2}$	1	0	0	$\frac{3}{2}$	-1	-12
S_4	0	600	0	450	0	1	0	-450	0	21600
$-C$	0	30000	0	25000	0	0	0	-25000	0	-1800000
z	0	-2	1	-2	1	0	0	3	0	-20

	x	y	S_1	S_2	S_3	S_4	A_1	A_2	A_3	$Capacity$
y	0	1	-1	$\frac{1}{2}$	0	0	1	$-\frac{1}{2}$	0	8
x	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	14
$-A_3$	0	0	-1	-1	1	0	1	1	-1	-4
S_4	0	0	600	150	0	1	-600	-150	0	16800
$-C$	0	0	30000	10000	0	0	-30000	-10000	0	-2040000
z	0	0	-1	-2	1	0	2	2	0	-4

	x	y	S_1	S_2	S_3	S_4	A_1	A_2	A_3	$Capacity$
y	0	1	$-\frac{3}{2}$	0	$\frac{1}{2}$	0	$\frac{3}{2}$	0	$-\frac{1}{2}$	6
x	1	0	1	0	$-\frac{1}{2}$	0	-1	0	$\frac{1}{2}$	16
S_2	0	0	1	1	-1	0	-1	-1	1	4
S_4	0	0	450	0	150	1	-450	0	-150	1620
$-C$	0	0	20000	0	10000	0	-20000	0	-10000	-2080000
z	0	0	0	0	0	0	1	1	1	0

Since $z = 0$, phase I is complete. We continue with phase II:

	x	y	S_1	S_2	S_3	S_4	$Capacity$
y	0	1	$-\frac{3}{2}$	0	$\frac{1}{2}$	0	6
x	1	0	1	0	$-\frac{1}{2}$	0	16
S_2	0	0	1	1	-1	0	4
S_4	0	0	450	0	150	1	1620
$-C$	0	0	20000	0	10000	0	-2080000

However, this tableau is optimal, so no further pivoting is required. The optimal solution is:

$$x = 16 \text{ townhouses}$$
$$y = 6 \text{ apartments}$$
$$C = \$2{,}080{,}000 \text{ (Minimized)}$$
$$S_1 = 0 \text{ surplus adults housed}, \quad M_1 = \$20{,}000/ \text{ adult}$$
$$S_2 = 4 \text{ surplus children housed}, \quad M_2 = 0$$
$$S_3 = 0 \text{ surplus people housed}, \quad M_1 = \$10{,}000/ \text{ person}$$
$$S_4 = 1{,}620 \text{ leftover square feet of land}, \quad M_4 = 0$$

This agrees with our previous solution

8. Consider the diet problem in Exercise 16 of Section 3.1. Suppose that Derek decides to ignore his constraints on carbohydrates and salt. Solve the modified problem, using the method of this section.

Solution: The setup of the modified problem is: Let s be the number of ounces of garden salad, v the number of ounces of grilled vegetables, p the number of ounces of pasta, m the number of ounces of meatballs, and x the number of ounces of curried chicken salad. Then Derek must:

$$\text{Minimize Cost } C = 25s + 40v + 30p + 50m + 60x$$

Subject to:

$$s + v + 3p + 6m + 4x \leq 40 \text{ (grams fat)}$$
$$v + 2p + 10m + 12x \geq 80 \text{ (grams protein)}$$
$$15s + 20v + 40p + 60m + 50x \leq 700 \text{ (Calories)}$$
$$s \geq 0, \quad v \geq 0, \quad p \geq 0, \quad m \geq 0, \quad x \geq 0$$

We must maximize $-C = -25s - 40v - 30p - 50m - 60x$. The protein constraint must be adjusted with an artificial variable:

$$-v - 2p - 10m - 12x + S_2 - A_2 = -80$$

This leads to

$$z = -A_2 = v + 2p + 10m + 12x - S_2 - 80$$

Phase I:

	s	v	p	m	x	S_1	S_2	S_3	A_2	$Capacity$
S_1	1	1	3	6	4	1	0	0	0	40
$-A_2$	0	-1	-2	-10	-12	0	1	0	-1	-80
S_3	15	20	40	60	50	0	0	1	0	700
$-C$	25	40	30	50	60	0	0	0	0	0
z	0	-1	-2	-10	-12	0	1	0	0	-80

	s	v	p	m	x	S_1	S_2	S_3	A_2	$Capacity$
S_1	1	$\frac{2}{3}$	$\frac{7}{3}$	$\frac{8}{3}$	0	1	$\frac{1}{3}$	0	$-\frac{1}{3}$	$\frac{40}{3}$
x	0	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{5}{6}$	1	0	$-\frac{1}{12}$	0	$\frac{1}{12}$	$\frac{20}{3}$
S_3	15	$\frac{95}{6}$	$\frac{95}{3}$	$\frac{55}{3}$	0	0	$\frac{25}{6}$	1	$-\frac{25}{6}$	$\frac{1100}{3}$
$-C$	25	35	20	0	0	0	5	0	-5	-400
z	0	0	0	0	0	0	0	0	1	0

This completes phase I. We continue with phase II:

	s	v	p	m	x	S_1	S_2	S_3	$Capacity$
S_1	1	$\frac{2}{3}$	$\frac{7}{3}$	$\frac{8}{3}$	0	1	$\frac{1}{3}$	0	$\frac{40}{3}$
x	0	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{5}{6}$	1	0	$-\frac{1}{12}$	0	$\frac{20}{3}$
S_3	15	$\frac{95}{6}$	$\frac{95}{3}$	$\frac{55}{3}$	0	0	$\frac{25}{6}$	1	$\frac{1100}{3}$
$-C$	25	35	20	0	0	0	5	0	-400

But this tableau represents an optimal point, so no further pivoting is required. The solution is:

$s = 0$ Oz. salad
$v = 0$ Oz. vegetables
$p = 0$ Oz. pasta
$m = 0$ Oz. meatballs
$x = \frac{20}{3} \approx 6.667$ Oz. curried chicken salad
$C = \$4.00$ (Minimized)
$S_1 = \frac{40}{3} \approx 13.33$ g fat below limit, $M_1 = 0$
$S_2 = 0$ g carbohydrates below limit, $M_2 = 5 = \$.05/\text{g}$
$S_3 = \frac{1100}{3} \approx 366.67$ calories below limit, $M_3 = 0$

9. Consider the diet problem in Exercise 18 of Section 3.1. Suppose that Phil decides to ignore the sugar constraint. Solve the modified problem, using the method of this section.

Solution: The setup of the modified problem is: Let s be the number of ounces of garden salad, v the number of ounces of grilled vegetables, p the number of ounces of pasta, m the number of ounces of meatballs, and x the number of ounces of curried chicken salad. Phil must

$$\text{Minimize Calories } Z = 15s + 20v + 40p + 60m + 50x$$
$$\text{Subject to:}$$
$$s + v + 3p + 6m + 4x \leq 20 \text{ (grams fat)}$$
$$3s + 4v + 12p + 6m + 3x \leq 50 \text{ (grams carbohydrates)}$$
$$25s + 40v + 30p + 50m + 60x \leq 600 \text{ (cost)}$$
$$s + v + p + m + x \geq 12 \text{ (Oz. food)}$$
$$s \geq 0, \quad v \geq 0, \quad p \geq 0, \quad m \geq 0, \quad x \geq 0$$

Let's denote the calories by C instead of Z to avoid confusion. So, we maximize $-C = -15s - 20v - 40p - 60m - 50x$. The fourth constraint needs an artificial variable:

$$-s - v - p - m - x + S_4 - A_4 = -12$$

This leads to

$$z = -A_4 - s + v + p + m + x - S_4 - 12$$

Phase I:

	s	v	p	m	x	S_1	S_2	S_3	S_4	A_4	$Capacity$
S_1	1	1	3	6	4	1	0	0	0	0	20
S_2	3	4	12	6	3	0	1	0	0	0	50
S_3	25	40	30	50	60	0	0	1	0	0	600
$-A_4$	-1	-1	-1	-1	-1	0	0	0	1	-1	-12
$-C$	15	20	40	60	50	0	0	0	0	0	0
z	-1	-1	-1	-1	-1	0	0	0	1	0	-12

	s	v	p	m	x	S_1	S_2	S_3	S_4	A_4	$Capacity$
S_1	0	0	2	5	3	1	0	0	1	-1	8
S_2	0	1	9	3	0	0	1	0	3	-3	14
S_3	0	15	5	25	35	0	0	1	25	-25	300
s	1	1	1	1	1	0	0	0	-1	1	12
$-C$	0	5	25	45	35	0	0	0	15	-15	-180
z	0	0	0	0	0	0	0	0	0	1	0

This completes phase I. We continue with phase II:

	s	v	p	m	x	S_1	S_2	S_3	S_4	$Capacity$
S_1	0	0	2	5	3	1	0	0	1	8
S_2	0	1	9	3	0	0	1	0	3	14
S_3	0	15	5	25	35	0	0	1	25	300
s	1	1	1	1	1	0	0	0	-1	12
$-C$	0	5	25	45	35	0	0	0	15	-180

However, this tableau represents an optimal point, so no further pivoting is required. The solution is:

$s = 12$ oz. salad
$v = 0$ oz. veggies
$p = 0$ oz. pasta
$m = 0$ oz. meatballs
$x = 0$ oz. curried chicken salad
$C(= Z) = 180$ calories (Minimized)
$S_1 = 8$ g (below limit) fat, $M_1 = 0$
$S_2 = 14$ g (below limit) carbohydrates, $M_2 = 0$
$S_3 = \$3.00$ (below limit), $M_3 = 0$
$S_4 = 0$ oz. surplus food, $M_4 = 15$ cal./oz. food

10. Solve the transportation problem in Exercise 20 of Section 3.1, using the method of this section. [**Remark**: Be very careful with your arithmetic! This problem is quite tedious. The tableaux are large - 7×15 matrices, and it takes 9 pivots (invoking Bland's rule several times time break ties) to arrive at the final tableau. But, the good news is the matrices have entries that are mostly 0,1, or -1. This makes it much easier to pivot than would be the case otherwise. All transportation problems share this unusual feature in the setup of the problem. I was able to do all the pivoting by hand without hurrying in about 45 minutes. On the other hand, this problem has only two supply points and three destinations. Most real-life transportation problems are much larger, making the pivoting by hand nearly impossible and even making the solution by computer slow and expensive. Because of this, special techniques have been developed for solving transportation problems which do not rely on the simplex algorithm. However, a downside is many of these other methods find a solution which is feasible and near optimal, but often not actually optimal. We do not cover these methods in this book. The interested reader can explore some of the more well-known of these methods, such as the Northwest Corner Method, the Least Cost Method, and Vogel's Approximation Method, in more specialized books on linear programming, including Calvert and Voxman (1989) or Loomba (1976). A search on-line will lead you to other methods for solving transportation problems, and even to video tutorials on how to implement these methods.]

Solution: Recall the setup: Let u be the number of kilograms shipped from Atlanta to Chicago, v the number of kilograms shipped from Atlanta to Denver, w the number of kilograms shipped from Atlanta to San Francisco, x the number of kilograms shipped from Boston to Chicago, y the number of kilograms shipped from Boston to Denver, and z the number of kilograms shipped from Boston to San Francisco. Then:

$$\text{Minimize Cost } C = 30u + 45v + 80w + 40x + 40y + 70z$$

Subject to:

$$u + v + w \leq 330 \text{ (kg. stored in Atlanta)}$$
$$x + y + z \leq 450 \text{ (kg. stored in Boson)}$$
$$u + x \geq 300 \text{ (kg. requested by Chicago)}$$
$$v + y \geq 120 \text{ (kg. requested by Denver)}$$
$$w + z \geq 360 \text{ (kg. requested by San Francisco)}$$
$$u \geq 0, \quad v \geq 0, \quad w \geq 0, \quad x \geq 0, \quad y \geq 0, \quad z \geq 0$$

We must maximize $-C = -30u - 45v - 80w - 40x - 40y - 70z$, and we modify the last three constraints to include artificial variables:

$$-u - x + S_3 - A_3 = -300$$
$$-v - y + S_4 - A_4 = -120$$
$$-w - z + S_5 - A_5 = -360$$

This leads to

$$Z = -A_3 - A_4 - A_5 = u + v + w + x + y + z - S_3 - S_4 - S_5 - 780$$

(where we are using the capital Z to distinguish it from the decision variable z.)

Phase I:

	u	v	w	x	y	z	S_1	S_2	S_3	S_4	S_5	A_3	A_4	A_5	*Capacity*
S_1	1	1	1	0	0	0	1	0	0	0	0	0	0	0	330
S_2	0	0	0	1	1	1	0	1	0	0	0	0	0	0	450
$-A_3$	−1	0	0	−1	0	0	0	0	1	0	0	−1	0	0	−300
$-A_4$	0	−1	0	0	−1	0	0	0	0	1	0	0	−1	0	−120
$-A_5$	0	0	−1	0	0	−1	0	0	0	0	1	0	0	−1	−360
$-C$	30	45	80	40	40	70	0	0	0	0	0	0	0	0	0
Z	−1	−1	−1	−1	−1	−1	0	0	1	1	1	0	0	0	−780

After six pivots, you should arrive at the following tableau:

	u	v	w	x	y	z	S_1	S_2	S_3	S_4	S_5	A_3	A_4	A_5	*Capacity*
y	0	1	0	0	1	0	0	0	0	−1	0	0	1	0	120
z	−1	−1	0	1	0	1	0	1	1	1	0	−1	−1	0	30
w	1	1	1	0	0	0	1	0	0	0	0	0	0	0	330
x	1	0	0	1	0	0	0	0	−1	0	0	1	0	0	300
$-A_5$	0	0	0	0	0	0	1	1	1	1	1	−1	−1	−1	0
$-C$	−20	−5	0	0	0	0	−80	−70	−30	−30	0	30	30	0	−45300
Z	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0

Since $Z = 0$, phase I is complete, except for driving the basic artificial variable A_5 out of solution. Deleting the appropriate rows and columns:

	u	v	w	x	y	z	S_1	S_2	S_3	S_4	S_5	*Capacity*
y	0	1	0	0	1	0	0	0	0	−1	0	120
z	−1	−1	0	1	0	1	0	1	1	1	0	30
w	1	1	1	0	0	0	1	0	0	0	0	330
x	1	0	0	1	0	0	0	0	−1	0	0	300
$-A_5$	0	0	0	0	0	0	1	1	1	1	1	0
$-C$	−20	−5	0	0	0	0	−80	−70	−30	−30	0	−45300

Pivot on the shaded cell to drive A_5 out of solution. Since it is in a unit column, none of the rows change – just the label on the fifth row:

	u	v	w	x	y	z	S_1	S_2	S_3	S_4	S_5	*Capacity*
y	0	1	0	0	1	0	0	0	0	−1	0	120
z	−1	−1	0	1	0	1	0	1	1	1	0	30
w	1	1	1	0	0	0	1	0	0	0	0	330
x	1	0	0	1	0	0	0	0	−1	0	0	300
S_5	0	0	0	0	0	0	1	1	1	1	1	0
$-C$	−20	−5	0	0	0	0	−80	−70	−30	−30	0	−45300

Now continue with phase II. After three more pivots, you should arrive at the optimal tableau:

	u	v	w	x	y	z	S_1	S_2	S_3	S_4	S_5	*Capacity*
y	0	0	−1	1	1	0	0	1	0	0	1	90
z	0	0	1	0	0	1	0	0	0	0	−1	360
v	0	1	1	−1	0	0	0	−1	0	−1	−1	30
u	1	0	0	1	0	0	0	0	−1	0	0	300
S_1	0	0	0	0	0	0	1	1	1	1	1	0
$-C$	0	0	5	15	0	0	0	5	30	45	75	−39150

The solution is:

$u = 300$ kg. shipped from Atlanta to Chicago
$v = 30$ kg. shipped from Atlanta to Denver
$w = 0$ kg. shipped from Atlanta to San Francisco
$x = 0$ kg. shipped from Boston to Chicago
$y = 90$ kg. shipped from Boston to Denver
$z = 360$ kg. shipped form Boston to San Francisco
$C = \$39{,}150$ (Minimized)
$S_1 = 0$ leftover kg, in Atlanta, $M_1 = 0$
$S_2 = 0$ leftover kg, in Boston, $M_2 = \$5/\text{kg}.$
$S_3 = 0$ surplus kg, shipped to Chicago, $M_3 = \$30/\text{kg}.$
$S_4 = 0$ surplus kg, shipped to Denver, $M_4 = \$45/\text{kg}.$
$S_5 = 0$ surplus kg, shipped to San Francisco, $M_5 = \$75/\text{kg}.$

This solution agrees with our previous solution (obtained using Mathematica in Chapter 5.)

Section 9.3 Sensitivity Analysis and the Simplex Algorithm

9.3.1 Changes in Capacity

Solutions to Exercises:

1. a) In the lemonade stand problem, compute the new optimal data if the number of lemons available decreases from 60 to 55.

Solution:

$$B^{-1}C = \begin{bmatrix} -\frac{1}{5} & 0 & \frac{4}{5} \\ \frac{2}{3} & 0 & -\frac{3}{5} \\ -\frac{3}{5} & 1 & \frac{2}{5} \end{bmatrix} \begin{bmatrix} 55 \\ 28 \\ 30 \end{bmatrix} = \begin{bmatrix} 13 \\ 4 \\ 7 \end{bmatrix}$$

Recall that the order of the basic variables in the final tableau was x, y, S_2. Thus, the revised optimal data is:

$$x = 13 \text{ glasses sweet lemonade}$$
$$y = 4 \text{ glasses tart lemonade}$$
$$R = 1.25(13) + 1.5(4) = \$22.25 \text{ (Maximized)}$$
$$S_1 = 0 \text{ leftover lemons}, \quad M_1 = \$.35/\text{lemon}$$
$$S_2 = 7 \text{ leftover limes}, \quad M_2 = \$0/\text{lime}$$
$$S_3 = 0 \text{ leftover tbsp. sugar}, \quad M_3 = \$.10/\text{tbsp}$$

b) What would you expect to happen if you kept the amount of lemons and sugar at their current levels of 60 and 30, respectively, but increased the supply of limes from 28 to 30? Verify your guess by using B^{-1}.

Solution: Since we already have surplus limes at the optimal point, we guess that adding more limes will not change anything about the optimal point except for having even more surplus limes. Indeed:

$$B^{-1}C = \begin{bmatrix} -\frac{1}{5} & 0 & \frac{4}{5} \\ \frac{2}{3} & 0 & -\frac{3}{5} \\ -\frac{3}{5} & 1 & \frac{2}{5} \end{bmatrix} \begin{bmatrix} 60 \\ 30 \\ 30 \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \\ 6 \end{bmatrix}$$

Thus, $(x, y) = (12,6)$, as expected, but $S_2 = 6$ so there are now 6 surplus limes instead of 4.

2. a) In Example 9.10, verify that pivoting leads to the final tableau indicated.

Solution: The initial tableau, taken from page 449 of the text:

	x	y	S_1	A_2	S_3	A_3	*Capacity*
S_1	1	3	1	0	0	0	48
A_2	1	6	0	1	0	0	66
$-A_3$	-5	-3	0	0	1	-1	-60
P	-5	-16	0	0	0	0	0
z	-6	-9	0	0	1	0	-126

	x	y	S_1	A_2	S_3	A_3	*Capacity*
S_1	$\frac{3}{6}$	0	1	$-\frac{3}{6}$	0	0	15
y	$\frac{1}{6}$	1	0	$\frac{1}{6}$	0	0	11
$-A_3$	$-\frac{27}{6}$	0	0	$\frac{3}{6}$	1	-1	-27
P	$-\frac{14}{6}$	0	0	$\frac{16}{6}$	0	0	176
z	$-\frac{27}{6}$	0	0	$\frac{9}{6}$	1	0	-27

	x	y	S_1	A_2	S_3	A_3	*Capacity*
S_1	0	0	1	$-\frac{4}{9}$	$\frac{1}{9}$	$-\frac{1}{9}$	12
y	0	1	0	$\frac{5}{27}$	$\frac{1}{27}$	$-\frac{1}{27}$	10
x	1	0	0	$-\frac{1}{9}$	$-\frac{2}{9}$	$\frac{2}{9}$	6
P	0	0	0	$\frac{65}{27}$	$-\frac{14}{27}$	$\frac{14}{27}$	190
z	0	0	0	1	0	1	0

This tableau represents a feasible point, so phase I is complete. It also agrees with the tableau given in the text on page 449.) Continuing with phase II (where we opt to keep all the columns for completeness):

	x	y	S_1	A_2	S_3	A_3	$Capacity$
S_1	0	0	1	$-\frac{4}{9}$	$\frac{1}{9}$	$-\frac{1}{9}$	12
y	0	1	0	$\frac{5}{27}$	$\frac{1}{27}$	$-\frac{1}{27}$	10
x	1	0	0	$-\frac{1}{9}$	$-\frac{2}{9}$	$\frac{2}{9}$	6
P	0	0	0	$\frac{65}{27}$	$-\frac{14}{27}$	$\frac{14}{27}$	190

	x	y	S_1	A_2	S_3	A_3	$Capacity$
S_3	0	0	9	-4	1	-1	108
y	0	1	$-\frac{1}{3}$	$\frac{1}{3}$	0	0	6
x	1	0	2	-1	0	0	30
P	0	0	$\frac{14}{3}$	$\frac{1}{3}$	0	0	246

This is the optimal tableau and agrees with the one given on page 449 of the text.

b) Find the revised optimal data if the capacity in the first constraint increases from 48 to 60.

Solution:

$$B^{-1}C = \begin{bmatrix} 9 & -4 & 1 \\ -\frac{1}{3} & \frac{1}{3} & 0 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 60 \\ 66 \\ -60 \end{bmatrix} = \begin{bmatrix} 216 \\ 2 \\ 54 \end{bmatrix}$$

Thus, the revised optimal data is:

$$x = 54$$
$$y = 2$$
$$P = 5(54) + 16(2) = 302 \text{ (Maximized)}$$
$$S_1 = 0, \quad M_1 = \frac{14}{3}$$
$$A_2 = 0, \quad M_2 = \frac{1}{3}$$
$$S_3 = 216, \quad M_3 = 0$$

c) Explicitly write down the matrix B and verify that $BB^{-1} = I_3$.

Solution:

$$BB^{-1} = \begin{bmatrix} 0 & 3 & 1 \\ 0 & 6 & 1 \\ 1 & -3 & -5 \end{bmatrix} \begin{bmatrix} 9 & -4 & 1 \\ -\frac{1}{3} & \frac{1}{3} & 0 \\ 2 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. Recall the following Exercise: Joni is putting her designing skills to work at the *Court and Sparkle Jewelry Emporium*. She makes two signature design bracelet models. Each Dawntreader bracelet uses 1 ruby, 6 pearls, and 10 opals. Each Hejira bracelet uses 3 rubies, 3 pearls, and 15 opals. She has 54 rubies, 120 pearls, and 300 opals to work with. If either model results in a profit of $1,800 for Joni, how many of each type should she make?

In Section 4.4, you answered the following questions using a graphical approach. Now answer them using the techniques of this section:

a) What is the revised optimal data if the number of pearls available is increased to 150?

Solution: The initial and final tableaux (See Exercise 12 in Section 5.2) are

	x	y	S_1	S_2	S_3	$Capacity$
S_1	1	3	1	0	0	54
S_2	6	3	0	1	0	120
S_3	10	15	0	0	1	300
P	-1800	-1800	0	0	0	0

	x	y	S_1	S_2	S_3	$Capacity$
S_1	0	0	1	$\frac{1}{4}$	$-\frac{1}{4}$	9
x	1	0	0	$\frac{1}{4}$	$-\frac{1}{20}$	15
y	0	1	0	$-\frac{1}{6}$	$\frac{1}{10}$	10
P	0	0	0	150	90	45000

Thus,

$$B^{-1}C = \begin{bmatrix} 1 & \frac{1}{4} & -\frac{1}{4} \\ 0 & \frac{1}{4} & -\frac{1}{20} \\ 0 & -\frac{1}{6} & \frac{1}{10} \end{bmatrix} \begin{bmatrix} 54 \\ 150 \\ 300 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{45}{2} \\ 5 \end{bmatrix}$$

The revised optimal data (allowing fractional answers) is

$$x = \frac{45}{2} = 22.5 \text{ Dawntreader bracelets}$$
$$y = 5 \text{ Hejira bracelets}$$
$$P = 1800\left(\frac{45}{2}\right) + 1800(5) = \$49{,}500 \text{ (Maximized)}$$
$$S_1 = \frac{3}{2} = 1.5 \text{ leftover rubies}, \quad M_1 = 0$$
$$S_2 = 0 \text{ leftover pearls}, \quad M_2 = \$150/\text{pearl}$$
$$S_3 = 0 \text{ leftover opals}, \quad M_3 = \$90/\text{opal}$$

4. Recall the following exercise: *Green's Heavy Metal Foundry* mixes three different alloys composed of copper, zinc, and iron. Each 100 lb. unit of Alloy I consists of 50 lbs. copper, 50 lbs. zinc, and no iron. Each 100 lb. unit of Alloy II consists of 30 lbs. copper, 30 lbs. zinc, and 40 lbs. iron. Each 100 lb. unit of Alloy III consists of 50 lbs. copper, 20 lbs. zinc, and 30 lbs. iron. Each unit of Alloy I generates $100 profit, each unit of Alloy II generates $80 profit, and each unit of Alloy III generates $40 profit. There are 12,000 lbs. copper, 10,000 lbs. zinc, and 12,000 lbs. iron available. The foundry has hired Gary Giante, and outside consultant from *Mortimore, Weathers, and Smith, Ltd*., to help them maximize their profit. What problem does Mr. Giante need to solve in order to advise the foundry?

In Section 5.2, you solved this problem via the simplex algorithm. Determine the revised optimal data if the amount of copper available increased to 13,000 lbs.

Solution: The initial and final tableaux (see Exercise 14 in Section 5.2) are

	x	y	z	S_1	S_2	S_3	*Capacity*
S_1	50	30	50	1	0	0	12000
S_2	50	30	20	0	1	0	10000
S_3	0	40	30	0	0	1	12000
P	-100	-80	-40	0	0	0	0

	x	y	z	S_1	S_2	S_3	$Capacity$
S_1	0	0	30	1	-1	0	2000
x	1	0	$\frac{1}{20}$	0	$\frac{1}{50}$	$-\frac{3}{200}$	20
y	0	1	$\frac{3}{4}$	0	0	$\frac{1}{40}$	300
P	0	0	15	0	2	$\frac{1}{2}$	26000

Thus,

$$B^{-1}C = \begin{bmatrix} 1 & -1 & 0 \\ 0 & \frac{1}{50} & -\frac{3}{200} \\ 0 & 0 & \frac{1}{40} \end{bmatrix} \begin{bmatrix} 13000 \\ 10000 \\ 12000 \end{bmatrix} = \begin{bmatrix} 3000 \\ 20 \\ 300 \end{bmatrix}$$

The revised optimal data is

$$x = 20 \text{ units Alloy I}$$
$$y = 300 \text{ units Alloy II}$$
$$z = 0 \text{ units Alloy III}$$
$$S_1 = 3000 \text{ lbs. leftover copper,} \quad M_1 = 0$$
$$S_2 = 0 \text{ lbs. leftover zinc,} \quad M_2 = \$2/\text{ lb}$$
$$S_3 = 0 \text{ lbs. leftover iron,} \quad M_3 = \$.50/\text{lb}$$

The solution did not change at all, except now there is more leftover copper. This should have been expected, since we increased a resource (copper), which was already leftover at the optimal point (so has marginal value 0.)

5. Recall the following exercise: The Spooky Boogie Costume Salon makes and sells four different Halloween costumes: the witch, the ghost, the goblin, and the werewolf. Each witch costume uses 3 yards material and takes 2 hours to sew. Each ghost costume uses 2 yards of material and takes 1 hour to sew. Each goblin costume uses 2 yards of material and takes 3 hours to sew. Each werewolf costume uses 2 yards of material and takes 4 hours to sew. The profits for each costume are as follows: \$10 for the witch, \$8 for the ghost, \$12 for the goblin, and \$16 for the werewolf. If they have 600 yards of material and 510 sewing hours available before the holiday, how many of each costume should they make in order to maximize profit, assuming they can sell everything they make?

In Section 5.2, you solved this problem via the simplex algorithm. Determine the revised optimal data under the following changes:

a) The amount of material available decreases from 600 yards to 420 yards due to a labor strike in the garment industry.

Solution: The initial and final tableaux (see Exercise 17 in Section 5.2) are

	w	x	y	z	S_1	S_2	$Capacity$
S_1	3	2	2	2	1	0	600
S_2	2	1	3	4	0	1	510
P	−10	−8	−12	−16	0	0	0

	w	x	y	z	S_1	S_2	$Capacity$
x	$\frac{4}{3}$	1	$\frac{1}{3}$	0	$\frac{2}{3}$	$-\frac{1}{3}$	230
z	$\frac{1}{6}$	0	$\frac{2}{3}$	1	$-\frac{1}{6}$	$\frac{1}{3}$	70
P	$\frac{10}{3}$	0	$\frac{4}{3}$	0	$\frac{8}{3}$	$\frac{8}{3}$	2960

Thus,

$$B^{-1}C = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{6} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 420 \\ 510 \end{bmatrix} = \begin{bmatrix} 110 \\ 100 \end{bmatrix}$$

The revised optimal data is:

$w = 0$ witch costumes
$x = 110$ ghost costumes
$y = 0$ goblin costumes
$z = 100$ werewolf costumes
$P = 8(110) + 16(100) = \$2{,}480$ (Maximized)
$S_1 = 0$ leftover yards material, $M_1 = \frac{8}{3} \approx \$2.67$/yard
$S_2 = 0$ leftover hours sewing time, $M_2 = \frac{8}{3} \approx \$2.67$/hr.

b) The amount of sewing hours increases from 510 hours to 540 hours due to the purchase of a new sewing machine.

Solution:

$$B^{-1}C = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{6} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 600 \\ 540 \end{bmatrix} = \begin{bmatrix} 220 \\ 80 \end{bmatrix}$$

The revised optimal data is:

$w = 0$ witch costumes
$x = 220$ ghost costumes
$y = 0$ goblin costumes
$z = 80$ werewolf costumes

$$P = 8(220) + 16(80) = \$3{,}040 \text{ (Maximized)}$$
$$S_1 = 0 \text{ leftover yards material,} \quad M_1 = \frac{8}{3} \approx \$2.67/\text{yard}$$
$$S_2 = 0 \text{ leftover hours sewing time,} \quad M_2 = \frac{8}{3} \approx \$2.67/\text{hr.}$$

6. Recall the following exercise with mixed constraints: The *Jefferson Plastic Fantastic Assembly Corporation* manufactures gadgets and widgets for airplanes and starships. Each case of gadgets uses 2 kg. steel and 5 kg. plastic. Each case of widgets uses 2 kg. steel and 3 kg. plastic. The profit for a case of gadgets is \$360 and the profit for a case of widgets is \$200. Suppose they have 80 kg. steel available and 150 kg. plastic available on a daily basis, and can sell everything they manufacture. How many cases of each should they manufacture if they are obligated to produce at least 10 cases of widgets per day? The objective is to maximize daily profit.

In Exercise 7 of Section 9.1, you solved this via the simplex algorithm. Now compute the revised optimal data if the amount of plastic increases 160 kg. per day.

Solution: From Exercise 7 of Section 9.1, we obtain the initial and final tableaux:

	x	y	S_1	S_2	S_3	A_3	$Capacity$
S_1	2	2	1	0	0	0	80
S_2	5	3	0	1	0	0	150
S_3	0	−1	0	0	1	−1	−10
P	−360	−200	0	0	0	0	0
z	0	−1	0	0	1	0	−10

	x	y	S_1	S_2	S_3	$Capacity$
S_1	0	0	1	$-\frac{2}{5}$	$\frac{4}{5}$	12
x	1	0	0	$\frac{1}{5}$	$\frac{3}{5}$	24
y	0	1	0	0	−1	10
P	0	0	0	72	16	10640

We have, as usual, deleted the column with the artificial variable because it did not affect B^{-1}, which appears in the shaded cells. Thus,

$$B^{-1}C = \begin{bmatrix} 1 & -\frac{2}{5} & \frac{4}{5} \\ 0 & \frac{1}{5} & \frac{3}{5} \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 80 \\ 160 \\ -10 \end{bmatrix} = \begin{bmatrix} 8 \\ 26 \\ 10 \end{bmatrix}$$

The revised optimal data is

$$x = 26 \text{ cases gadgets}$$
$$y = 10 \text{ cases widgets}$$
$$P = 360(26) + 200(10) = \$11{,}360 \text{ (Maximized)}$$
$$S_1 = 8 \text{ kg. leftover steel}, \quad M_1 = 0$$
$$S_2 = 0 \text{ kg. leftover plastic}, \quad M_2 = \$72/\text{kg}$$
$$S_3 = 0 \text{ surplus cases widgets}, \quad M_3 = \$16/\text{case}$$

7. In Exercise 5 of Section 9.1, you solved the following problem with mixed constraints:

Maximize $P = x + 4y$
Subject to:
$$4x + 7y \le 140$$
$$5x + 22y = 228$$
$$3x + 2y \ge 36$$
$$x \ge 0, \quad y \ge 0$$

Find the revised optimal data under the following changes:

a) The first constraint decreases to 120.
b) The second constraint increases to 236.

Solution: In Exercise 5 of Section 9.1, we wrote the initial tableau as:

	x	y	S_1	S_3	A_2	A_3	$Capacity$
S_1	4	7	1	0	0	0	140
$-A_2$	-5	-22	0	0	-1	0	-228
$-A_3$	-3	-2	0	1	0	-1	-36
P	-1	-4	0	0	0	0	0
z	-8	-24	0	1	0	0	264

However, as pointed out in this section of the text, in order to find B^{-1} for studying changes in capacity, we need to reorder the columns and not negate the exact equality constraint A_2 in the second row, so that the initial tableau will have an identity matrix in the appropriate columns (shaded below). Thus, we'd write the initial tableau this way:

	x	y	S_1	A_2	S_3	A_3	***Capacity***
S_1	4	7	1	0	0	0	140
A_2	5	22	0	1	0	0	228
$-A_3$	-3	-2	0	0	1	-1	-36
P	-1	-4	0	0	0	0	0
z	-8	-24	0	0	1	0	264

Notice that these changes do not affect the location of the first pivot. It follows that the fist pivot row is still the A_2 row, and the first step in the pivot process is to put a 1 in the pivot position. Thus, the NPR (normalized pivot row) is the same whether or not the row is negated in setting up the tableau. It follows that all subsequent tableaux will look exactly the same as they did in Exercise 5, except with the A_2 and S_3 columns interchanged.

Thus, without having to pivot the problem over, we can write down the final tableau we seek by just interchanging those two columns on the final tableau in Exercise 5 of Section 9.1. Also, we need to keep the A_2 column at the end of phase I because it involves B^{-1}. In fact, when we pivoted Exercise 5 in Section 9.1, we retained all the columns. So the final tableaux we seek is:

	x	y	S_1	A_2	S_3	A_3	***Capacity***
S_1	0	0	1	$-\frac{13}{56}$	$\frac{53}{56}$	$-\frac{53}{56}$	53
y	0	1	0	$\frac{3}{56}$	$\frac{5}{56}$	$-\frac{5}{56}$	9
x	1	0	0	$-\frac{2}{56}$	$-\frac{22}{56}$	$\frac{22}{56}$	6
P	0	0	0	$\frac{10}{56}$	$\frac{2}{56}$	$\frac{2}{56}$	42

The matrix B^{-1} appears in the shaded cells. Now we can answer parts (a) and (b) simultaneously by making C have two columns- one for the modifications in part (a) and one for part (b):

$$B^{-1}C = \begin{bmatrix} 1 & -\frac{13}{56} & \frac{53}{56} \\ 0 & \frac{3}{56} & \frac{5}{56} \\ 0 & -\frac{2}{56} & -\frac{22}{56} \end{bmatrix} \begin{bmatrix} 120 & 140 \\ 228 & 236 \\ -36 & -36 \end{bmatrix} = \begin{bmatrix} 33 & \frac{358}{7} \\ 9 & \frac{66}{7} \\ 6 & \frac{40}{7} \end{bmatrix}$$

Thus, the revised optimal data is:

Part (a): (First constraint decreases to 120):

$$x = 6$$
$$y = 9$$
$$P = 42 \text{ (Maximized)}$$
$$S_1 = 33, \quad M_1 = 0$$
$$A_2 = 0, \quad M_2 = \frac{10}{56} = \frac{5}{28}$$
$$A_3 = 0, \quad M_3 = \frac{2}{56} = \frac{1}{28}$$

Part (b): (second constraint increases to 236):

$$x = \frac{40}{7} \approx 5.7143$$
$$y = \frac{66}{7} \approx 9.4286$$
$$P = \frac{40}{7} + 4\left(\frac{66}{7}\right) = \frac{304}{7} \approx 43.429 \text{ (Maximized)}$$
$$S_1 = \frac{358}{7}, \quad M_1 = 0$$
$$A_2 = 0, \quad M_2 = \frac{10}{56} = \frac{5}{28}$$
$$A_3 = 0, \quad M_3 = \frac{2}{56} = \frac{1}{28}$$

8. a) Solve the following problem:

$$\text{Maximize } P = 48x + 40y$$
$$\text{Subject to:}$$
$$3x + y \leq 120$$
$$x + 3y \leq 120$$
$$x + y \leq 66$$
$$7x + 5y \leq 350$$
$$x \geq 0, \quad y \geq 0$$

Solution: This is a standard form maximization, so phase I is not necessary. Phase II:

	x	y	S_1	S_2	S_3	S_4	$Capacity$
S_1	3	1	1	0	0	0	120
S_2	1	3	0	1	0	0	120
S_3	1	1	0	0	1	0	66
S_4	7	5	0	0	0	1	350
P	-48	-40	0	0	0	0	0

The reader should verify that after two pivots, we arrive at the final tableau:

	x	y	S_1	S_2	S_3	S_4	$Capacity$
x	1	0	1	$-\frac{5}{16}$	0	$\frac{3}{16}$	$\frac{225}{8}$
S_1	0	0	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	5
S_3	0	0	0	$-\frac{1}{8}$	1	$-\frac{1}{8}$	$\frac{29}{4}$
y	0	1	0	$\frac{7}{16}$	0	$-\frac{1}{16}$	$\frac{245}{8}$
P	0	0	0	$\frac{5}{2}$	0	$\frac{13}{2}$	2575

So, the optimal data is:

$$x = \frac{225}{8} = 28.125$$
$$y = \frac{245}{8} = 30.625$$
$$P = 2575 \text{ (Maximized)}$$
$$S_1 = 5, \quad M_1 = 0$$
$$S_2 = 0, \quad M_2 = \frac{5}{2}$$
$$S_3 = \frac{29}{4} = 7.25, \quad M_3 = 0$$
$$S_4 = 0, \quad M_4 = \frac{13}{2}$$

b) Find the revised optimal data if the capacity in the fourth constraint increases from 350 to 500

Solution:

$$B^{-1}C = \begin{bmatrix} 0 & -\frac{5}{16} & 0 & \frac{3}{16} \\ 1 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{8} & 1 & -\frac{1}{8} \\ 0 & \frac{7}{16} & 0 & -\frac{1}{16} \end{bmatrix} \begin{bmatrix} 120 \\ 120 \\ 66 \\ 500 \end{bmatrix} = \begin{bmatrix} \frac{225}{4} \\ -70 \\ -\frac{23}{2} \\ \frac{85}{4} \end{bmatrix}$$

Notice we obtained two negative entries. This indicates that the change to 500 was outside the stable range, so more pivoting is required. We need to write the entire "final" tableau with the modified values:

	x	y	S_1	S_2	S_3	S_4	*Capacity*
x	1	0	1	$-\frac{5}{16}$	0	$\frac{3}{16}$	$\frac{225}{4}$
S_1	0	0	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	-70
S_3	0	0	0	$-\frac{1}{8}$	1	$-\frac{1}{8}$	$-\frac{23}{2}$
y	0	1	0	$\frac{7}{16}$	0	$-\frac{1}{16}$	$\frac{85}{4}$
P	0	0	0	$\frac{5}{2}$	0	$\frac{13}{2}$	Type equation here.

Note that the only the capacity column changes, and the entry in the lower right hand corner is the value of the objective function $48\left(\frac{225}{4}\right) + 40\left(\frac{85}{4}\right) = 3550$. As is usual in this situation, we must use phase I via either algorithm 9.3 of algorithm 9.5. We will follow algorithm 9.5 here, and leave the other approach to the reader. Pivoting as indicated, the next tableau is:

	x	y	S_1	S_2	S_3	S_4	*Capacity*
x	1	0	$\frac{3}{8}$	$-\frac{1}{8}$	0	0	30
S_4	0	0	-2	-1	0	1	140
S_3	0	0	$-\frac{1}{4}$	$-\frac{1}{4}$	1	0	6
y	0	1	$-\frac{1}{8}$	$\frac{3}{8}$	0	0	30
P	0	0	13	9	0	0	2640

Thus, the revised optimal data is:

$$x = 30$$
$$y = 30$$
$$P = 2640 \text{ (Maximized)}$$
$$S_1 = 0, \quad M_1 = 13$$
$$S_2 = 0, \quad M_2 = 9$$
$$S_3 = 6, \quad M_3 = 0$$
$$S_4 = 140, \quad M_4 = 0$$

Notice that the marginal values have changed because this corner corresponds to a different set o basic variables than in the solution for part (a). Again, this happened because the change was outside the stable range.

9.3.2 Changes in Objective Coefficients

Solutions to Exercises and more music references:

1. In the lemonade stand problem, determine the revised optimal data if the girls decide to lower the price of tart lemonade to $1.30 per glass.

Solution: As noted in the text, we proceed assuming the change is within the stable range, and obtain the matrix:

$$\hat{B}^{-1} = \begin{bmatrix} -\frac{1}{5} & 0 & \frac{4}{5} & 0 \\ \frac{2}{5} & 0 & -\frac{3}{5} & 0 \\ -\frac{3}{5} & 1 & \frac{2}{5} & 0 \\ M_1 & M_2 & M_3 & 1 \end{bmatrix}$$

We find the marginal values, using $\hat{B}\hat{B}^{-1} = I_4$ as follows. The matrix $\hat{B}$ is

$$\begin{bmatrix} 3 & 4 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ -1.25 & -1.3 & 0 & 1 \end{bmatrix}$$

Thus

$$\hat{B}\hat{B}^{-1} = \begin{bmatrix} 3 & 4 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ -1.25 & -1.3 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{5} & 0 & \frac{4}{5} & 0 \\ \frac{2}{5} & 0 & -\frac{3}{5} & 0 \\ -\frac{3}{5} & 1 & \frac{2}{5} & 0 \\ M_1 & M_2 & M_3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ M_1 - .27 & M_2 & M_3 - .22 & 1 \end{bmatrix}$$

Setting this equal to I_4, we see $M_1 = \$.27$/lemon, $M_2 = 0$/lime, and $M_3 = \$.22$/tbsp sugar. Since they all came out to be nonnegative, the price change was indeed within the stable range. We compute the revised revenue as follows:

$$M \cdot C = [.27 \quad 0 \quad .22] \begin{bmatrix} 60 \\ 28 \\ 30 \end{bmatrix} = 22.8$$

The values of the basic variables are unchanged. Thus, the revised optimal data is:

$$x = 12 \text{ glasses sweet lemonade}$$
$$y = 6 \text{ glasses tart lemonade}$$
$$R = \$22.80 \text{ (Maximized)}$$
$$S_1 = 0 \text{ leftover lemons,} \quad M_1 = \$.27$$
$$S_2 = 4 \text{ leftover limes,} \quad M_2 = \$.0$$
$$S_3 = 0 \text{ leftover Tbsp sugar,} \quad M_3 = \$.22$$

2. In the lemonade stand problem, determine the revised optimal data if the girls deci to raise the price of tart lemonade to $1.60 per glass.

Solution: The calculations are just like Exercise 1, except with a $\widehat{B}$ that differs in one entry (the price of a glass of tart lemonade):

$$\widehat{B}\widehat{B}^{-1} = \begin{bmatrix} 3 & 4 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ -1.25 & -1.6 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{5} & 0 & \frac{4}{5} & 0 \\ \frac{2}{5} & 0 & -\frac{3}{5} & 0 \\ -\frac{3}{5} & 1 & \frac{2}{5} & 0 \\ M_1 & M_2 & M_3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ M_1 - .39 & M_2 & M_3 - .04 & 1 \end{bmatrix}$$

Setting this equal to the identity matrix gives us the marginal values: $M_1 = .39, M_2 = 0, M_3 = .40$. Since these are nonnegative, the change is within the stable range. For the revised revenue:

$$R = M \cdot C = [.39 \quad 0 \quad .04] \begin{bmatrix} 60 \\ 28 \\ 30 \end{bmatrix} = 24.6$$

The revised optimal data is:

$$x = 12 \text{ glasses sweet lemonade}$$
$$y = 6 \text{ glasses tart lemonade}$$
$$R = \$24.60 \text{ (Maximized)}$$
$$S_1 = 0 \text{ leftover lemons,} \quad M_1 = \$.39$$
$$S_2 = 4 \text{ leftover limes,} \quad M_2 = \$.0$$
$$S_3 = 0 \text{ leftover Tbsp sugar,} \quad M_3 = \$.04$$

3. In the lemonade stand problem, determine the revised optimal data if the girls decide to raise the price of tart lemonade to $2.00 per glass.

Solution: We proceed as in the previous exercises:

$$\hat{B}\hat{B}^{-1} = \begin{bmatrix} 3 & 4 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ -1.25 & -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{5} & 0 & \frac{4}{5} & 0 \\ \frac{2}{5} & 0 & -\frac{3}{5} & 0 \\ -\frac{3}{5} & 1 & \frac{2}{5} & 0 \\ M_1 & M_2 & M_3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ M_1 - .55 & M_2 & M_3 + .2 & 1 \end{bmatrix}$$

When we set this to the identity matrix, note that M_3 comes out negative, which indicates that the change went outside the stable range for the price of tart lemonade. Thus, we write down the current tableau (with the modified data), and continue pivoting from there. The revenue at the current (no longer optimal) point is:

$$M \cdot C = [.55 \quad 0 \quad -.2] \begin{bmatrix} 60 \\ 28 \\ 30 \end{bmatrix} = 27$$

Thus, we are at the following tableau:

	x	y	S_1	S_2	S_3	$Capacity$
x	1	0	$-\frac{1}{5}$	0	$\frac{4}{5}$	12
y	0	1	$\frac{2}{5}$	0	$-\frac{3}{5}$	6
S_2	0	0	$-\frac{3}{5}$	1	$\frac{2}{5}$	4
R	0	0	$\frac{55}{100}$	0	$-\frac{1}{5}$	27

Pivot one more time, where indicated, to obtain:

	x	y	S_1	S_2	S_3	$Capacity$
x	1	0	1	-2	0	4
y	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	0	12
S_3	0	0	$-\frac{3}{2}$	$\frac{5}{2}$	1	10
R	0	0	$\frac{1}{4}$	$\frac{1}{2}$	0	29

The tableau is optimal, so the revised data is:

$$x = 4 \text{ glasses sweet lemonade}$$
$$y = 12 \text{ glasses tart lemonade}$$
$$R = \$29.00 \text{ (Maximized)}$$
$$S_1 = 0 \text{ leftover lemons,} \quad M_1 = \$.25/\text{lemon}$$
$$S_2 = 0 \text{ leftover limes,} \quad M_2 = \$.50/\text{lime}$$
$$S_3 = 10 \text{ leftover Tbsp sugar,} \quad M_3 = \$0/\text{Tbsp sugar}$$

4. In the Spooky Boogie Halloween Costume problem, determine the revised optimal data if the profit of a goblin costume increases to \$14.

Solution: In this case, as noted in the text, the marginal values (and hence $\hat{B}^{-1}$) do not change when we change the profit for a goblin costume because the variable y is nonbasic:

$$\hat{B}^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{6} & \frac{1}{3} & 0 \\ \frac{8}{3} & \frac{8}{3} & 1 \end{bmatrix}$$

All of the other entries in the bottom row of the final tableau do not change except for that in the y column. The new value is

$$\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{6} & \frac{1}{3} & 0 \\ \frac{8}{3} & \frac{8}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -14 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix}$$

Thus, the current tableau is:

	w	x	y	z	S_1	S_2	$Capacity$
x	1	1	$\frac{1}{3}$	0	$\frac{2}{3}$	$-\frac{1}{3}$	230
z	$\frac{1}{2}$	0	$\frac{2}{3}$	1	$-\frac{1}{6}$	$\frac{1}{3}$	70
P	6	0	$-\frac{2}{3}$	0	$\frac{8}{3}$	$\frac{8}{3}$	2960

One more pivot is required:

	w	x	y	z	S_1	S_2	$Capacity$
x	$\frac{3}{4}$	1	0	$-\frac{1}{2}$	$\frac{3}{4}$	$-\frac{1}{2}$	195
y	$\frac{3}{4}$	0	1	$\frac{3}{2}$	$-\frac{1}{4}$	$\frac{1}{2}$	105
P	$\frac{13}{2}$	0	0	1	$\frac{5}{2}$	3	3030

This is the optimal point, so the revised data is:

$w = 0$ witch costumes
$x = 195$ ghost costumes
$y = 105$ goblin costumes
$z = 0$ werewolf costumes
$P = \$3,030$ (Maximized)
$S_1 = 0$ leftover yards material, $M_1 = \$2.50/$ yard
$S_2 = 0$ leftover hours sewing time, $M_2 = \$3$/hour

5. In the Spooky Boogie Halloween Costume problem, determine the revised optimal data if the profit of a werewolf costume decreases to $12.

Solution: This time, we are changing the price for a variable which is a basic variable in solution, so the marginal values do change. We compute them as usual:

$$\hat{B}\hat{B}^{-1} = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 4 & 0 \\ -8 & -12 & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{6} & \frac{1}{3} & 0 \\ M_1 & M_2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ M_1 - \frac{10}{3} & M_2 - \frac{4}{3} & 1 \end{bmatrix}$$

So, the marginal values are nonnegative, but remember we must check that every entry in the bottom row remain nonnegative before we can conclude the change is within the stable range. Invoking Theorem 9.2b, we find the current tableau by multiplying

$$\hat{B}^{-1} \cdot (Initial\ tableau) = Current\ Tableau$$

$$\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{6} & \frac{1}{3} & 0 \\ M_1 & M_2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 2 & 2 & 1 & 0 & 600 \\ 3 & 1 & 3 & 4 & 0 & 1 & 510 \\ -10 & -8 & -12 & -12 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & \frac{1}{3} & 0 & \frac{2}{3} & -\frac{1}{3} & 230 \\ \frac{1}{2} & 0 & \frac{2}{3} & 1 & -\frac{1}{6} & \frac{1}{3} & 70 \\ 4 & 0 & -\frac{4}{3} & 0 & \frac{10}{3} & \frac{4}{3} & 2680 \end{bmatrix}$$

Since the last row contains a negative entry, the change was indeed outside of the stable range, and we must continue pivoting from the current tableau:

	w	x	y	z	S_1	S_2	$Capacity$
x	1	1	$\frac{1}{3}$	0	$\frac{2}{3}$	$-\frac{1}{3}$	230
z	$\frac{1}{2}$	0	$\frac{2}{3}$	1	$-\frac{1}{6}$	$\frac{1}{3}$	70
P	4	0	$-\frac{4}{3}$	0	$\frac{10}{3}$	$\frac{4}{3}$	2680

One more pivot leads to:

	w	x	y	z	S_1	S_2	$Capacity$
x	$\frac{3}{4}$	1	0	$-\frac{1}{2}$	$\frac{3}{4}$	$-\frac{1}{2}$	195
y	$\frac{3}{4}$	0	1	$\frac{3}{2}$	$-\frac{1}{4}$	$\frac{1}{2}$	105
P	5	0	0	2	3	2	2820

This tableau is optimal, so the revised data is:

$w = 0$ witch costumes
$x = 195$ ghost costumes
$y = 105$ goblin costumes
$z = 0$ werewolf costumes
$P = \$2,820$ (Maximized)
$S_1 = 0$ leftover yards material, $M_1 = \$3/$ yard
$S_2 = 0$ leftover hours sewing time, $M_2 = \$2/$hour

6. Recall the following problem: Joni is putting her designing skills to work at the *Court and Sparkle Jewelry Emporium*. She makes two signature design bracelet models. Each Dawntreader bracelet uses 1 ruby, 6 pearls, and 10 opals. Each Hejira bracelet uses 3 rubies, 3 pearls, and 15 opals. She has 54 rubies, 120 pearls, and 300 opals to work with. If either model results in a profit of $1,800 for Joni, how many of each type should she make?

In earlier sections of the text, you solved this problem via the simplex algorithm. Suppose Joni's profit for a Hejira bracelet increases to $2,000. Find the revised optimal data.

Solution: The initial and final tableaux (See Exercise 12 in Section 5.2) are

	x	y	S_1	S_2	S_3	$Capacity$
S_1	1	3	1	0	0	54
S_2	6	3	0	1	0	120
S_3	10	15	0	0	1	300
P	-1800	-1800	0	0	0	0

	x	y	S_1	S_2	S_3	$Capacity$
S_1	0	0	1	$\frac{1}{4}$	$-\frac{1}{4}$	9
x	1	0	0	$\frac{1}{4}$	$-\frac{1}{20}$	15
y	0	1	0	$-\frac{1}{6}$	$\frac{1}{10}$	10
P	0	0	0	150	90	45000

If the profit for a Hejira bracelet increases to $2,000 we find the marginal values as usual:

$$\hat{B}\hat{B}^{-1} = \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & 6 & 3 & 0 \\ 0 & 10 & 15 & 0 \\ 0 & -1800 & -2000 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{4} & -\frac{1}{20} & 0 \\ 0 & -\frac{1}{6} & \frac{1}{10} & 0 \\ M_1 & M_2 & M_3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ M_1 & M_2 - \frac{350}{3} & M_3 - 110 & 1 \end{bmatrix}$$

The marginal values are nonnegative, and this time were the only entries in the bottom row that could change, so the change is within the stable range. The current tableau remains the optimal one, with revised profit:

$$P = M \cdot C = \begin{bmatrix} 0 & \frac{350}{3} & 110 \end{bmatrix} \begin{bmatrix} 54 \\ 120 \\ 300 \end{bmatrix} = \$47{,}000$$

The revised data is:

$x = 15$ Dawntreader bracelets
$y = 10$ Hejira bracelets
$P = \$47{,}000$ (Maximized)
$S_1 = 9$ leftover rubies, $M_1 = 0$
$S_2 = 0$ leftover pearls, $M_2 = \frac{350}{3} \approx 101.67/\text{pearl}$
$S_3 = 0$ leftover opals, $M_3 = \$110/\text{opal}$

7. The *Brain One Lighting and Heavenly Music Corporation* designs sophisticated ambient lighting and music for airports and other public spaces. Each Airport contract requires 6 weeks with the design team, 4 weeks with the technical team, and a 2 week review with the legal team to check local building codes, etc. The profit for an Airport contract is \$8,000,000. Each Museum contract requires 5 weeks with the design team, 4 weeks with the technical team, and 1 week with the legal team and generates \$6,000,000 profit. Each Theater contract requires 3 weeks with the design team, 3 weeks with the technical team, and 1 week with the legal team and generates \$5,000,000 profit. The design team works 48 weeks per year. The technical team works 44 weeks per year (and spends 4 weeks per year in research and development.) The legal team works 30 weeks per year (and spends the rest of the year doing pro bono work lobbying to keep arts funding in public government and education.) How many of each type of contract should they take on each year in order to maximize profits?

Music homage: "Brain One" is an anagram of Brian Eno, who was a member of the band Roxy Music before he became a solo artist, producer, and pioneer of ambient music. Eno collaborated with many artists, including Robert Fripp of King Crimson. In 1973 Fripp and Eno released an album No Pussyfooting, which contained the track 'The Heavenly Music Corporation.' In 1978 Eno released an album of ambient music called 'Ambient I: Music for Airports.' (Website: https://www.enoshop.co.uk/)

a) Solve this problem via the simplex algorithm.

Solution: Let x be the number of Airport contracts, y the number of Museum contracts, and z the number of Theater contracts. Then, measuring profit in units of \$1,000,000, we must:

Maximize $P = 8x + 6y + 5z$

Subject to:

$6x + 5y + 3z \leq 48$ (weeks of design team time)

$4x + 4y + 3z \leq 44$ (weeks of technical team time)

$2x + y + z \leq 30$ (weeks of legal team time)

$x \geq 0, \quad y \geq 0, \quad z \geq 0$

This is a standard form maximization problem, so phase I is not required. Phase II:

	x	y	z	S_1	S_2	S_3	$Capacity$
S_1	6	5	3	1	0	0	48
S_2	4	4	3	0	1	0	44
S_3	2	1	1	0	0	1	30
P	-8	-6	-5	0	0	0	0

After two pivots, you should arrive at the following tableau:

	x	y	z	S_1	S_2	S_3	$Capacity$
x	1	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	2
z	0	$\frac{2}{3}$	1	$-\frac{2}{3}$	1	0	12
S_3	0	$-\frac{2}{3}$	0	$-\frac{1}{3}$	0	1	14
P	0	$\frac{4}{3}$	0	$\frac{2}{3}$	1	0	76

The optimal data is

$x = 2$ Airport contracts
$y = 0$ Museum contracts
$z = 12$ theater contracts
$P = \$76{,}000{,}000$ (Maximized)
$S_1 = 0$ weeks leftover for design team, $M_1 = \frac{2}{3} \approx \$666{,}666.67$ /week
$S_2 = 0$ weeks leftover for technical team, $M_2 = \$1{,}000{,}000$ /week
$S_3 = 14$ weeks leftover for design team, $M_3 = 0$

b) Suppose the profit for a theater contract decreased to $3,000,000. Determine the revised optimal solution.

Solution: We determine the marginal values:

$$\hat{B}\hat{B}^{-1} = \begin{bmatrix} 6 & 3 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ -8 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{2}{3} & 1 & 0 & 0 \\ -\frac{1}{3} & 0 & 1 & 0 \\ M_1 & M_2 & M_3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ M_1 - 2 & M_2 + 1 & M_3 & 1 \end{bmatrix}$$

Notice M_2 is negative, so more pivoting is needed. We determine the current tableau by invoking Theorem 9.2, part b:

$$\hat{B}^{-1} \cdot (Initial\ tableau) = Current\ Tableau$$

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{2}{3} & 1 & 0 & 0 \\ -\frac{1}{3} & 0 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & 5 & 3 & 1 & 0 & 0 & 48 \\ 4 & 4 & 3 & 0 & 1 & 0 & 44 \\ 2 & 1 & 1 & 0 & 0 & 1 & 30 \\ -8 & -6 & -3 & 0 & 0 & 0 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 2 \\ 0 & \frac{2}{3} & 1 & -\frac{2}{3} & 1 & 0 & 12 \\ 0 & -\frac{2}{3} & 0 & -\frac{1}{3} & 0 & 1 & 14 \\ 0 & 0 & 0 & 2 & -1 & 0 & 52 \end{bmatrix}$$

So, the current tableau is

	x	y	z	S_1	S_2	S_3	$Capacity$
x	1	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	2
z	0	$\frac{2}{3}$	1	$-\frac{2}{3}$	1	0	12
S_3	0	$-\frac{2}{3}$	0	$-\frac{1}{3}$	0	1	14
P	0	0	0	2	−1	0	52

One more pivot yields:

	x	y	z	S_1	S_2	S_3	$Capacity$
x	1	$\frac{5}{6}$	$\frac{1}{2}$	$\frac{1}{6}$	0	0	8
S_2	0	$\frac{2}{3}$	1	$-\frac{2}{3}$	1	0	12
S_3	0	$-\frac{2}{3}$	0	$-\frac{1}{3}$	0	1	14
P	0	$\frac{2}{3}$	1	$\frac{4}{3}$	0	0	64

The tableau is optimal, so the revised data is:

$x = 8$ Airport contracts
$y = 0$ Museum contracts
$z = 0$ theater contracts
$P = \$64{,}000{,}000$ (Maximized)
$S_1 = 0$ weeks leftover for design team, $M_1 = \frac{4}{3} \approx \$1{,}333{,}333.33$ /week
$S_2 = 12$ weeks leftover for technical team, $M_2 = 0$
$S_3 = 14$ weeks leftover for design team, $M_3 = 0$

8. Recall the following problem with mixed constraints: The *Jefferson Plastic Fantastic Assembly Corporation* manufactures gadgets and widgets for airplanes and starships. Each case of gadgets uses 2 kg. steel and 5 kg. plastic. Each case of widgets uses 2 kg. steel and 3 kg. plastic. The profit for a case of gadgets is \$360 and the profit for a case of widgets is \$200. Suppose they have 80 kg. steel available and 150 kg. plastic available on a daily basis, and can sell everything they manufacture. How many cases of each should they manufacture if they are obligated to produce at least 10 cases of widgets per day? The objective is to maximize daily profit.

In Exercise 7 of Section 9.1 you solved this via the simplex algorithm. Now find the revised optimal data if the profit for a case of widgets drops to \$120.

Solution: From Exercise 7 of Section 9.1, we obtain the initial and final tableaux:

	x	y	S_1	S_2	S_3	A_3	$Capacity$
S_1	2	2	1	0	0	0	80
S_2	5	3	0	1	0	0	150
S_3	0	-1	0	0	1	-1	-10
P	-360	-200	0	0	0	0	0
z	0	-1	0	0	1	0	-10

	x	y	S_1	S_2	S_3	$Capacity$
S_1	0	0	1	$-\frac{2}{5}$	$\frac{4}{5}$	12
x	1	0	0	$\frac{1}{5}$	$\frac{3}{5}$	24
y	0	1	0	0	-1	10
P	0	0	0	72	16	10640

Since there is no z row in the final tableau, we ignore the z row in the initial tableau in constructing $\hat{B}$. (All the calculations will still be valid just as using B instead of $\hat{B}$ gives the correct values for all the rows except the missing row – the objective row…) Thus

$$\hat{B}\hat{B}^{-1} = \begin{bmatrix} 1 & 2 & 2 & 0 \\ 0 & 5 & 3 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -360 & -120 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{2}{5} & \frac{4}{5} & 0 \\ 0 & \frac{1}{5} & \frac{3}{5} & 0 \\ 0 & 0 & -1 & 0 \\ M_1 & M_2 & M_3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ M_1 & M_2 - 72 & M_3 - 96 & 1 \end{bmatrix}$$

Since all the marginal values are nonnegative, the change was in the stable range. The current tableau will be the final one:

$$\begin{bmatrix} 1 & -\frac{2}{5} & \frac{4}{5} & 0 \\ 0 & \frac{1}{5} & \frac{3}{5} & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 72 & 96 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 & 0 & 0 & 80 \\ 5 & 3 & 0 & 1 & 0 & 150 \\ 0 & -1 & 0 & 0 & 1 & -10 \\ -360 & -120 & 0 & 0 & 0 & 0 \end{bmatrix} =$$

	x	y	S_1	S_2	S_3	$Capacity$
S_1	0	0	1	$-\frac{2}{5}$	$\frac{4}{5}$	12
x	1	0	0	$\frac{1}{5}$	$\frac{3}{5}$	24
y	0	1	0	0	-1	10
P	0	0	0	72	16	9840

Thus, the revised optimal data is:

$x = 24$ cases gadgets
$y = 10$ cases widgets
$P = \$9{,}840$ (Maximized)
$S_1 = 12$ kg. leftover steel, $M_1 = 0$
$S_2 = 0$ kg. leftover plastic, $M_2 = \$72/\text{kg}$
$S_3 = 0$ surplus cases widgets, $M_3 = \$96/\text{case}$

9.3.3 Stable Ranges

Solutions to Exercises:

1. In the Spooky Boogie Halloween Costume problem, find the stable range of the capacity for material.

Solution: The relevant matrices are taken from the text:

$$\hat{B} = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 4 & 0 \\ -8 & -16 & 1 \end{bmatrix}$$

$$\hat{B}^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{6} & \frac{1}{3} & 0 \\ M_1 & M_2 & 1 \end{bmatrix}$$

Of course, since we are discussing changes in the capacity column, we could work simply with B and its inverse rather than $\hat{B}$. We have

$$B = \begin{bmatrix} 2 & 2 \\ 1 & 4 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{6} & \frac{1}{3} \end{bmatrix}$$

Thus, we replace the capacity column with $\begin{bmatrix} u \\ 510 \end{bmatrix}$, multiply by B^{-1}, and set the entries ≥ 0. We obtain

$$\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{6} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} u \\ 510 \end{bmatrix} = \begin{bmatrix} \frac{2}{3}u - 170 \\ 170 - \frac{1}{6}u \end{bmatrix}$$

So,

$$\frac{2}{3}u - 170 \geq 0$$

$$\frac{2}{3}u \geq 170$$

$$u \geq \frac{3}{2}(170) = 255$$

And

$$170 - \frac{1}{6}u \geq 0$$

$$170 \geq \frac{u}{6}$$
$$u \leq 6(170) = 1020$$

Thus, the stable range for material is

$$255 \leq u \leq 1020$$

Square yards of material. In interval notation: $[255,1020]$.

2. In the Spooky Boogie Halloween Costume problem, find the stable range of the capacity for sewing time.

Solution: Replace the capacity column by $\begin{bmatrix} 600 \\ u \end{bmatrix}$, and proceed as in Exercise 1:

$$\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{6} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 600 \\ u \end{bmatrix} = \begin{bmatrix} 400 - \frac{1}{3}u \\ \frac{1}{3}u - 100 \end{bmatrix}$$

So

$$400 - \frac{1}{3}u \geq 0$$
$$400 \geq \frac{u}{3}$$
$$u \leq 1200$$

And

$$\frac{1}{3}u - 100 \geq 0$$
$$\frac{u}{3} \geq 100$$
$$u \geq 300$$

Thus, the stable range for sewing time is

$$300 \leq u \leq 1200 \text{ hours}$$

Or $[300,1200]$ in interval notation.

3. In the Spooky Boogie Halloween Costume problem, find the stable range for the profit generated by a goblin costume.

Solution: As noted in the text, when finding the stable range for the objective function coefficient of a nonbasic decision variable, we need to keep the variable nonbasic, which means the marginal values, and hence $\hat{B}^{-1}$ does not change. Furthermore, the only column that changes is the column of that decision variable, so that means we need only compute $\hat{B}^{-1}$ times that one column:

$$\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{6} & \frac{1}{3} & 0 \\ \frac{8}{3} & \frac{8}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -u \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{40}{3} - u \end{bmatrix}$$

So

$$\frac{40}{3} - u \geq 0$$

$$u \leq \frac{40}{3}$$

Therefore, the stable range for the price of a goblin costume is $\left(-\infty, \frac{40}{3}\right]$.

4. In the Spooky Boogie Halloween Costume problem, find the stable range for the profit generated by a werewolf costume.

Solution: As in the example in the text, we multiply $\hat{B}^{-1}$ times the entire initial tableau to see the full set of conditions on u. We must find $\hat{B}^{-1}$ first in terms of u:

$$\hat{B}\hat{B}^{-1} = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 4 & 0 \\ -8 & -u & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{6} & \frac{1}{3} & 0 \\ M_1 & M_2 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{6}u + M_1 - \frac{16}{3} & M_2 - \frac{1}{3}u + \frac{8}{3} & 1 \end{bmatrix}$$

Thus, setting this equal to the identity matrix, we can express the marginal values in terms of u:

$$M_1 = \frac{16}{3} - \frac{u}{6}$$

$$M_2 = \frac{u}{3} - \frac{8}{3}$$

So $\hat{B}^{-1}$ times the initial tableau is:

$$\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{6} & \frac{1}{3} & 0 \\ \frac{16}{3}-\frac{u}{6} & \frac{u}{3}-\frac{8}{3} & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 2 & 2 & 1 & 0 & 600 \\ 3 & 1 & 3 & 4 & 0 & 1 & 510 \\ -10 & -8 & -12 & -u & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & \frac{1}{3} & 0 & \frac{2}{3} & -\frac{1}{3} & 230 \\ \frac{1}{2} & 0 & \frac{2}{3} & 1 & -\frac{1}{6} & \frac{1}{3} & 70 \\ \frac{1}{2}u-2 & 0 & \frac{2}{3}u-\frac{28}{3} & 0 & \frac{16}{3}-\frac{1}{6}u & \frac{1}{3}u-\frac{8}{3} & 70u+1840 \end{bmatrix}$$

Thus, u must satisfy

$$\frac{1}{2}u-2 \geq 0$$
$$\frac{2}{3}u-\frac{28}{3} \geq 0$$
$$\frac{16}{3}-\frac{1}{6}u \geq 0$$
$$\frac{1}{3}u-\frac{8}{3} \geq 0$$

These reduce to:

$$u \geq 4$$
$$u \geq 14$$
$$u \leq 32$$
$$u \geq 8$$

The common solution (the intersection of the solution sets) is the stable range: $14 \leq u \leq 32$ for the price of a werewolf costume, or in interval notation: $[14,32]$.

5. In Exercise 1 in Section 4.2, you found the stable ranges for the profit of each type of bracelet in the *Court and Sparkle Jewelry Emporium*. In this problem, recall that Joni makes two signature design bracelet models. Each Dawntreader bracelet uses 1 ruby, 6 pearls, and 10 opals. Each Hejira bracelet uses 3 rubies, 3 pearls, and 15 opals. She has 54 rubies, 120 pearls, and 300 opals to work with. If either model results in a profit of \$1,800 for Joni, how many of each type should she make?
Now find the stable ranges for the profit of each type of bracelet, using the techniques of this section.

Solution: The initial and final tableaux (See Exercise 12 in Section 5.2) are

	x	y	S_1	S_2	S_3	*Capacity*
S_1	1	3	1	0	0	54
S_2	6	3	0	1	0	120
S_3	10	15	0	0	1	300
P	-1800	-1800	0	0	0	0

	x	y	S_1	S_2	S_3	*Capacity*
S_1	0	0	1	$\frac{1}{4}$	$-\frac{1}{4}$	9
x	1	0	0	$\frac{1}{4}$	$-\frac{1}{20}$	15
y	0	1	0	$-\frac{1}{6}$	$\frac{1}{10}$	10
P	0	0	0	150	90	45000

Thus,

$$\hat{B} = \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & 6 & 3 & 0 \\ 0 & 10 & 15 & 0 \\ 0 & -1800 & -1800 & 1 \end{bmatrix}$$

$$\hat{B}^{-1} = \begin{bmatrix} 1 & \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{4} & -\frac{1}{20} & 0 \\ 0 & -\frac{1}{6} & \frac{1}{10} & 0 \\ 0 & 150 & 90 & 1 \end{bmatrix}$$

If we want the stable range for the price of a Dawntreader bracelet, we replace the 1800 by u, then express the marginal values in terms of u:

$$I = \hat{B}\hat{B}^{-1} = \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & 6 & 3 & 0 \\ 0 & 10 & 15 & 0 \\ 0 & -u & -1800 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{4} & -\frac{1}{20} & 0 \\ 0 & -\frac{1}{6} & \frac{1}{10} & 0 \\ M_1 & M_2 & M_3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ M_1 & M_2 - \frac{u}{4} + 300 & \frac{u}{20} + M_3 - 180 & 1 \end{bmatrix}$$

Thus

$$M_1 = 0$$
$$M_2 = \frac{u}{4} - 300$$
$$M_3 = 180 - \frac{u}{20}$$

Now, with these values, multiply $\hat{B}^{-1}$ times the initial tableau:

$$\begin{bmatrix} 1 & \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{4} & -\frac{1}{20} & 0 \\ 0 & -\frac{1}{6} & \frac{1}{10} & 0 \\ 0 & \frac{u}{4} - 300 & 180 - \frac{u}{20} & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 & 0 & 0 & 54 \\ 6 & 3 & 0 & 1 & 0 & 120 \\ 10 & 15 & 0 & 0 & 1 & 300 \\ -u & -1800 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{4} & 9 \\ 1 & 0 & 0 & \frac{1}{4} & -\frac{1}{20} & 15 \\ 0 & 1 & 0 & -\frac{1}{6} & \frac{1}{10} & 10 \\ 0 & 0 & 0 & \frac{u}{4} - 300 & 180 - \frac{u}{20} & 15u + 18000 \end{bmatrix}$$

The values in the bottom row (except for the objective function in the corner) must be nonnegative, so

$$\frac{u}{4} - 300 \geq 0$$
$$180 - \frac{u}{20} \geq 0$$

These reduce to the stable range we seek:

$$1200 \leq u \leq 3600$$

Or in int4erval notation [1200,3600].

Repeating the calculations for the stable range of a Hejira bracelet:

$$I = \hat{B}\hat{B}^{-1} = \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & 6 & 3 & 0 \\ 0 & 10 & 15 & 0 \\ 0 & -1800 & -u & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{4} & -\frac{1}{20} & 0 \\ 0 & -\frac{1}{6} & \frac{1}{10} & 0 \\ M_1 & M_2 & M_3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ M_1 & M_2 + \frac{u}{6} - 450 & M_3 - \frac{u}{10} + 90 & 1 \end{bmatrix}$$

This time

$$M_1 = 0$$
$$M_2 = 450 - \frac{u}{6}$$
$$M_3 = \frac{u}{10} - 90$$

Thus

$$\begin{bmatrix} 1 & \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{4} & -\frac{1}{20} & 0 \\ 0 & -\frac{1}{6} & \frac{1}{10} & 0 \\ 0 & 450 - \frac{u}{6} & \frac{u}{10} - 90 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 & 0 & 0 & 54 \\ 6 & 3 & 0 & 1 & 0 & 120 \\ 10 & 15 & 0 & 0 & 1 & 300 \\ -1800 & -u & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & \frac{1}{4} & -\frac{1}{4} & 9 \\ 1 & 0 & 0 & \frac{1}{4} & -\frac{1}{20} & 15 \\ 0 & 1 & 0 & -\frac{1}{6} & \frac{1}{10} & 10 \\ 0 & 0 & 0 & 450 - \frac{u}{6} & \frac{u}{10} - 90 & 10u + 27000 \end{bmatrix}$$

Therefore

$$450 - \frac{u}{6} \geq 0$$
$$\frac{u}{10} - 90 \geq 0$$

This reduces to the stable range we week:

$$900 \leq u \leq 2700, \quad \text{or } [900,2700].$$

6. In the Court and Sparkle Jewelry Emporium problem, find the stable range for the capacity of ech of the three resources.

Solution: For changes in capacity, we can work with B^{-1}, rather than with $\hat{B}^{-1}$. We set each entry in $B^{-1}C \geq 0$. First, we find the stable range for rubies:

$$B^{-1}C = \begin{bmatrix} 1 & \frac{1}{4} & -\frac{1}{4} \\ 0 & \frac{1}{4} & -\frac{1}{20} \\ 0 & -\frac{1}{6} & \frac{1}{10} \end{bmatrix} \begin{bmatrix} u \\ 120 \\ 300 \end{bmatrix} = \begin{bmatrix} u-45 \\ 15 \\ 10 \end{bmatrix}$$

The only condition on u is $u - 45 \geq 0$, so $u \geq 45$; that is, the stable range for rubies is $[45, \infty)$.

Next for pearls:

$$B^{-1}C = \begin{bmatrix} 1 & \frac{1}{4} & -\frac{1}{4} \\ 0 & \frac{1}{4} & -\frac{1}{20} \\ 0 & -\frac{1}{6} & \frac{1}{10} \end{bmatrix} \begin{bmatrix} 54 \\ u \\ 300 \end{bmatrix} = \begin{bmatrix} \frac{u}{4} - 21 \\ \frac{u}{4} - 15 \\ 30 - \frac{u}{6} \end{bmatrix}$$

Thus, u must satisfy:

$$\frac{u}{4} - 21 \geq 0$$
$$\frac{u}{4} - 15 \geq 0$$
$$30 - \frac{u}{6} \geq 0$$

These reduce to:

$$u \geq 84$$
$$u \geq 60$$
$$u \leq 180,$$

The common solution is:

$$84 \leq u \leq 180, \quad \text{or } [84,180]$$

Finally, for opals

$$B^{-1}C = \begin{bmatrix} 1 & \frac{1}{4} & -\frac{1}{4} \\ 0 & \frac{1}{4} & -\frac{1}{20} \\ 0 & -\frac{1}{6} & \frac{1}{10} \end{bmatrix} \begin{bmatrix} 54 \\ 120 \\ u \end{bmatrix} = \begin{bmatrix} 84 - \frac{u}{4} \\ 30 - \frac{u}{20} \\ \frac{u}{10} - 20 \end{bmatrix}$$

The conditions on u are:

$$84 - \frac{u}{4} \geq 0$$
$$30 - \frac{u}{20} \geq 0$$
$$\frac{u}{10} - 20 \geq 0$$

These conditions reduce to

$$u \leq 336$$
$$u \leq 600$$
$$u \geq 200$$

The common solution is the stable range for opals:

$200 \leq u \leq 336$, or [200,336].

7. In Section 4.2, you solved the following problem based on graphical arguments. Now solve the problem using the techniques of this chapter.

Kerry's Kennel is mixing two commercial brands of dog food for its canine guests. A bag of *Dog's Life Canine Cuisine* contains 3 lbs. fat, 2 lbs. carbohydrates, 5 lbs. protein, and 3 oz. vitamin C. A giant size bag of *Way of Life Healthy Mix* contains 1 lb. fat, 5 lbs. carbohydrates, 10 lbs. protein, and 7 oz. vitamin C. The requirements for a weeks' supply of food for the kennel are that there should be at most 21 lbs. fat, at mist 40 lbs. carbohydrates, and at least 21 oz. vitamin C. How many bags of each type of food should be mixed in order to design a diet that maximizes protein?

a) Suppose the recipe for Dog's Life Canine Cuisine changes so that the protein content of each bag increases to 12 lbs. Find the new optimal data.

Solution: From our previous solution, the initial and final tableaux are

	x	y	S_1	S_2	S_3	A_3	$Capacity$
S_1	3	1	1	0	0	0	21
S_2	2	5	0	1	0	0	40
$-A_3$	-3	-7	0	0	1	-1	-21
P	-5	-10	0	0	0	0	0
z	-3	-7	0	0	1	0	-21

and

	x	y	S_1	S_2	S_3	$Capacity$
x	1	0	$\frac{5}{13}$	$-\frac{1}{13}$	0	5
S_3	0	0	$\frac{1}{13}$	$\frac{18}{13}$	1	36
y	0	1	$-\frac{2}{13}$	$\frac{3}{13}$	0	6
P	0	0	$\frac{5}{13}$	$\frac{25}{13}$	0	85

Then

$$I = \hat{B}\hat{B}^{-1} = \begin{bmatrix} 3 & 0 & 1 & 0 \\ 2 & 0 & 5 & 0 \\ -3 & 1 & -7 & 0 \\ -12 & 0 & -10 & 1 \end{bmatrix} \begin{bmatrix} \frac{5}{13} & -\frac{1}{13} & 0 & 0 \\ \frac{1}{13} & \frac{18}{13} & 1 & 0 \\ -\frac{2}{13} & \frac{3}{13} & 0 & 0 \\ M_1 & M_2 & M_3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ M_1 - \frac{40}{13} & M_2 - \frac{18}{13} & M_3 & 1 \end{bmatrix}$$

Thus,

$$M_1 = \frac{40}{13}$$
$$M_2 = \frac{18}{13}$$
$$M_3 = 0$$

Since they are nonnegative, the change is within the stable range for protein in Dog's Life Canine Cuisine. Multiply by initial tableau (ignoring the z row).

$$\begin{bmatrix} \frac{5}{13} & -\frac{1}{13} & 0 & 0 \\ \frac{1}{13} & \frac{18}{13} & 1 & 0 \\ -\frac{2}{13} & \frac{3}{13} & 0 & 0 \\ \frac{40}{13} & \frac{18}{13} & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 & 0 & 0 & 0 & 21 \\ 2 & 5 & 0 & 1 & 0 & 0 & 40 \\ -3 & -7 & 0 & 0 & 1 & -1 & -21 \\ -12 & -10 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$
= \begin{bmatrix} 1 & 0 & \frac{5}{13} & -\frac{1}{13} & 0 & 0 & 5 \\ 0 & 0 & \frac{1}{13} & \frac{18}{13} & 1 & -1 & 36 \\ 0 & 1 & -\frac{2}{13} & \frac{3}{13} & 0 & 0 & 6 \\ 0 & 0 & \frac{40}{13} & \frac{18}{13} & 0 & 0 & 120 \end{bmatrix}
$$

The revised optimal solution is:

$x = 5$ bags Dog's life Canine Cuisine
$y = 6$ bags Way of Life Healthy Mix
$P = 120$ lbs. Protein (Maximized)
$S_1 = 0$ lbs. fat, $M_1 = \frac{40}{13}$ lbs. protein/lb. fat
$S_2 = 0$ lbs. carbohydrates, $M_1 = \frac{18}{13}$ lbs. protein/lb. carbohydrate
$S_3 = 36$ oz. surplus vitamin C, $M_3 = 0$ lbs. protein/oz. vitamin C

b) Find the stable range for the protein in a bag of Dog's Life Canine Cuisine.

Solution: Repeat the calculations of part (a), with u in place of 12, and set the entries in the last row ≥ 0:

$$
I = \hat{B}\hat{B}^{-1} = \begin{bmatrix} 3 & 0 & 1 & 0 \\ 2 & 0 & 5 & 0 \\ -3 & 1 & -7 & 0 \\ -u & 0 & -10 & 1 \end{bmatrix} \begin{bmatrix} \frac{5}{13} & -\frac{1}{13} & 0 & 0 \\ \frac{1}{13} & \frac{18}{13} & 1 & 0 \\ -\frac{2}{13} & \frac{3}{13} & 0 & 0 \\ M_1 & M_2 & M_3 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ M_1 - \frac{5}{13}u + \frac{20}{13} & M_2 - \frac{30}{13} + \frac{u}{13} & M_3 & 1 \end{bmatrix}
$$

Thus,

$$
M_1 = \frac{5}{13}u - \frac{20}{13}
$$
$$
M_2 = \frac{30}{13} - \frac{u}{13}
$$
$$
M_3 = 0
$$

Now multiply by the initial tableau:

$$\begin{bmatrix} \frac{5}{13} & -\frac{1}{13} & 0 & 0 \\ \frac{1}{13} & \frac{18}{13} & 1 & 0 \\ -\frac{2}{13} & \frac{3}{13} & 0 & 0 \\ \frac{5}{13}u - \frac{20}{13} & \frac{30}{13} - \frac{u}{13} & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 & 0 & 0 & 0 & 21 \\ 2 & 5 & 0 & 1 & 0 & 0 & 40 \\ -3 & -7 & 0 & 0 & 1 & -1 & -21 \\ -u & -10 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \frac{5}{13} & -\frac{1}{13} & 0 & 0 & 5 \\ 0 & 0 & \frac{1}{13} & \frac{18}{13} & 1 & -1 & 36 \\ 0 & 1 & -\frac{2}{13} & \frac{3}{13} & 0 & 0 & 6 \\ 0 & 0 & \frac{5}{13}u - \frac{20}{13} & \frac{30}{13} - \frac{u}{13} & 0 & 0 & 5u + 60 \end{bmatrix}$$

The only conditions on u come from making the marginal values nonnegative. We could have predicted this knowing that the final row for the tableau for this corner only had nonzero entries in the marginal values. Hence, we could have just computed $\hat{B}^{-1}C$ instead of multiplying by the entire tableau. In any case,

$$\frac{5}{13}u - \frac{20}{13} \geq 0$$
$$\frac{30}{13} - \frac{u}{13} \geq 0$$

The intersection of these conditions is the desired stable range:

$$4 \leq u \leq 30, \text{ or } [4,30]$$

c) Find the stable range for the protein in a bag of Way of Life Healthy Mix.

Solution: Follow the same steps as part (b), except put the u in the objective row in the y column instead of the x column:

$$I = \hat{B}\hat{B}^{-1} \begin{bmatrix} 3 & 0 & 1 & 0 \\ 2 & 0 & 5 & 0 \\ -3 & 1 & -7 & 0 \\ -5 & 0 & -u & 1 \end{bmatrix} \begin{bmatrix} \frac{5}{13} & -\frac{1}{13} & 0 & 0 \\ \frac{1}{13} & \frac{18}{13} & 1 & 0 \\ -\frac{2}{13} & \frac{3}{13} & 0 & 0 \\ M_1 & M_2 & M_3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ M_1 + \frac{2}{13}u - \frac{25}{13} & M_2 + \frac{5}{13} - \frac{3}{13}u & M_3 & 1 \end{bmatrix}$$

Thus,

$$M_1 = \frac{25}{13} - \frac{2}{13}u$$
$$M_2 = \frac{3}{13}u - \frac{5}{13}$$
$$M_3 = 0$$

Next,

$$\begin{bmatrix} \frac{5}{13} & -\frac{1}{13} & 0 & 0 \\ \frac{1}{13} & \frac{18}{13} & 1 & 0 \\ -\frac{2}{13} & \frac{3}{13} & 0 & 0 \\ \frac{25}{13} - \frac{2}{13}u & \frac{3}{13}u - \frac{5}{13} & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 & 0 & 0 & 0 & 21 \\ 2 & 5 & 0 & 1 & 0 & 0 & 40 \\ -3 & -7 & 0 & 0 & 1 & -1 & -21 \\ -5 & -u & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & \frac{5}{13} & -\frac{1}{13} & 0 & 0 & 5 \\ 0 & 0 & \frac{1}{13} & \frac{18}{13} & 1 & -1 & 36 \\ 0 & 1 & -\frac{2}{13} & \frac{3}{13} & 0 & 0 & 6 \\ 0 & 0 & \frac{25}{13} - \frac{2}{13}u & \frac{3}{13}u - \frac{5}{13} & 0 & 0 & 6u + 25 \end{bmatrix}$$

Thus,

$$\frac{25}{13} - \frac{2}{13}u \geq 0$$
$$\frac{3}{13}u - \frac{5}{13} \geq 0$$

This reduces to the stable range in question:

$$\frac{5}{3} \leq u \leq \frac{25}{2}$$

8. a) Consider the Brain One Lighting and Heavenly Music Corporation problem (Exercise 7 in Section 9.3.2) The company is considering hiring more designers. Compute the stable range for the weeks available per year for the design team (the capacity number that is currently 48.) Repeat for the weeks available with the technical team (the capacity number which is currently 44.)

Solution: From our previous solution, the initial and final tableaux are:

	x	y	z	S_1	S_2	S_3	$Capacity$
S_1	6	5	3	1	0	0	48
S_2	4	4	3	0	1	0	44
S_3	2	1	1	0	0	1	30
P	-8	-6	-5	0	0	0	0

And

	x	y	z	S_1	S_2	S_3	$Capacity$
x	1	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	2
z	0	$\frac{2}{3}$	1	$-\frac{2}{3}$	1	0	12
S_3	0	$-\frac{2}{3}$	0	$-\frac{1}{3}$	0	1	14
P	0	$\frac{4}{3}$	0	$\frac{2}{3}$	1	0	76

Since we are finding the stable range of a capacity column entry we can use B^{-1}, rather than $\hat{B}^{-1}$. We solve for both teams at once:

$$B^{-1}C = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{2}{3} & 1 & 0 \\ -\frac{1}{3} & 0 & 1 \end{bmatrix} \begin{bmatrix} u & 48 \\ 44 & v \\ 30 & 30 \end{bmatrix} = \begin{bmatrix} \frac{u}{2} - 22 & 24 - \frac{v}{2} \\ 44 - \frac{2}{3}u & v - 32 \\ 30 - \frac{u}{3} & 14 \end{bmatrix}$$

The first column contains the conditions on u, which determines the stable range for the design team, the second column contains conditions on v and gives the stable range for the technical team.

$$\frac{u}{2} - 22 \geq 0$$
$$44 - \frac{2}{3}u \geq 0$$
$$30 - \frac{u}{3} \geq 0$$

These conditions reduce to the stable range in weeks for the design team:

$$44 \leq u \leq 66$$

For the technical team:

$$24 - \frac{v}{2} \geq 0$$
$$v - 32 \geq 0$$

These conditions reduce to the stable range in weeks for the technical team:

$$32 \leq v \leq 48$$

b) Find these stable ranges by solving the problem with Excel, as we did in Chapter 5, and verify that your solution agrees with what you got in part (a).

Solution: Left for the reader to check.

9. a) In the *Brain One Lighting and Heavenly Music Corporation* problem, determine the stable range for the profit generated by an airport contract, the profit generated by a museum contract, and the profit generated by a theater contract.

Solution: These are the objective coefficients, for which we need all of $\hat{B}^{-1}$. For airport contracts:

$$I = \hat{B}\hat{B}^{-1} = \begin{bmatrix} 6 & 3 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ -u & -5 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{2}{3} & 1 & 0 & 0 \\ -\frac{1}{3} & 0 & 1 & 0 \\ M_1 & M_2 & M_3 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ M_1 - \frac{u}{2} + \frac{10}{3} & M_2 - 5 + \frac{u}{2} & M_3 & 1 \end{bmatrix}$$

Thus,

$$M_1 = \frac{u}{2} - \frac{10}{3}$$
$$M_2 = 5 - \frac{u}{2}$$
$$M_3 = 0$$

Therefore, with these values in $\hat{B}^{-1}$, multiply by the initial tableau:

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{2}{3} & 1 & 0 & 0 \\ -\frac{1}{3} & 0 & 1 & 0 \\ \frac{u}{2}-\frac{10}{3} & 5-\frac{u}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & 5 & 3 & 1 & 0 & 0 & 48 \\ 4 & 4 & 3 & 0 & 1 & 0 & 44 \\ 2 & 1 & 1 & 0 & 0 & 1 & 30 \\ -u & -6 & -5 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 2 \\ 0 & \frac{2}{3} & 1 & -\frac{2}{3} & 1 & 0 & 12 \\ 0 & -\frac{2}{3} & 0 & -\frac{1}{3} & 0 & 1 & 14 \\ 0 & \frac{u}{2}-\frac{8}{3} & 0 & \frac{u}{2}-\frac{10}{3} & 5-\frac{u}{2} & 0 & 2u+60 \end{bmatrix}$$

We see u satisfies conditions beyond that the marginal values are nonnegative, since there is a u is the second column, bottom row. The conditions are

$$\frac{u}{2}-\frac{8}{3} \geq 0$$
$$\frac{u}{2}-\frac{10}{3} \geq 0$$
$$5-\frac{u}{2} \geq 0$$

This reduces to the stable range we seek for airport contracts (in units of millions of dollars):

$$\frac{20}{3} \leq u \leq 10$$

For museum contracts, which are not in solution, note that y is a nonbasic variable. Thus, the y column is not part of $\hat{B}$ so this matrix and its inverse do not change with u. We already know the result of multiplying $\hat{B}^{-1}$ by all the columns in the initial tableau, except for the y column containing the variable u. Thus

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{2}{3} & 1 & 0 & 0 \\ -\frac{1}{3} & 0 & 1 & 0 \\ \frac{2}{3} & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \\ 1 \\ -u \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{2}{3} \\ -\frac{2}{3} \\ \frac{22}{3} - u \end{bmatrix}$$

It follows that the stable range for the profit generated by a museum contract is $-\infty \leq u \leq \frac{22}{3}$, or the interval $\left(-\infty, \frac{22}{3}\right]$.

For the profit generated by a theater contract:

$$I = \hat{B}\hat{B}^{-1} \begin{bmatrix} 6 & 3 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ -8 & -u & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{2}{3} & 1 & 0 & 0 \\ -\frac{1}{3} & 0 & 1 & 0 \\ M_1 & M_2 & M_3 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ M_1 + \frac{2}{3}u - 4 & M_2 - u + 4 & M_3 & 1 \end{bmatrix}$$

Thus,

$$M_1 = 4 - \frac{2}{3}u$$
$$M_2 = u - 4$$
$$M_3 = 0$$

Multiplying by the initial tableau:

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{2}{3} & 1 & 0 & 0 \\ -\frac{1}{3} & 0 & 1 & 0 \\ 4 - \frac{2}{3}u & u - 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & 5 & 3 & 1 & 0 & 0 & 48 \\ 4 & 4 & 3 & 0 & 1 & 0 & 44 \\ 2 & 1 & 1 & 0 & 0 & 1 & 30 \\ -8 & -6 & -u & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 2 \\ 0 & \frac{2}{3} & 1 & -\frac{2}{3} & 1 & 0 & 12 \\ 0 & -\frac{2}{3} & 0 & -\frac{1}{3} & 0 & 1 & 14 \\ 0 & \frac{2}{3}u - 2 & 0 & 4 - \frac{2}{3}u & u - 4 & 0 & 12u + 16 \end{bmatrix}$$

The conditions on u (including a condition other than on the marginal values) are:

$$\frac{2}{3}u - 2 \geq 0$$
$$4 - \frac{2}{3}u \geq 0$$
$$u - 4 \geq 0$$

This reduces to the stable range for the profit generated by a theater contract (in uits of millions of dollars):

$4 \leq u \leq 6$, or [4,6].

b) Find these stable ranges by solving the problem with Excel, as we did in Chapter 5, and verify that your solution agrees with what you got in part (a).

Solution: Left for the reader to check.

10. Consider the following minimization problem, which you have solved earlier in the text:

Solve the following scheduling problem. The *Poseidon's Wake* petroleum company operates two refineries. The *Cadence Refinery* can produce 40 units of low grade oil, 10 units medium grade oil, and 10 units high grade oil in a single day. (Each unit is 1000 barrels.) The *Cascade Refinery* can produce 10 units low grade oil, 10 units medium grade oil, and 30 units high grade oil in a single day. They receive an order from the Mars Triangle Oil Retailers for at least 80 units low grade oil, at least 50 units medium grade oil, and at least 90 units high grade oil. If it costs Poseidon's Wake $1,800 to operate the Cadence refinery for a day, and $2,000 to operate the Cascade Refinery for a day, how many days should they operate each refinery to fill the order at least cost?

In an exercise in Chapter 4, some sensitivity analysis was done on this problem based on graphical techniques. In this exercise, we use the techniques of this chapter, while exploring the connections between stables ranges and duality.

a) Solve the problem using phase I pivoting as we did in Section 9.2, or look up your previous solution to Exercise 5 in Section 9.2.

Solution: From our solution in Section 9.2, the initial and final tableaux are:

	x	y	S_1	S_2	S_3	A_1	A_2	A_3	$Capacity$
$-A_1$	-40	-10	1	0	0	-1	0	0	-80
$-A_2$	-10	-10	0	1	0	0	-1	0	-50
$-A_3$	-10	-30	0	0	1	0	0	-1	-90
$-C$	1800	2000	0	0	0	0	0	0	0
z	-60	-50	1	1	1	0	0	0	-220

And

	x	y	S_1	S_2	S_3	$Capacity$
x	1	0	0	$-\frac{3}{20}$	$\frac{1}{20}$	3
S_1	0	0	1	$-\frac{11}{2}$	$\frac{3}{2}$	60
y	0	1	0	$\frac{1}{20}$	$-\frac{1}{20}$	2
$-C$	0	0	0	170	10	-9400

b) Instead, solve the problem using duality without phase I, or look u your previous solution to Exercise 5 in Section 5.3.

Solution: From our solution in Section 5.3, the initial and final tableau are:

	u	v	w	T_1	T_2	$Capacity$
T_1	40	10	10	1	0	1800
T_2	10	10	30	0	1	2000
P	-80	-50	-90	0	0	0

And:

	u	v	w	T_1	T_2	$Capacity$
v	$\frac{11}{2}$	1	0	$\frac{3}{20}$	$-\frac{1}{20}$	170
w	$-\frac{3}{2}$	0	1	$-\frac{1}{20}$	$\frac{1}{20}$	10
P	60	0	0	3	2	9400

c) We are interested in finding the table range in the number of units of high-grade oil which comprises part of the order from the Mars Triangle Oil Retailers (this number, 90, is currently a capacity number in the primal problem, and also an objective coefficient in the dual problem.) Find the stable range for this capacity number in the primal problem.

Solution: As an entry in the capacity column in the primal problem, we can find its stable range using B^{-1}:

$$B^{-1}C = \begin{bmatrix} 0 & -\frac{3}{20} & \frac{1}{20} \\ 1 & -\frac{11}{2} & \frac{3}{2} \\ 0 & \frac{1}{20} & -\frac{1}{20} \end{bmatrix} \begin{bmatrix} -80 \\ -50 \\ -u \end{bmatrix} = \begin{bmatrix} \frac{15}{2} - \frac{u}{20} \\ 195 - \frac{3}{2}u \\ \frac{u}{20} - \frac{5}{2} \end{bmatrix}$$

Thus:

$$\frac{15}{2} - \frac{u}{20} \geq 0$$
$$195 - \frac{3}{2}u \geq 0$$
$$\frac{u}{20} - \frac{5}{2} \geq 0$$

These conditions reduce to the stable range for u:

$50 \leq u \leq 130$, or [50,130]

d) Instead, use the dual problem and compute its stable range as an objective coefficient. Compare your answer to that of part (c).

Solution: We need $\hat{B}^{-1}$ for the dual problem since this number now appears as an objective coefficient.

$$I = \hat{B}\hat{B}^{-1} = \begin{bmatrix} 10 & 10 & 0 \\ 10 & 30 & 0 \\ -50 & -u & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{20} & -\frac{1}{20} & 0 \\ -\frac{1}{20} & \frac{1}{20} & 0 \\ M_1 & M_2 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ M_1 + \frac{u}{20} - \frac{15}{2} & M_2 - \frac{u}{20} + \frac{5}{2} & 1 \end{bmatrix}$$

Thus, $M_1 = \frac{15}{2} - \frac{u}{20}$ and $M_2 = \frac{u}{20} - \frac{5}{2}$. Now multiply $\hat{B}^{-1}$ times the initial tableau:

$$\begin{bmatrix} \frac{3}{20} & -\frac{1}{20} & 0 \\ -\frac{1}{20} & \frac{1}{20} & 0 \\ \frac{15}{2}-\frac{u}{20} & \frac{u}{20}-\frac{5}{2} & 1 \end{bmatrix} \begin{bmatrix} 40 & 10 & 10 & 1 & 0 & 1800 \\ 10 & 10 & 30 & 0 & 1 & 2400 \\ -80 & -50 & -u & 0 & 0 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} \frac{11}{2} & 1 & 0 & \frac{3}{20} & -\frac{1}{20} & 170 \\ -\frac{3}{2} & 0 & 1 & -\frac{1}{20} & \frac{1}{20} & 10 \\ 195-\frac{3}{2}u & 0 & 0 & \frac{15}{2}-\frac{u}{20} & \frac{u}{20}-\frac{5}{2} & 10u+8500 \end{bmatrix}$$

Thus, u must satisfy:

$$195-\frac{3}{2}u \geq 0$$
$$\frac{15}{2}-\frac{u}{20} \geq 0$$
$$\frac{u}{20}-\frac{5}{2}$$

These are exactly the same inequalities as in part (c), so we obtain the same stable range (which we hope the reader expected to be the case!):

$$50 \leq u \leq 130$$

e) Suppose Mars Triangle Oil lowers that part of its order from 90 units to 60 units. Determine the revised optimal data. (You can use either the primal or the dual to answer this.)

Solution: The change is within the stable range. If we simply plug in $u = 60$ in the final tableau for the dual which we computed in part (d), we obtain the current tableau, which we know will be optimal:

$$\begin{bmatrix} \frac{11}{2} & 1 & 0 & \frac{3}{20} & -\frac{1}{20} & 170 \\ -\frac{3}{2} & 0 & 1 & -\frac{1}{20} & \frac{1}{20} & 10 \\ 105 & 0 & 0 & \frac{9}{2} & \frac{1}{2} & 9100 \end{bmatrix}$$

We can now read off the revised optimal data (remember, this tableau is for the dual problem, so the solution appears in the bottom row):

$x = \frac{9}{2}$ days to operate Cadence refinery

$y = \frac{1}{2}$ day to operate Cascade refinery

$C = \$9{,}100$ (Minimized)

$S_1 = 105$ units surplus low grade oil, $M_1 (= u) = 0$

$S_2 = 0$ units surplus medium grade oil, $M_2 (= v) = \$170/\text{unit}$

$S_3 = 0$ units surplus high grade oil, $M_3 (= w) = \$10/\text{unit}$

9.3.4 Integer Programming: Branch and Bound Method

Music allusions in the text:

The appliance store example (Example 9.11) alludes to the band Dire Straits (The name of the store, Dryer States, is a sort of spoonerism of the band name – switching the 'r' instead of the first letters of each word.) The band released songs 'Sultans of Swing' (1978) and 'Money for Nothing' (1985.) The latter song was a major MTV video hit, featuring workers who complained they had to "move refrigerators" or "color TVs" – hard work compared to rock musicians who got paid to do virtually nothing and filled the airwaves with "Hawaiian noises."

The Glasswork Monolith Arts example (Example 9.12) contains several musical references. Overall the, homage is to minimalist composer Philip Glass, who released many works including a 1982 album entitled Glassworks. The example also alludes to songs by other artists who have released songs with the word 'glass' in the title. The 'Battle of Glass Tears' is a section of the Lizard Suite from the 1971 King Crimson album Lizard. 'Heart of Glass' is the title of a 1978 song by new wave/punk band Blondie. 'Glass Onion' is a song from the 1968 eponymous album by the Beatles (the White album.) (Websites: https://philipglass.com, https://www.blondie.net/)

Solutions to Exercises and more music references:

1. Check all the pivoting in Example 9.11, the *Money For Nothing Rebate Sale* problem, and verify the solutions at each node.

Solution: Details left for the reader.

2. Re-solve the *Money For Nothing* problem, where at node 1, the branching is done by adding constraints that involve the variable y instead of x. Verify that you obtain the same answer as in Example 9.11

Solution: Recall the setup of the problem: Let x stand for the number of refrigerators and y stand for the number of color TVs. Then we must:

$$\text{Maximize } P = 500c + 400y$$
$$\text{Subject to:}$$
$$x + y \leq 60 \text{ (contract requirement)}$$
$$8x + 5y \leq 400 \text{ (space requirement)}$$
$$x \geq 0, \quad y \geq 0$$

The solution (ignoring the values of the slack variables and the marginal values) for node 1 is:

$$x = \frac{100}{3} \approx 33.33$$
$$y = \frac{80}{3} \approx 27.33$$
$$P = \frac{82000}{3} \approx \$26{,}666.67$$

We add constraints that force y to be an integer. For node 2 we have:

Maximize $P = 500c + 400y$
Subject to:
$x + y \leq 60$ (contract requirement)
$8x + 5y \leq 400$ (space requirement)
$y \leq 26$
$x \geq 0, \quad y \geq 0$

And for node 3 we have:

Maximize $P = 500c + 400y$
Subject to:
$x + y \leq 60$ (contract requirement)
$8x + 5y \leq 400$ (space requirement)
$y \geq 27$
$x \geq 0, \quad y \geq 0$

Solving the problem at node 2:
Initial tableau:

	x	y	S_1	S_2	S_3	$Capacity$
S_1	1	1	1	0	0	60
S_2	8	5	0	1	0	400
S_3	0	1	0	0	1	26
P	-500	-400	0	0	0	0

After two pivots, you should arrive at the optimal tableau:

	x	y	S_1	S_2	S_3	$Capacity$
S_1	0	0	1	$-\frac{1}{8}$	$-\frac{3}{8}$	$\frac{1}{4}$
x	1	0	0	$\frac{1}{8}$	$-\frac{5}{8}$	$\frac{135}{4}$
y	0	1	0	0	1	26
P	0	0	0	$\frac{125}{2}$	$\frac{175}{2}$	27,275

The solution (ignoring slack variables and marginal values) is:

$$x = \frac{135}{4} = 33.75$$
$$y = 26$$
$$P = 27{,}275$$

For node 3, we have to use phase I. Write the third constraint as

$$-y + S_3 - A_3 = -27$$

So that

$$z = -A_3 = y - S_3 - 27$$

Thus, the initial tableau is

	x	y	S_1	S_2	S_3	A_3	$Capacity$
S_1	1	1	1	0	0	0	60
S_2	8	5	0	1	0	0	400
$-A_3$	0	-1	0	0	1	-1	-27
P	-500	-400	0	0	0	0	0
z	0	-1	0	0	1	0	-27

After one pivot, we arrive at the tableau:

	x	y	S_1	S_2	S_3	A_3	$Capacity$
S_1	1	0	1	0	1	−1	33
S_2	8	0	0	1	5	−5	265
y	0	1	0	0	−1	1	27
P	−500	0	0	0	−400	400	10,800
z	0	0	0	0	0	1	0

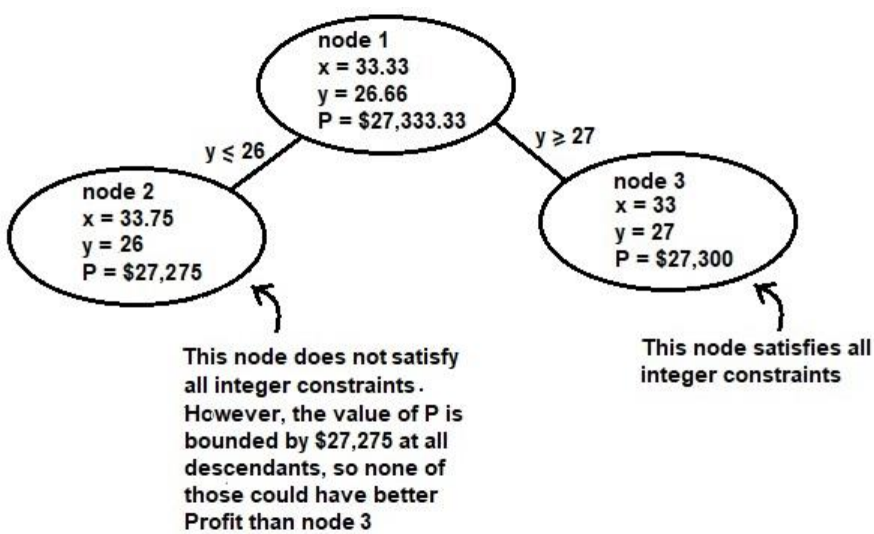
This completes phase I. Deleting the z row and the A_3 column, we continue with phase II: One more pivot yields:

	x	y	S_1	S_2	S_3	$Capacity$
x	1	0	1	0	1	33
S_2	0	0	−8	1	−3	1
y	0	1	0	0	−1	27
P	0	0	500	0	100	27,300

The solution (ignoring the slack and marginal values) is:

$$x = 33$$
$$y = 27$$
$$P = 27{,}300$$

We note that the solution is in integers, so no further branching from node 3 is needed. What about node 2? Any further branching from this node will produce solutions where P is bounded above by 27,275, which is less than 27,300, so they could not possibly improve on the solution at node 3. Thus, node 2 is terminal as well. The solution is the one at node 3, which does agree with the one in the text. The following picture summarizes the situation:



3. Check all the pivoting in Example 9.12, the Glassworks Monolith Art Studio Shop problem, and verify the solutions at each node.

Solution: Like Exercise 1, this is just more practice with the simplex algorithm. Details are left for the reader.

4. This exercise illustrates that sometimes we can apply the bounding part of the process slightly more generally than explained in the examples above. The goal is to solve the following problem:

$$\begin{gathered}\text{Maximize } P = 3x + 6y \\ \text{Subject to:} \\ 20x + 45y \leq 350 \\ 8x + 5y \leq 90 \\ x \geq 0, \;\; y \geq 0, \quad \text{and both variables integer}\end{gathered}$$

a) Solve the problem without the integer constraints to obtain the solution at node 1. You should get the solution $x = \frac{115}{13} = 8\frac{11}{13}$, $y = \frac{50}{13} = 3\frac{11}{13}$, and $P = \frac{645}{13} = 49\frac{8}{13}$.

Solution: The initial tableau is

	x	y	S_1	S_2	$Capacity$
S_1	20	45	1	0	350
S_2	8	5	0	1	90
P	-3	-6	0	0	0

After two pivots, we arrive at the final tableau:

	x	y	S_1	S_2	$Capacity$
y	0	1	$\frac{2}{65}$	$-\frac{1}{13}$	$\frac{50}{13}$
x	1	0	$-\frac{1}{52}$	$\frac{9}{52}$	$\frac{115}{13}$
P	0	0	$\frac{33}{260}$	$\frac{3}{52}$	$\frac{645}{13}$

This is the solution for node 1.

b) Add the constraint $x \leq 8$ to obtain the problem for node 2, and the constraint $x \geq 9$ to obtain the problem for node 3. Solve both problems. Verify that the solutions are: $x = 8$, $y = \frac{38}{9} = 4\frac{2}{9}$, and $P = \frac{148}{3} = 49\frac{1}{3}$ at node 2, and $x = 9$, $y = \frac{18}{5} = 3\frac{3}{5}$, and $P = \frac{243}{5} = 48.6$ at node 3.

Solution: For node 2:

	x	y	S_1	S_2	S_3	$Capacity$
S_1	20	45	1	0	0	350
S_2	8	5	0	1	0	90
S_3	1	0	0	0	1	8
P	-3	-6	0	0	0	0

After two pivots, we arrive at the final tableau:

	x	y	S_1	S_2	S_3	$Capacity$
y	0	1	$\frac{1}{45}$	0	$-\frac{4}{9}$	$\frac{38}{9}$
S_2	0	0	$-\frac{1}{9}$	1	$-\frac{52}{9}$	$\frac{44}{9}$
x	1	0	0	0	1	8
P	0	0	$\frac{2}{15}$	0	$\frac{1}{3}$	$\frac{148}{3}$

The solution is:

$$x = 8$$
$$y = \frac{38}{9} \approx 4.222$$
$$P = \frac{148}{3} \approx 49.333$$

(confirming what was claimed for node 2.)

For node 3, we need phase I: Write the constraint $x \geq 9$ as $-x + S_3 - A_3 = -9$, and $z = -A_3 = x - S_3 - 9$. Thus

	x	y	S_1	S_2	S_3	A_3	$Capacity$
S_1	20	45	1	0	0	0	350
S_2	8	5	0	1	0	0	90
$-A_3$	-1	0	0	0	1	-1	-9
P	-3	-6	0	0	0	0	0
z	-1	0	0	0	1	0	-9

After one pivot, we arrive at the end of phase I:

	x	y	S_1	S_2	S_3	A_3	$Capacity$
S_1	0	45	1	0	20	-20	170
S_2	0	5	0	1	8	-8	18
x	1	0	0	0	-1	1	9
P	0	-6	0	0	-3	3	27
z	0	0	0	0	1	0	0

Delete the appropriate row and column and continue with phase II. One more pivot yields:

	x	y	S_1	S_2	S_3	$Capacity$
S_1	0	0	1	-9	-52	8
y	0	1	0	$\frac{1}{5}$	$\frac{8}{5}$	$\frac{18}{5}$
x	1	0	0	0	-1	9
P	0	0	0	$\frac{6}{5}$	$\frac{33}{5}$	$\frac{243}{5}$

The solution is

$$x = 9$$
$$y = \frac{18}{5} = 3.6$$
$$P = \frac{243}{5} = 48.6$$

(confirming what was claimed for node 3.)

c) Now branch at node 2. Add the constraint $y \leq 4$ to obtain the problem for node 4, and $y \geq 5$ to obtain the problem for node 5. Solve both problems. Verify that the solutions are: $x = 8,\ y = 4$, and $P = 48$ at node 4, and $x = \frac{25}{4} = 6.25,\ y = 5,$ and $P = 48.75$ at node 5.

Solution: For node 4:

	x	y	S_1	S_2	S_3	S_4	$Capacity$
S_1	20	45	1	0	0	0	350
S_2	8	5	0	1	0	0	90
S_3	1	0	0	0	1	0	8
S_4	0	1	0	0	0	1	4
P	-3	-6	0	0	0	0	0

After two pivots, we arrive at the final tableau:

	x	y	S_1	S_2	S_3	S_4	$Capacity$
S_1	0	0	1	0	-20	-45	10
S_2	0	0	0	1	-8	-5	6
x	1	0	0	0	1	0	8
y	0	1	0	0	0	1	4
P	0	0	0	0	3	6	48

The solution is:

$$x = 8$$
$$y = 4$$
$$P = 48$$

(Conforming what was claimed for node 4.)

For node 5, we need phase I: the fourth constraint becomes $-y + S_4 - A_4 = -5$, and so $z = -A_4 = y - S_4 - 5$. Thus:

	x	y	S_1	S_2	S_3	S_4	A_4	$Capacity$
S_1	20	45	1	0	0	0	0	350
S_2	8	5	0	1	0	0	0	90
S_3	1	0	0	0	1	0	0	8
$-A_4$	0	-1	0	0	0	1	-1	-5
P	-3	-6	0	0	0	0	0	0
z	0	-1	0	0	0	1	0	-5

Phase I ends after one pivot:

	x	y	S_1	S_2	S_3	S_4	A_4	$Capacity$
S_1	20	0	1	0	0	45	-45	125
S_2	8	0	0	1	0	5	-5	65
S_3	1	0	0	0	1	0	0	8
y	0	1	0	0	0	-1	1	5
P	-3	0	0	0	0	-6	6	30
z	0	0	0	0	0	0	1	0

Delete the appropriate row and column and continue with phase II.

	x	y	S_1	S_2	S_3	S_4	$Capacity$
S_1	20	0	1	0	0	45	125
S_2	8	0	0	1	0	5	65
S_3	1	0	0	0	1	0	8
y	0	1	0	0	0	-1	5
P	-3	0	0	0	0	-6	30

After two more pivots, we arrive at the final tableau:

	x	y	S_1	S_2	S_3	S_4	$Capacity$
x	1	0	$\frac{1}{20}$	0	0	$\frac{9}{4}$	$\frac{25}{4}$
S_2	0	0	$\frac{2}{5}$	1	0	-13	15
S_3	0	0	$-\frac{1}{20}$	0	1	$-\frac{9}{4}$	$\frac{7}{4}$
y	0	1	0	0	0	-1	5
P	0	0	$\frac{3}{20}$	0	0	$\frac{3}{4}$	$\frac{195}{4}$

The solution is:

$$x = \frac{25}{4} = 6.25$$
$$y = 5$$
$$P = \frac{195}{4} = 48.75$$

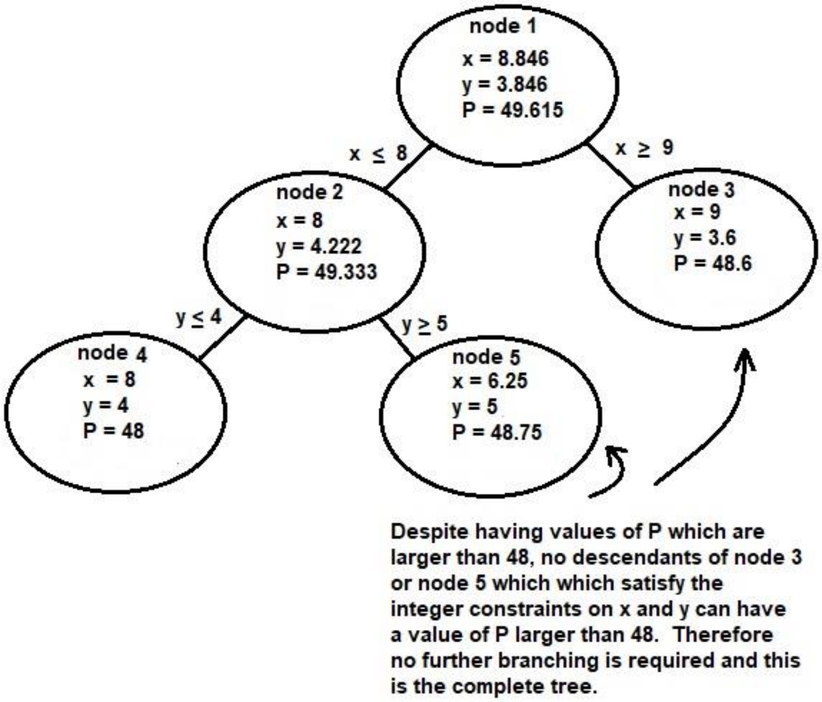
(conforming what was claimed for node 5.)

d) The value of the objective function at node 5, $P = 48.75$, is greater than the value at the terminal node 4, where $P = 48$. So it would appear that further branching is required from node 5. However, explain why any node descended from node 5 which satisfies all the integer constraints cannot have a value of P higher than 48. Therefore, no further branching is necessary at node 5. [**Hint**: The objective coefficients are integers!]

Solution: The descendants of node 5 all have their values of P bounded by 48.75. However, the objective coefficients are integers, so when P is evaluated at any point (x, y) with integer coordinates, then P itself must be an integer. Hence if any descendants of node 5 have integer solutions, the value of P will be at most the greatest integer less than 48.75, which is 48, the value of P at node 4. Thus, no descendant of node 5 (satisfying the integer constraints) could possibly improve on node 4, so no further branching is necessary from node 5.

e) Similarly, no node satisfying the integer constraints which are descendants of node 3 can hae a value higher than 48, even though the value of P at node 3 is higher than 48. Therefore, no further branching is required. Draw the binary tree for this problem and state the final answer which satisfies the integer constraints.

Solution: The binary tree is:



Thus, the optimal solution is at node 4:

$$x = 8$$
$$y = 4$$
$$P = 48$$

f) Verify that $x = 6, y = 5,$ and $P = 48$ is also a solution to this problem which satisfies the integer constraints. Since $P = 48$, which we already know is maximized by part (e), this is a different optimal solution. This illustrates that integer programming problems can have non-unique solutions. (Had we actually carried out the branching from node 5 which we decided was unnecessary, we would have found that solution as well.)

Solution: Indeed, if $x = 6$ and $y = 5$, then $P = 3x + 6y = 18 + 30 = 48$. Furthermore, (6,5) is a feasible point since it satisfies all the constraints:

$$20(6) + 45(3) = 345 < 350$$
$$8(6) + 5(3) = 73 < 90$$

And x, y are integers. Since we already know $P = 48$ is the maximal possible value, this is indeed another optimal solution.

5. The branch and bound method also works with minimization problems. The only difference is that when a branch constraint is added, to a problem and the new feasible set is a proper subset of the feasible set of the problem at the parent node, the value of the objective function must go up (or stay the same) instead of going down, so when you decide to ignore descendants of a node by the bounding

part of the process, you bound from below rather than from above. In other words, if at node A all the integer constraints are satisfied, and at node B they are not, but the values of the objective function at node A is less than or equal to the value at node B, you need not perform further branches from node B, since none of the descendant nodes will have an objective value lower than that at A. The following problem illustrates this:

Buttercup Builders is contracted by the *White Stripes Foundation* to build an apartment complex. Each studio apartment will cost 8 units to build, each one-bedroom apartment will cost 11 units to build, and each two-bedroom apartment will cost 12 units to build, where a unit of cost is $10,000. The contract stipulates that that at least 30 apartments must be included in the project, and at least $\frac{1}{4}$ of the total number of apartments must be two-bedroom apartments. How many of each type should be built in order to minimize the total cost?

Music References to: The Foundations, who released a pop song in 1967 entitled 'Build Me Up, Buttercup.' The example also alludes to the White Stripes, who released a song in 2000 entitled 'Let's Build a Home.' (Websites: https://en.wikipedia.org/wiki/Build_Me_Up_Buttercup and https://www.whitestripes.com/)

a) Set this up as an integer programming problem, and solve it without the integer constraints in order to find node 1.

Solution: Let x be the number of studio apartments, y the number of one-bedroom apartments, and z the number of two-bedroom apartments. Then:

$$\text{Minimize Cost } C = 8x + 11y + 12z$$
$$\text{Subject to:}$$
$$x + y + z \geq 30$$
$$x + y - 3z \leq 0$$
$$x \geq 0,\ \ y \geq 0,\ \text{ and integers}$$

Being a minimization problem, we use phase I: Maximize $-C = -8x - 11y - 12z$, and rewrite the first constraint as $-x - y - z + S_1 - A_1 = -30$, so that $Z = -A_1 = x + y + z - S_1 - 30$. We use a capital Z to distinguish it from z which already has a meaning. Thus:

	x	y	z	S_1	S_2	A_1	$Capacity$
$-A_1$	-1	-1	-1	1	0	-1	-30
S_2	1	1	-3	0	1	0	0
$-C$	8	11	12	0	0	0	0
Z	-1	-1	-1	1	0	0	-30

After two pivots, we arrive at the following tableau:

	x	y	z	S_1	S_2	A_1	$Capacity$
z	0	0	1	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{15}{2}$
x	1	1	0	$-\frac{3}{4}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{45}{2}$
$-C$	0	3	0	9	1	-9	-270
Z	0	0	0	0	0	1	0

Since $Z = 0$, phase I is complete. Delete the appropriate row and column and continue with phase II:

	x	y	z	S_1	S_2	$Capacity$
z	0	0	1	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{15}{2}$
x	1	1	0	$-\frac{3}{4}$	$\frac{1}{4}$	$\frac{45}{2}$
$-C$	0	3	0	9	1	-270

However, this table is optimal so no further pivoting is required. The solution at node 1 is:

$x = 22.5$ studio apartments
$y = 0$ one-bedroom apartments
$z = 7.5$ two-bedroom apartments
$C = 270$ units $=$ \$2,700,000 (Minimized)

b) Your solution to part (a) should be $x = 22.5$ (studio apartments), $y = 0$ (one-bedroom apartments), and $z = 7.5$ (two-bedroom apartments), with a cost of $C = 270$ units (which is \$2,700,000.) Branch on the variable x, so add $x \leq 22$ to create node 2 and $x \geq 23$ to create node 3. Solve both nodes.

Solution: For node 2 we add $x \leq 22$. Now $Z = -A_1 = x + y + z - S_1 - 30$ as before, so the only change to the initial tableau is a new row and a new column coming from the new slack variable S_3 that goes with the added constraint:

Type equation here.

	x	y	z	S_1	S_2	S_3	A_1	$Capacity$
$-A_1$	-1	-1	-1	1	0	0	-1	-30
S_2	1	1	-3	0	1	0	0	0
S_3	1	0	0	0	0	1	0	22
$-C$	18	11	12	0	0	0	0	0
Z	-1	-1	-1	1	0	0	0	-30

	x	y	z	S_1	S_2	S_3	A_1	$Capacity$
y	0	1	0	$-\frac{3}{4}$	$\frac{1}{4}$	-1	$\frac{3}{4}$	$\frac{1}{2}$
x	1	0	0	0	0	1	0	22
z	0	0	1	$-\frac{1}{4}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{15}{2}$
$-C$	0	0	0	$\frac{45}{4}$	$\frac{1}{4}$	3	$-\frac{45}{4}$	$-\frac{543}{2}$
Z	0	0	0	0	0	0	1	0

Since $Z = 0$, phase I is complete. Delete the appropriate row and column to continue with phase II:

	x	y	z	S_1	S_2	S_3	$Capacity$
y	0	1	0	$-\frac{3}{4}$	$\frac{1}{4}$	-1	$\frac{1}{2}$
x	1	0	0	0	0	1	22
z	0	0	1	$-\frac{1}{4}$	$-\frac{1}{4}$	0	$\frac{15}{2}$
$-C$	0	0	0	$\frac{45}{4}$	$\frac{1}{4}$	3	$-\frac{543}{2}$

However, this tableau is already optimal, so no further pivoting is required. The solution for node 2 is:

$x = 22$ studio apartments
$y = \frac{1}{2}$ one-bedroom apartments
$z = \frac{15}{2}$ two-bedroom apartments
$C = \frac{543}{2} = 271.5$ units, or \$2,715,000

For node 3, we add $x \geq 23$, which we write as $-x + S_3 - A_3 = -23$. Now, $Z = -A_1 - A_3 = 2x + y + z - S_1 - S_3 - 53$. Thus:

	x	y	z	S_1	S_2	S_3	A_1	A_3	$Capacity$
$-A_1$	-1	-1	-1	1	0	0	-1	0	-30
S_2	1	1	-3	0	1	0	0	0	0
$-A_3$	-1	0	0	0	0	1	0	-1	-23
$-C$	8	11	12	0	0	0	0	0	0
Z	-2	-1	-1	1	0	1	0	0	-53

After three pivots, we arrive at the following tableau:

	x	y	z	S_1	S_2	S_3	A_1	A_3	$Capacity$
z	0	$-\frac{1}{3}$	1	0	$-\frac{1}{3}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{23}{3}$
x	1	0	0	0	0	-1	0	1	23
S_1	0	$-\frac{4}{3}$	0	1	$-\frac{1}{3}$	$-\frac{4}{3}$	-1	$\frac{4}{3}$	$\frac{2}{3}$
$-C$	0	15	0	0	4	12	0	-12	-276
Z	0	0	0	0	0	0	1	1	0

Phase I is complete. Delete the Z row and the two columns with artificial variables for phase II:

	x	y	z	S_1	S_2	S_3	$Capacity$
z	0	$-\frac{1}{3}$	1	0	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{23}{3}$
x	1	0	0	0	0	-1	23
S_1	0	$-\frac{4}{3}$	0	1	$-\frac{1}{3}$	$-\frac{4}{3}$	$\frac{2}{3}$
$-C$	0	15	0	0	4	12	-276

However, this tableau is already optimal, so no further pivoting is required. The solution at node 3 is:

$x = 23$ studio apartments
$y = 0$ one-bedroom apartments
$z = \frac{25}{3} \approx 7.667$ two-bedroom apartments
$C = 276$ units, or \$2,760,000

c) Your solution to node 2 should be $x = 22, y = .5,\ z = 7.5$, and $C = 271.5$ and your solution to node 3 should be $x = 23, y = 0,\ z = 7.6666$, and $C = 276$. Neither node satisfies the integer constraints, but since this is a minimization problem, we choose to branch on node 2 and same node 3 for later. We will branch on the variable y. So create node 4 by adding the constraint $y \leq 0$ and create node 5 by adding the constraint $y \geq 1$. Solve both nodes.

Solution: For node 4 we have:

	x	y	z	S_1	S_2	S_3	S_4	A_1	$Capacity$
$-A_1$	-1	-1	-1	1	0	0	0	-1	-30
S_2	1	1	-3	0	1	0	0	0	0
S_3	1	0	0	0	0	1	0	0	22
S_4	0	1	0	0	0	0	1	0	0
$-C$	8	11	12	0	0	0	0	0	0
Z	-1	-1	-1	1	0	0	0	0	-30

After four pivots, we arrive at the tableau:

	x	y	z	S_1	S_2	S_3	S_4	A_1	$Capacity$
S_2	0	0	0	-3	1	-4	-4	3	2
x	1	0	0	0	0	1	0	0	22
z	0	0	1	-1	0	-1	-1	1	8
y	0	1	0	0	0	0	1	0	0
$-C$	8	0	0	12	0	4	1	-12	-272
Z	0	0	0	0	0	0	0	1	0

Phase I is complete. After deleting the Z row and the A_3 column, what remains is an optimal tableau for phase II, so no further pivoting is required. The solution for node 4 is:

$x = 22$ studio apartments
$y = 0$ one-bedroom apartments
$z = 8$ two-bedroom apartments
$C = 272$ units, or \$2,720,000

For node 5 we add the constraint $y \geq 1$, which we write as $-y + S_4 - A_4 = -1$. Now $Z = -A_1 - A_4 = x + 2y + z - S_1 - S_4 - 31$. Thus:

	x	y	z	S_1	S_2	S_3	S_4	A_1	A_4	$Capacity$
$-A_1$	-1	-1	-1	1	0	0	0	-1	0	-30
S_2	1	1	-3	0	1	0	0	0	0	0
S_3	1	0	0	0	0	1	0	0	0	22
$-A_4$	0	-1	0	0	0	0	1	0	-1	-1
$-C$	8	11	12	0	0	0	0	0	0	0
Z	-1	-2	-1	1	0	0	1	0	0	-31

After three pivots we arrive at the following tableau:

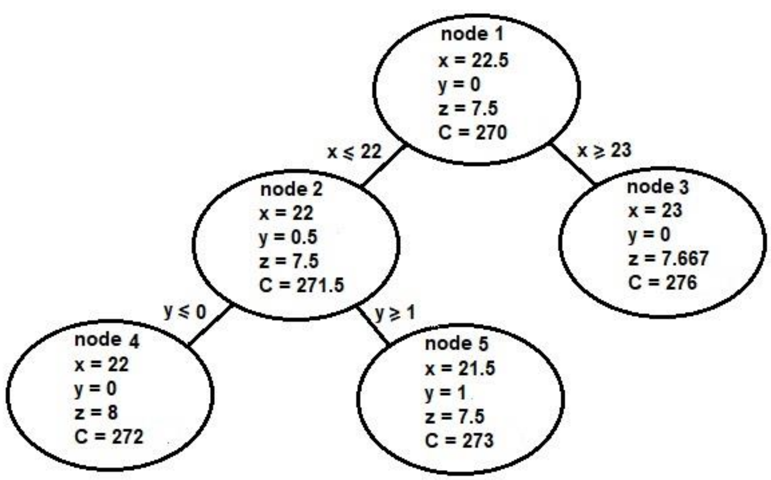
	x	y	z	S_1	S_2	S_3	S_4	A_1	A_4	$Capacity$
x	1	0	0	$-\frac{3}{4}$	$\frac{1}{4}$	0	1	$\frac{3}{4}$	-1	$\frac{43}{2}$
y	0	1	0	0	0	0	-1	0	1	1
S_3	0	0	0	$\frac{3}{4}$	$-\frac{1}{4}$	1	-1	$-\frac{3}{4}$	1	$\frac{1}{2}$
z	0	0	1	$-\frac{1}{4}$	$-\frac{1}{4}$	0	0	$\frac{1}{4}$	0	$\frac{15}{2}$
$-C$	0	0	0	9	1	0	3	-9	-3	-273
Z	0	0	0	0	0	0	0	1	1	0

Phase I is complete. Delete the Z row and the columns headed by artificial variables for phase II. But the tableau that remains is already optimal, so no further pivoting is required. The solution for node 5 is:

$$x = \frac{43}{2} = 21.5 \text{ studio apartments}$$
$$y = 1 \text{ one-bedroom apartments}$$
$$z = \frac{15}{2} = 7.5 \text{ two-bedroom apartments}$$
$$C = 273 \text{ units, or \$2,730,000}$$

d) Your solution for node 4 should be $x = 22, y = 0, z = 8$ with $C = 272$. This is a terminal node. Your solution to node 5 should be $x = 21.5, y = 1, z = 7.5$ with $C = 273$. Note that all nodes descending form node 5 will have a cost of $C \geq 273$, so no such descendant can have a better cost than node 4. Similarly, every descendant of node 3 will have a cost $C \geq 276$, so no further branching is necessary. Draw the binary tree for this problem, and state the optimal integer solution. Can you explain why it was not even necessary to solve node 5?

Solution: The binary tree is



Any descendant of node 3 has C bounded below by 276, so none could be better than node 4. Similarly, any descendant of node 5 has C bounded below by 273, so could not improve upon node 4. In fact, because the objective coefficients are integers, we could have predicted node 5 would have $C \geq 272$, since C would have to be an integer number of units. Thus node 4 is the optimal solution:

$x = 22$ studio apartments
$y = 0$ one-bedroom apartments
$z = 8$ two-bedroom apartments
$C = 272$ units, or $2,720,000

6. To solve an integer programming problem using *Excel*, proceed as follows. Ener the data on the spreadsheet as usual. In the step where you are adding the constraints, you can add a constraint that a decision variable must be an integer by (after opening the 'add constraint' pop-up) clicking on the cell which contains the value of that decision variable, then in the drop-down menu in the middle of the 'add constraint' pop-up, select "INT". Then solve the problem with the usual steps. Use this method to solve the integer programming problems:

a) The *Money for Nothing* problem (Exercise 1)

Solution: Left for the reader.

b) The *Glasswork Monolith Art Studio* shop problem (Exercise 2)

Solution: Left for the reader.

c) The *Buttercup Builders* problem (Exercise 5)

Solution: Left for the reader.

Appendix: A Rapid Review of Sets and Probability

Section A.1 A Review of Sets

Music References in the text: In the discussions of the distributive law for unions over intersections, three songs are alluded to: 'Blueberry Hill' was a song written in 1940 and released by a number of artists, including Glenn Miller, Gene Krupa, Louis Armstrong, Elvis Presley, and Elton John. The song is best known from the 1956 release by Fats Domino. 'Cherry Hill Park' is a 1969 song released by Billy Joe Royal. 'Strawberry Fields Forever' is a 1967 song by the Beatles. (Websites: https://en.wikipedia.org/wiki/Blueberry_Hill, https://en.wikipedia.org/wiki/Cherry_Hill_Park, and https://en.wikipedia.org/wiki/Strawberry_Fields_Forever)

Furthermore, in the discussion of DeMorgan's laws, more songs and albums are alluded to. Eat a Peach is the title of a 1972 album by the Allman Brothers. Other references include 'Brown Sugar', the 1971 song by the Rolling Stones, and 'A Taste of Honey', a song based on a TV series and movie of the same name in the early 1960s, and covered by several artists, including the Beatles on their first album in 1963, and Herb Alpert & the Tijuana Brass in 1965. (Websites: https://allmanbrothersband.com/, https://en.wikipedia.org/wiki/A_Taste_of_Honey)

Solutions to exercises (and more music references):

1. Let $A = \{a, b, c, d, e\}$, $B = \{a, c, e, f, g\}$, and $C = \{b, d, g, h\}$, and suppose $U = \{a, b, c, d, e, f, g, h\}$. Find the following sets:

a) $A \cup B$ and $A \cap B$.

Solution:

$$A \cup B = \{a, b, c, d, e, f, g\}$$

$$A \cap B = \{a, c, e\}$$

b) A^c, B^c, and C^c.

Solution:

$$A^c = \{f, g, h\}$$

$$B^c = \{b, d, h\}$$

$$C^c = \{a, c, e, f\}$$

c) $(A \cap B) \cup C$

Solution:

$$(A \cap B) \cup C = \{a, b, c, d, e, g, h\}$$

d) $A^c \cap B^c$ and $A^c \cup B^c$

Solution:

$$A^c \cap B^c = (A \cup B)^c = \{h\}$$

$$A^c \cup B^c = (A \cap B)^c = \{b, d, f, g, h\}$$

e) $A \cap B \cap C$

Solution:

$$A \cap B \cap C = \emptyset$$

f) $A \cap B^c \cap C$

Solution:

$$A \cap B^c \cap C = \{b, d\}$$

2. Suppose $U = \{1,2,3,4,5,6,7,8,9\}$, $E = \{x \in U | x \text{ is even}\}$, $F = \{x \in U | x \leq 7\}$, and $G = \{x \in U | x \leq 2 \ or \ x \geq 8\}$.

a) Find $E \cup G$ and $E \cap G$

Solution:

$$E \cup G = \{1,2,4,6,8,9\}$$

$$E \cap G = \{2,8\}$$

b) Find $E \cap F$ and $F \cap G$

Solution:

$$E \cap F = \{2,4,6\}$$

$$F \cap G = \{1,2\}$$

c) Find $E^c \cup F^c$

Solution:

$$E^c \cup F^c = (E \cap F)^c = \{1,3,5,7,8,9\}$$

d) Find $(E \cup F) \cap G$

Solution:

$$(E \cup F) \cap G = \{1.2.8\}$$

e) Express G as the union of two sets with simpler definitions.

Solution: Let $X = \{x \in U | x \leq 2\} = \{1,2\}$, and let $Y = \{x \in U | x \geq 8\} = \{8,9\}$. Then $G = X \cup Y$.

3. Suppose $U = \{\text{all living musicians}\}$, $S = \{\text{singers in rock and roll bands}\}$, $G = \{\text{guitar players}\}$, $K = \{\text{keyboard and piano players}\}$, and $D = \{\text{drummers}\}$. Give verbal descriptions of the following sets:

a) $S \cap (K \cup D)$

Solution: The set of all singers in rock and roll bands who also play either keyboards and pianos or drums.

b) $S \cap (G \cup K \cup D)$

Solution: The set of all singers in in rock and roll bands who also play either guitar, keyboards and piano, or drums.

c) $(S \cap D) \cup (G \cap K)$

Solution: The set of all living musicians who are either singers in rock and roll bands and drummers, or they are guitarists who also play keyboards and piano.

d) $G \cap K^c \cap D$

Solution: The set of all living musicians who play guitar and drums, but do not play keyboards and piano.

e) $S \cap G^c \cap K^c \cap D^c$

Solution: The set of all singers in rock and roll bands who do not play guitar, keyboards, piano, or drums.

4. Let $A = \{a, b, c\}$ and $B = \{x, y, z\}$. List the elements of $A \times B$ and $B \times A$.

Solution:

$$A \times B = \{(a, x), (a, y), (a, z), (b, x), (b, y), (b, z), (c, x), (c, y), (c, z)\}$$

$$B \times A = \{(x, a), (x, b), (x, c), (y, a), (y, b), (y, c), (z, a), (z, b), (z, c)\}$$

5. The *King of Hollywood Auto Showroom* is selling new 2021 model four-wheel drive SUV's called the *California Eagle*. The Eagle comes in the following colors: sunrise orange, witchy white, gone green, or peaceful purple. The following options for sound systems are available: Internet Radio, CD player, or both.

Music allusion to: The California based band the Eagles. The 'King of Hollywood' is a song from their 1979 album The Long Run. The car colors also refer to Eagles songs: 'Tequila Sunrise' (from Desperado, 1973), 'Witchy Woman' and 'Peaceful, Easy Feeling' from their 1972 eponymous debut album), and 'Already Gone' (from On the Border, 1974) (Website: https://eagles.com/)

a) Write down all the different possible choices as ordered pairs, with a color in the first coordinate and a sound system choice in the second. Thus, a Cartesian product represents all possible choices.

Solution: Abbreviate the colors by O (sunrise orange), W (Witchy White), G (Gone Green), and P (Peaceful Purple), and abbreviate the sound system choices by 1 (Internet radio), 2 (CD player), and 3 (both). Then the possible choices are

$$\left\{\begin{matrix}(O, 1), (O, 2), (O, 3), (W, 1), (W, 2), (W, 3) \\ (G, 1), (G, 2), (G, 3), (P, 1), (P, 2), (P, 3)\end{matrix}\right\}$$

If $C = \{O, W, G, P\}$ is the set of colors and $S = \{1,2,3\}$ is the set of sound system choices, then the above solution can be written more simply as $C \times S$.

b) Suppose that in addition to a color and a sound system, the buyer may also choose between a six-cylinder engine, an eight-cylinder engine, or a hybrid gas/electric engine. Describe how the different models can now be described as a Cartesian product.

Solution: Now each choice is codified by an ordered triple, with the first coordinate a color, the second a sound system choice, and the third an engine choice. If E is the set of engine choices, then the set of these ordered triples is simply $C \times S \times E$.

c) If the hybrid model does not come in the sunrise orange color, could you still use a Cartesian product to describe all the possible models?

Solution: No, because several elements of the Cartesian product $C \times S \times E$ are not actual car choices – namely, those with first coordinate O and third coordinate 'hybrid'. (At a somewhat deeper level, there is no possible Cartesian product of any sets in this case where one of the coordinates correspond to a color. Because if you included orange as a color, you would pick up some orange hybrids among the choices, which are not allowed. On the other hand, if you omitted orange to avoid these disallowed choices, then you also would miss all the other orange cars so your Cartesian product would be missing some possible models.)

6. Let $A = \{1,2,3,4\}$. Find $\mathcal{P}(A)$, the power set of A.

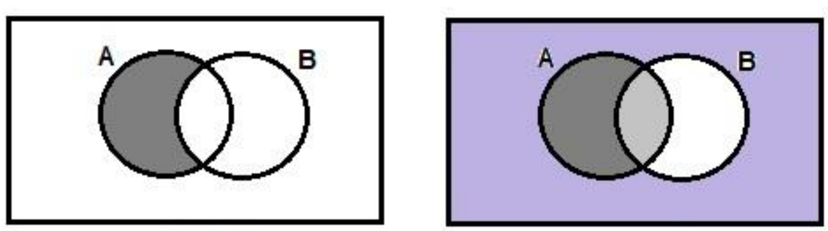
Solution: The power set is

$$\mathcal{P}(A) = \left\{ \begin{matrix} \emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \\ \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\} \\ \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\} \end{matrix} \right\}$$

7. There are other set operations, but the most important ones are easily expressed in terms of unions, intersections, and compliments. For example, the *set difference* is defined as $A - B = \{x \in A | x \notin B\}$.

a) Use a Venn diagram to show that $A - B = A \cap B^c$.

Solution:



A − B   A ∩ B^{c}

In the diagram on the left, we shade everything inside A, but outside B, the definition of $A - B$. In the diagram on the right, we shaded in A lighty, and shaded in B^c – everything outside of B, slightly darker. Where the two shaded regions overlap, it is shaded darkest – and it agrees with the picture on the left.

b) Rewrite $A - (B - C)$ and $(A - B) - C$ in terms of unions, intersections, and complements. Are they the same?

Solution:

$$A - (B - C) = A - (B \cap C^c)$$
$$= A \cap (B \cap C^c)^c,$$

by the definition of the set difference used twice. Next apply DeMorgan's law (recall $C^{c^c} = C$):

$$= A \cap (B^c \cup C)$$
$$= (A \cap B^c) \cup (A \cap C)$$

by the distributive law.

Similarly,

$$(A - B) - C = (A - B) \cap C^c$$
$$= A \cap B^c \cap C^c$$

They are not the same. Note that going further by DeMorgan's law, this is the same as $A \cap (B \cup C)^c = A - (B \cup C)$.

c) Let $U = \{1,2,3,4,5,6,7,8\}, A = \{2,4,6,8\}$, and $B = \{1,2,3\}$. Find $A - B$ and $B - A$.

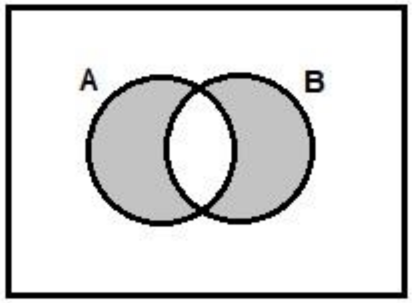
Solution:

$$A - B = \{4,6,8\}$$
$$B - A = \{1,3\}$$

8. The symmetric difference of sets is defined as $A \triangle B = (A - B) \cup (B - A)$.

a) Use a Venn diagram to show that $A \triangle B = (A \cup B) - (A \cap B)$. (Thus, the symmetric difference is the set theoretic connective that goes with the English 'exclusive or'.)

Solution: Both pictures reduce to the same shaded region:



b) Find $A \triangle B$ for the sets in Exercise 7c.

Solution:

$$A \triangle B = \{1,3,4,6,8\}$$

c) For any set, explain why $A \triangle A = \emptyset$.

Solution: We know by part (a) that

$$A \triangle A = (A \cup A) - (A \cap A)$$

But $A \cup A = A = A \cap A$, so

$$A \triangle A = A - A = \emptyset$$

Section A.2 Enumeration

Music References in the Text: The Art School Example (Example A.14) is a reference to American singer/songwriter Don McLean. In addition to his signature song 'American Pie', he also released a hit song 'Vincent' (Starry Starry Night)' about artist Vincent Van Gogh. (Website: https://donmclean.com/)

The horse race example (Example A.18) alludes to several songs and artists. The bettor Neil, and well as one of the horses 'Young and Crazy' allude to Neil Young's collaborations with the band Crazy Horse. 'Wildfire' is a song (about a horse) released in 1975 by Michael Murphey, and 'No-name' refers to the 1971 song 'Horse with No Name' by the band America. (Websites: https://en.wikipedia.org/wiki/Crazy_Horse, https://en.wikipedia.org/wiki/Wildfire, https://www.venturahighway.com/)

The committee in a monastery example (Example A.21) alludes to the pop band the Monkees, who had a hot song in 1967 entitled 'Pleasant Valley Sunday'. The names of the committee members are the names of the band members. (Website: https://www.monkees.com)

The Science Dept. example, (Example A.24) alludes to the 1966 song 'Eight Miles High' by the Byrds. (Website: https://en.wikipedia.org/wiki/Eight_Miles_High)

Solutions to exercises and more music references:

1. If $n(A) = 11, n(A \cup B) = 15$, and $n(A \cap B) = 4$, find $n(B)$.

Solution:

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$
$$15 = 11 + n(B) - 4$$
$$n(B) = 8$$

2. In the text, we showed that part (b) of Proposition A.13 follows from part (a). Show that part (a) follows from part (b), so that the two parts are actually equivalent. [**Hint**: Consider a Venn diagram.]

Solution: We assume part (b) holds. Notice, either from your Venn diagram or otherwise, that the union $A = (A - B) \cup (A \cap B)$ is disjoint, so $n(A) = n(A - B) + n(A \cap B)$. Similarly, $n(B) = n(B - A) + n(A \cap B)$, and so $n(B - A) = n(B) - n(A \cap B)$. Finally, notice that $A \cup B = A \cup (B - A)$, and the union on the right is disjoint, so

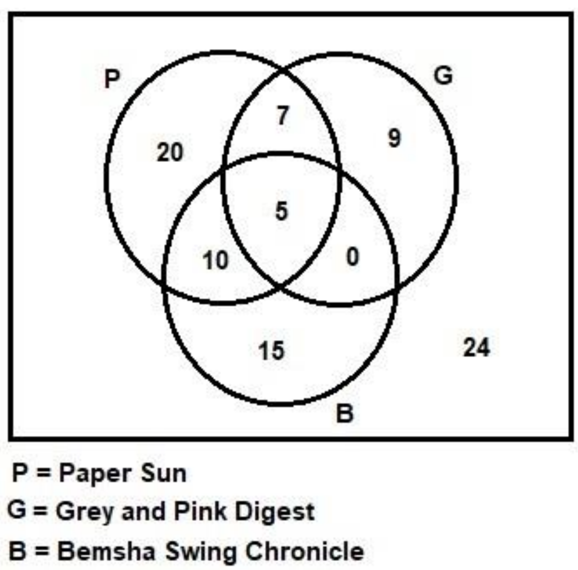
$$n(A \cup B) = n(A) + n(B - A) = n(A) + n(B) - n(A \cap B),$$

Which is exactly part (a).

3. *Caravan Country Club* is conducting a survey to determine the magazine subscription habits of their members. They receive a response from 90 golfers. 42 of them subscribe to *Paper Sun*, 21 of them subscribe to the *Grey and Pink Golf Digest*, and 30 of them subscribe to the *Bemsha Swing Chronicle*. Also 12 subscribe to both Paper Sun and the Golf Digest, 5 subscribe to both the Golf Digest and the Swing Chronicle, and 15 subscribe to both the Paper Sun and the Swing Chronicle. Finally, 5 subscribe to all three magazines. Answer the following questions.

Music References: to three different songs. The first is 'The Land of Grey and Pink', the title track from the 1971 album by the British band Caravan (which also included the song 'Golf Girl'.) The second is to the song 'Paper Sun', the first hit song from the British band Traffic in 1967. The third song is the 1952 jazz standard 'Bemsha Swing' from Thelonious Monk. (Websites: https://officialcaravan.co.uk/home/, https://en.wikipedia.org/wiki/Paper_Sun, https://en.wikipedia.org/wiki/Thelonious_Monk)

Solution: All questions can be answered using the Venn diagram:



a) How many subscribe to just the Paper Sun?

Solution: 20 (In symbols, it's $n\big(P-(G\cup B)\big).$)

b) How many subscribe to exactly one of the magazines?

Solution: From the diagram it's $20+9+15=44$ that subscribe to just one magazine. In symbols, it's $n\Big(\big(P-(G\cup B)\big)\cup\big(G-(P\cup B)\big)\cup\big(B-(P\cup G)\big)\Big)$, if you are curious.

c) How many subscribe to exactly two of the magazines?

Solution: $7+10+0=17$ subscribe to exactly two.

d) How many do not subscribe to any of the three?

Solution: 24 members.

4. a) Derive a formula for $n(A \cup B \cup C)$ in terms of cardinalities of the sets A, B, C and the various intersections of those sets. [**Hint**: $A \cup B \cup C = D \cup C$, where $D = A \cup B$.]

Solution: Following the hint,

$$n(A \cup B \cup C) = n(D \cup C)$$

$$= n(D) + n(C) - n(D \cap C)$$

$$= n(A \cup B) + n(C) - n(D \cap C)$$

$$= n(A) + n(B) + n(C) - n(A \cap B) - n(D \cap C)$$

However, the distributive law yields

$$D \cap C = (A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$

Thus,

$$n((D \cap C) = n\big((A \cap C) \cup (B \cap C)\big)$$

$$= n(A \cap C) + n(B \cap C) - n(A \cap B \cap C),$$

because $(A \cap C) \cap (B \cap C) = A \cap B \cap C$. Substituting this into the above equation, we have:

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

b) Can you guess how this formula generalizes to four or more sets?

Solution: Left of the reader to think about. If you want to look it up, its called the principle of inclusion/exclusion.

5. The *Night Hawks Diner* is running a dinner special. For a single price, Tom can choose one of three appetizers (clam chowder, shrimp cocktail, or oysters), one of five entrees (Shrimp Scampi, Flaming Scallops, Lobster a la Hopper, Strip Steak, od Surf and Turf), one of four side dishes (rice, red potatoes, mashed potatoes, or stir-fried veggie mix), and one of three desserts (cheesecake, chocolate cream pie, or apple crisp.) If he eats there once every day and orders a different dinner every night, how many·4 days must Tom wait before he must order a duplicate?

Music reference to: singer/songwriter Tom Waits. Waits released an album in 1975 called Nighthawks at the Diner (which was partly an homage to the painting Nighthawks by Edward Hopper.) (Website: https://www.tomwaits.com/news/)

Solution: Represent each dinner as an ordered quadruple with the appetizer in the first coordinate, the entrée in the second coordinate, the side dish n the third coordinate, and the dessert in the fourth coordinate. Then, by the multiplication principle, the total number of different possible meals is

$$3 \cdot 5 \cdot 4 \cdot 3 = 180$$

So, Tom can order 180 different dinner specials without duplicating any – a different dinner every night for almost 6 months.

6. The *Holistic House of Four Doors Healing Center* has a staff of 11 therapists. In how many different ways can they select 4 of their staff to line up for a photograph for an article in the *Sardonicus Twelve Dreams Newsletter*?

Music allusions to: The Moody Blues (their 1868 album In Search of the Lost Chord contained a song entitled 'House of Four Doors'), and to the American roc band Spirit (who released an album in 1970 entitled the Twelve Dreams of Dr. Sardonicus.) (Website: https://en.wikipedia.org/wiki/Spirit)

Solution: $P(11,4) = \frac{11!}{7!} = 7920$

7. A license plate has 3 letters followed by 5 digits. The digit 0 is not allowed because it can be confused with the letter O. If the letters cannot be repeated, but the digits can, how many different license plates can be formed?

Solution: $P(26,3) \cdot 9^5 = \frac{26!}{23!} \cdot 9^5 = 921{,}164{,}400$

8. In Exercise 5, how does the answer change if Tom is allowed to choose 2 different side dishes out of the four instead of just one?

Solution: There are $C(4,2) = 6$ different choices now for the pair of side dishes in the third coordinate. There are no other changes to the answer is $2 \cdot 5 \cdot 6 \cdot 3 = 270$ distinct dinners.

9. The Four Doors Healing Center from Exercise 6 must form some staff committees. The therapists include 4 medical doctors, 5 psychotherapists, and 2 acupuncturists.

a) How many different committees of 5 can be formed?

Solution: $C(11,5) = \frac{11!}{6!5!} = 462$

b) How many different committees of 5 can be formed with 2 doctors, 2 psychotherapists, and 1 acupuncturist?

Solution: $C(4,2) \cdot C(5,2) \cdot C(2,1) = 6 \cdot 10 \cdot 2 = 120$

c) How many committees of 5 can be formed with exactly two doctors?

Solution: $C(4,2) \cdot C(7,3) = 6 \cdot \frac{7!}{3!4!} = 210$

d) How many committees of 5 can be formed with 1 psychotherapist and at least 3 doctors?

Solution:

$$C(5,1) \cdot C(4,3) \cdot C(2,1) + C(5,1) \cdot C(4,4) \cdot C(2,0) = 5 \cdot 4 \cdot 2 + 5 \cdot 1 \cdot 1 = 45$$

10. A Voting system has 20 members. How many different coalitions are there?

Solution: A coalition is any subset of the voters, so the collection of all coalitions is the same as the collection of all subsets of the voters; that is, it is the power set of the set of voters. Thus the number of coalitions is $2^{20} = 1{,}048{,}576$.

11. a) In Exercise 10, how many coalitions of exactly k voters are there, if $k = 2,3,4,19,20$?

Solution: The answers are given by combinations: $C(20,2) = 190$ coalitions of 2 voters, $C(20,3) = 1140$ coalitions of 3 voters, $C(20,4) = 4845$ coalitions of 4 voters, $C(20,19) = 20$ coalitions of 19 voters, and $C(20,20) = 1$ coalition of 20 voters (the grand coalition.)

b) Based on your answers in part (a), can you prove the following identity about combinations?

$$C(n,0) + C(n,1) + C(n,2) + \cdots + C(n,n) = 2^n$$

Solution: Indeed, the number on the left side is the sum of the numbers of coalitions of size 0 plus the number of size 1, plus the number of size 2, etc. That is, it is the sum of the number of coalitions of all possible sizes. When summed, one obtains the total number of all coalitions, which we already know is 2^n.

Remark: Reader's familiar with the binomial theorem and Pascals's triangle will recognize that the left side is the sum of all the numbers in one row of Pascal's triangle. In fact, one can use the binomial theorem to prove the identity.

The next exercises are for readers who have already studied Section 8.5. Recall the definition of the Shapley-Shubik Power index. In the next few exercises, we will compute the SSI for the voters in the following voting system: *Red Lizard Enterprises* manufactures mellotrons, pianos, and other keyboard instruments. The company is controlled by a Steering Board, which is a weighted voting system. The weights of the voters are given by the table below, and the quota is set at $q = 12$.

voter	weight
Robert	4
Ian	4
Greg	3
Michael	3
Peter	1

Music allusion to: the band King Crimson, who released albums entitled Red (1974) and Lizard (1970). The four voters are the names of the original lineup of the band in 1969: Robert (Fripp), Ian (McDonald), Greg (Lake), Michael (Giles), and Peter (Sinfield).

12. a) How many orderings of the voters are possible?

Solution: $P(5,5) = 5! = 120$.

b) Consider the voter Robert. Explain why Robert is not pivotal unless he is fourth or fifth in an ordering.

Solution: No voter can be pivotal if they are first, second, or third. Indeed, to be pivotal in third, there are only two voters preceding you, with at most $4 + 4 = 8$ votes before you join. But then you would need 4 or more points yourself, which is impossible since there are only 2 voters with 4 points. Thus, for anyone, including Robert, to be pivotal, they would have to be fourth or fifth.

c) How many different orderings are there with Robert listed fifth? Why is Robert pivotal in every such ordering?

Solution: There are $P(4,4) = 4! = 24$ orderings with Robert fifth. Robert is pivotal in all of them because the sum of the weights of all other voters is 11.

d) If Robert is fourth, consider who is listed fifth. For each of the other voters A , find how many orderings there are with Robert pivotal and A fifth.

Solution: Suppose Ian is fifth. Since Ian has weight 4, the sum of all the other weights (including Robert) is 11, so the coalition is losing until Ian joins, and so Robert is not pivotal in any such ordering. Next, suppose Greg is fifth. Since the sum of the rest of the weights is 12, Robert is pivotal (in fourth place) in any such ordering, and there are $P(3,3) = 3! = 6$ such orderings. Similarly, the 6 orderings with Robert fourth and Michael fifth are also orderings where Robert is pivotal. Finally, if Peter is fifth, the sum is 10 before Robert joins, so Robert is pivotal in all of these as well. Again, there are six such orderings. In total, there are $0 + 6 + 6 + 6 = 18$ orderings where Robert is pivotal in fourth place.

e) Add up all the orderings where Robert is pivotal, and determine $SSI(Robert)$.

Solution: The number of orderings where Robert is pivotal is $24 + 18 = 42$. Thus

$$SSI(Robert) = \frac{42}{120} = \frac{7}{20} = .35$$

13. a) Consider the voter Greg. Explain why Greg is not pivotal unless he is fourth or fifth in an ordering.

Solution: As explained in Exercise 12a, no voter is pivotal unless they are fourth or fifth.

b) Is Greg always pivotal when he is fifth? Count how many orderings there are in which Greg is fifth and pivotal.

Solution: The total weight of the voters without Greg is $4 + 4 + 3 + 1 = 12$, so if Greg joins fifth, the coalition was already a winning coalition before he joined, so there are no orderings for Greg where he is both fifth and pivotal.

c) Count how many orderings there are where Greg is fourth and pivotal. (The count depends on who is fifth.)

Solution: If Robert or Ian is fifth, then when Greg joins, the total is 11, which is still losing, so Greg is never pivotal in such an ordering. If Michael is fifth, then the weight is 9 before Greg joins and 12 after, so Greg is indeed pivotal in all six such orderings. Similarly, if Peter is fifth, the weight is 11 before Greg joins fourth, and 14 after he joins, so Greg is also pivotal in all six of those orderings.

d) What is $SSI(Greg)$?

Solution: The number of orderings where Greg is pivotal is then 0 (when he is fifth), plus $0 + 0 + 6 + 6 = 12$ (when he is fourth), or 12 total orderings where Greg is pivotal. Thus.

$$SSI(Greg) = \frac{12}{120} = \frac{1}{10} = 0.1$$

14. a) Consider the voter Peter. Explain why Peter can only be pivotal if he is listed fourth.

Solution: The total weight without Peter is 14, which is already winning, so Peter is never pivotal when he is fifth. And we already know no voter can be pivotal if they are first, second, or third. Thus, Peter can only be pivotal if he is listed fourth.

b) Count all the orderings where Peter is pivotal.

Solution: If Robert or Ian joins fifth, the weight is 10 before Peter joins and 11 after, which is still losing, so Peter is not pivotal in any of those orderings. If Michael or Greg joins fifth, then the total is 11 before Peter joins, and 12 after, so Peter is pivotal in all of those orderings, and there are six where Michael joins fifth, and six when Greg joins fifth, for a total of 12. Thus, there are a total of 12 orderings where Peter is pivotal.

c) Find $SSI(Peter)$, and show that, despite having only one vote, he has equal power with Greg and Michael.

Solution:

$$SSI(Peter) = \frac{12}{120} = 0.1,$$

The same as for Greg or Michael.

15. The following example is taken from Taylor & Pacelli (2008). In 1958, the *Treaty of Rome* established the *European Economic Community*, a body that promoted common economic interests to its member countries. Later, this body was modified several times, until the *European Union* was established. However, back in 1958, there were only six member countries. Each country had a weight given by the table below, and the quota was set at $q = 12$.

Country	**Weight**
France	4
Germany	4
Italy	4
Belgium	2
The Netherlands	2
Luxembourg	1

a) Explain why Luxembourg is never pivotal and conclude it is a dummy voter.

Solution: Since Luxembourg has one vote, in order to be pivotal in any coalition, there would have to be exactly eleven votes already in the coalition preceding Luxembourg when it joins.

This is impossible, because all the other countries weights are even numbers, so cannot sum to 11, an odd number. Therefore, Luxembourg is never pivotal and so has no power.

b) Using arguments similar to the previous three exercises, determine the Shapley-Shubik index of each voter. [**Remark**: As an alternative to determining which place a voter is located in an ordering, you could argue by considering how many votes there are preceding the voter, before that voter joins the coalition. That is how is it done in Taylor and Pacelli (2008). For example, for Belgium to be pivotal, there must be exactly 10 or 11 votes preceding Belgium. Count each case separately, by breaking into sub-cases depending on which countries precede it.]

Solution: Consider France (the calculation is the same for Germany and Italy.) For France to be pivotal, there must be 8,9,10, or 11 votes already in the coalition when France joins.

Case 1: 8 votes before France joins. There are two ways this could happen. One is both Germany and Italy join first, so the ordering looks like _ _ F _ _ _, with both G & I in front of F There are $2! = 2$ ways to order these two countries, as well as $3! = 6$ ways to order the three that come after France. Thus, there are $2!\,3! = 12$ such orderings. The other way (a second subcase) to have 8 votes in front of France is if is to have both Belgium and the Netherlands join ahead of France, along with either Germany or Italy. Thus, the ordering looks like _ _ _ F _ _, $3!\,2! = 12$ ways of ordering the countries in the blanks. So 12 when G joins before F and 12 more when I joins before F. Thus we have a total of $12 + 12 + 12 = 36$ orderings in case 1.

Case 2: 9 votes before France joins. The only way this can happen is if Luxembourg is among the countries which join before France, since the total is an odd number. Thus, we must take one of the cases where there already are 8 before France, and move Luxembourg from behind France to ahead of it. Thus, for example, the ordering could look like _ _ _ F _ _ , with both G & I as well as L in front of F (coming from the first subcase of Case 1.) There are$3!\,2! = 12$ of those. The two coming from the second and third subcases of case 1 have orderings that look like _ _ _ _ F _, with both B and N, as well as L, in front of France along with one of the big weights – either Germany or Italy. (Another way to say it is either G or I must follow F.) For each of these two subcases there are $4! = 24$ orderings. Thus, case 2 has a total of $12 + 24 + 24 = 60$ orderings.

Case 3. 10 votes in front of F. So G & I must both join before France, along with either B or N. Each of these subcases has an ordering that looks like _ _ _ F _ _, and there are $3!\,2! = 12$ orderings in each subcase. Thus, we have a total of $12 + 12 = 24$ orderings I Case 3.

Case 4: 11 votes in front of F. There are two subcases. One looks like _ _ _ _ F B, and the other is _ _ _ _ F N. There are $4! = 24$ ways to order each subcase, so there is a total of $24 + 24 = 48$ orderings in case 4.

Summing the cases, the total number of orderings where F is pivotal is then

$$36 + 60 + 24 + 48 = 168$$

It follows that $SSI(F) = SSI(G) = SSI(I) = \frac{168}{720} = \frac{7}{30} \approx .2333$.

Next, observe that $SSI(L) = 0$ by part (a). But the power indices of all the voters must add up to 1. Observe B and N have the same weight, so they have the same power. Denote $x = SSI(B) = SSI(N)$. Then:

$$\frac{7}{30} + \frac{7}{30} + \frac{7}{30} + x + x + 0 = 1$$

$$2x = \frac{9}{30}$$

$$x = \frac{9}{60} = \frac{3}{20} = .15$$

This completes the calculation. We could have found $SSI(B)$ the same way we did for $SSI(F)$. To be pivotal, there must be either 10 or 11 votes in the coalition before B joins. Thus, there are just two cases instead of four. We leave it as an exercise for the reader to complete that approach.

Section A.3 Experiments, Sample Spaces, and Probability Models

A.3.1 Experiments and Sample Spaces

Exercises after next subsection.

A.3.2 Probability Models

Solutions to exercises:

1. Suppose a sample space $S = \{w, x, y, z\}$. How many different events are there? List them all.

Solution: The set of possible events is the set $\mathcal{P}(S)$ of all subsets of S – the power set of S. So there are $2^4 = 16$ events. Namely:

$$\emptyset, \{w\}, \{x\}, \{y\}, \{z\}$$

$$\{w,x\}, \{w,y\}, \{w,z\}, \{x,y\}, \{x,z\}, \{y,z\}$$

$$\{w,x,y\}, \{w,x,z\}, \{w,y,z\}, \{x,y,z\}, S$$

2. A penny, a nickel, and a dime are flipped, and it is noted whether the coins show heads H or tails T. Show that $\{H,T\}^3$ is the sample space (Cartesian product). List all the elements.

Solution: Each outcome is an ordered triple with the result of the flip of the penny (either H or T) in the first coordinate, the result of the flip of the nickel in the second coordinate, and the result of the flip of the dime in the third. This proves the sample space is

$$\{H,T\} \times \{H,T\} \times \{H,T\} = \{H,T\}^3$$

$$= \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

(We are writing the ordered triples without parenthesis or commas for simplicity – thus, "THT" is short for (T,H,T).)

3. In the example of Exercise 2, list the outcomes in the event "the penny shows H" and also the outcomes in the event "there are at least two coins showing T"

Solution: The penny shows heads is the event $E = \{HHH,\ HHT,\ HTH,\ HTT\}$. At least two coins show T is the event $\{HTT,\ THT,\ TTH,\ TTT\}$.

4. A jar contains lemon candies, mint candies, and chocolate candies. A candy is drawn at random and its flavor noted. Write down the sample space and write down the outcomes in the event ‘the candy drawn is not lemon’.

Solution: $S = \{L, M, C\}$, and the event ‘the candy drawn is not lemon’ is $E = \{M, C\}$.

5. a) In the candy jar of Exercise 4, suppose the experiment is to draw two candies in order. Now list the outcomes of the sample space.

Solution: Because order matters, we can represent the outcomes as ordered pairs. Therefore, the sample space is $S^2 = S \times S = \{LL,\ LM,\ LC,\ ML,\ MM,\ MC,\ CL,\ CM,\ CC\}$. (Again, omitting commas and parenthesis in the ordered pairs.)

b) This part is an example where multisets are useful instead of sets. Same question as part a, except we do not draw the candies in order – we just pull out two candies at once.

Solution. The outcomes are “unordered pairs”, or multisets of size 2. That is, $\{L, M\}$ instead of (L, M). (It’s a multiset instead of a set because both ‘coordinates’ could be the same; e.g. if you draw two mint candies, the outcomes is the multiset $\{M, M\}$.) In writing the outcomes, if we agree to drop the commas and set braces, analogous to what we did with ordered pairs, we can abbreviate the outcome $\{L, M\}$ by LM, except that since order doesn’t matter LM and ML are the same multiset, and the same outcome. Now the sample space is

$$\{LL, LM, LC, MM, MC, CC\}$$

6. Suppose $S = \{a, b, c, d, e\}$, and it is known that $p(a) = p(b) = p(c)$, but d is twice as likely as c and e is just as likely as $\{c, d\}$. Determine the probability model. [**Hint**: You are being asked to find the probabilities of each of the five possible outcomes. Let $p(a) = x$. Write all the probabilities in terms of x, then solve for x using axiom 3.]

Solution: Let $x = p(a) = p(b) = p(c)$.The $p(d) = 2x$ and $p(e) = p(\{c, d\}) = p(c) + p(d) = x + 2x = 3x$. Since the probabilities must sum to 1, we have:

$$x + x + x + 2x + 3x = 1$$

$$8x = 1$$

$$x = \frac{1}{8}$$

Thus, the probability model is the function $p(a) = p(b) = p(c) = \frac{1}{8}$, $p(d) = \frac{1}{4}$, and $p(e) = \frac{3}{8}$.

7. Suppose S is a sample space and E and F are events. Suppose we know $p(E \cup F) = .8$, $p(E) = .5$, and $p(F^c) = .6$. Find $p(E \cap F)$.

Solution: Since $p(F^c) = .6$, it follows that $p(F) = .4$. Therefore:

$$p(E \cup F) = p(E) + p(F) - p(E \cap F)$$

$$.8 = .5 + .4 - p(E \cap F)$$

$$p(E \cap F) = .9 - .8 = .1$$

8. Suppose S is a sample space and E and F are events. Suppose we know $p(E) = .15, p(F) = .65$, and $p((E \cup F)^c) = .4$. Find $p(E \cap F)$.

Solution: Since $p((E \cup F)^c) = .4$, it follows that $p(E \cup F) = .6$. Therefore:

$$p(E \cup F) = p(E) + p(F) - p(E \cap F)$$

$$.6 = .15 + .65 - p(E \cap F)$$

$$p(E \cap F) = .8 - .6 = .2$$

9. In the example of the loaded die given in the text above, find $p(outcome\ is\ at\ most\ 5)$ and $p(outcome\ is\ 1\ or\ 6)$.

Solution: Let A be the set of outcomes of at most 5. Then

$$p(A) = 1 - p(6) = 1 - .3 = .7$$

Similarly, if B is the event of a 1 or 6, $B = \{1,6\}$, then

$$p(B) = p(1) + p(6) = .1 + .3 = .4$$

10. a) In the example of rolling two fair dice, find the probabilities of the following events: $A = \{the\ sum\ is\ at\ most\ 4\},\ B = \{the\ numbers\ on\ the\ two\ dice\ differ\ by\ 1\}, and\ C = \{the\ number\ showing\ on\ the\ white\ die\ is\ at\ least\ 3\}$.

Solution: $p(A) = p(1,1) + p(1,2) + p(2,1) + p(1,3) + p(2,2) + p(3,1) = \frac{1}{6} + \frac{1}{6} + \cdots + \frac{1}{6}$ (6 summands) $= \frac{6}{36} = \frac{1}{6}$. $p(B) = p(1,2) + p(2,3) + p(3,4) + p(4,5) + p(5,6) + p(6,5) + p(5,4) + p(4,3) + p(3,2) + p(2,1) = \frac{10}{36} = \frac{5}{18}$.

b) Also find $p(A \cap B), p(A \cup B), p(B \cup C)$, and $p(A \cap B \cap C)$.

Solution: $p(A \cap B) = p(1,2) + p(2,1) = \frac{2}{36} = \frac{1}{18}$.

$$p(A \cup B) = p(A) + p(B) - p(A \cap B) = \frac{6}{36} + \frac{10}{36} - \frac{2}{36} = \frac{14}{36} = \frac{7}{18}.$$

Observe that $p(B \cap C) = p(4,3) + p(2,3) + p(5,4) + p(3,4) + p(6,5) + p(5,6) = \frac{6}{36}$

Also, $p(C) = \frac{24}{36} = \frac{2}{3}$ (why?). So:

$$p(B \cup C) = p(B) + p(C) - p(B \cap C)$$

$$= \frac{10}{36} + \frac{24}{36} - \frac{6}{36} = \frac{28}{36} = \frac{7}{9}$$

Section A.4 Uniform Sample Spaces

Solutions to exercises and more music references:

1. A fair coin is flipped 10 times. a) What is the probability of obtaining exactly 6 heads?

Solution: The sample space is ordered 10-tuples with entries H or T; that is, $S = \{S, T\}^{10}$. Thus, there are $2^{10} = 1{,}024$ possible outcomes. The successes are those where exactly 6 of the 10 coordinates say H. Since the other coordinates must say T, to count these we need only to enumerate the number of ways of selecting 6 of the 10 coordinates to be filled with 'H'. Since the order of selection does not matter (the same 6 coordinates selected in any order yield the same 10-tuple once the H's and T's are filled in), the answer is a combination: $C(10,6) = \frac{10!}{4!6!} = 210$. Thus, the probability of exactly 6 heads is

$$\frac{C(10,6)}{2^{10}} = \frac{210}{1024} \approx 0.20508$$

b) What is the probability of obtaining exactly 2 or 3 heads?

Solution: The event is the disjoint union of the event A of exactly 2 heads, and the event B of exactly 3 heads. By the addition principle, the answer we seek is $p(A) + p(B)$. But each event is calculated as in part (a), so the probability is:

$$\frac{(10,2)}{2^{10}} + \frac{C(10,3)}{2^{10}} = \frac{45}{1024} + \frac{120}{1024} = \frac{165}{1024} \approx 0.16113$$

c) What is the probability of obtaining at least two heads?

Solution: We solve this just like part (b), since the event "at least 2 heads" is the disjoint union of "exactly 2 heads" or "exactly 3 heads" or "exactly 4 heads", etc. up to "exactly 10 heads". Thus:

$$\frac{C(10,2)}{2^{10}} + \frac{C(10,3)}{2^{10}} + \frac{C(10,4)}{2^{10}} + \cdots + \frac{C(10,10)}{2^{10}}$$

A slightly easier approach uses complements. The event of "at least two heads" is the compliment of the event "at most one head". Therefore, the probability we seek is

$$1 - p(at\ most\ one\ head)$$

$$= 1 - \left(\frac{C(10,0)}{2^{10}} + \frac{C(10,1)}{2^{10}}\right) = 1 - \left(\frac{1}{1024} + \frac{10}{1024}\right) = \frac{1013}{1024} \approx .9893$$

2. Steven has a model of the solar system with a lightbulb sun. The bulb has burned out, so he reaches into a box of 8 bulbs for a replacement. He remembers that 3 of the bulbs in the box are defective, so he draws 2 bulbs out at random, just in case. What is the probability that light is restored to his model of the solar system?

Music allusion to: Steven Wilson's band Porcupine Tree. In 2000, they released an album entitled Lightbulb Sun. (Website: http://www.porcupinetree.com/)

Solution: Since he is drawing the bulbs at random, it is a uniform space. The total number of elements in S is the total number of ways of drawing 2 items from a set of 8 when order is not important. That is, $n(S) = C(8,2) = 28$. In order for light to be restored, Steven needs to draw at least one good lightbulb in his draw. It's easier to compute the probability of the complementary event – that light is NOT restored, which happens only if both bulbs are defective. Since there are 3 defective bulbs, the number of ways for this to occur is $C(3,2) = 3$. Therefore, the probability that light is restored is $1 - \frac{C(3,2)}{C(8,2)} = 1 - \frac{3}{28} = \frac{25}{28} \approx 0.89286$.

3. A jar contains 20 marbles, of equal size and weight, of which 10 are white, 7 are green, and 3 are red. Without looking at the jar, 4 marbles are drawn at random.

a) What is the probability that they are all white?

Solution: Since the sample space is uniform (why?), we have $p(4\ white\ marbles) = \frac{C(10,4)}{C(20,4)} = \frac{14}{323} \approx 0.043344$.

b) What is the probability that one is white, one is green, and two are red?

Solution: By the multiplication principle, we have the probability is

$$\frac{C(10,1) \cdot C(7,1) \cdot C(3,2)}{C(20,4)} = \frac{14}{323} \approx 0.043344.$$

c) What is the probability that at least one green, but no red marbles are drawn?

Solution: Splitting the event "at least one" into disjoint subcases, we an add each one, obtaining

$$\frac{C(7,1) \cdot C(10,3) + C(7,2) \cdot C(10,2) + C(7,3) \cdot C(10,1) + C(7,4) \cdot C(10,0)}{C(20,4)}$$

$$= \frac{(7 \cdot 120 + 21 \cdot 45 + 35 \cdot 10 + 35 \cdot 1) \cdot 4!}{20 \cdot 19 \cdot 18 \cdot 17} = \frac{2170 \cdot 24}{20 \cdot 19 \cdot 18 \cdot 17}$$

$$= \frac{217 \cdot 12}{19 \cdot 18 \cdot 17} 3 = \frac{217 \cdot 2}{19 \cdot 3 \cdot 17} = \frac{434}{969} \approx 0.44788.$$

d) What is the probability that all four marbles are one color?

Solution: Since there are only 3 red marbles, the color must be white or green. Thus:

$$\frac{C(10,4)+C(7,4)}{C(20,4)}=\frac{210+35}{C(20,4)}=\frac{49}{969}\approx 0.050568.$$

e) What is the probability that all three colors are obtained?

Solution: This means either 2 white (and one of each other color), or 2 green (and one of each other color), or 2 red (and one of each other color.) Since these events are mutually exclusive (disjoint), we can find the probability in each case and add, which is equivalent to:

$$\frac{C(10,2)\cdot C(7,1)\cdot C(3,1)+C(10,1)\cdot C(7,2)\cdot C(3,1)+C(10,1)\cdot C(7,1)\cdot C(3,2)}{C(20,4)}$$

$$=\frac{(45\cdot 7\cdot 3+10\cdot 21\cdot 3+10\cdot 7\cdot 3)}{C(20,4)}=\frac{1785\cdot 4!}{20\cdot 19\cdot 18\cdot 17}$$

$$=\frac{1785}{5\cdot 19\cdot 3\cdot 17}=\frac{357}{19\cdot 3\cdot 17}=\frac{21}{19\cdot 3}=\frac{7}{19}\approx 0.3684.$$

4. a) A full house in a game of poker is three of a kind plus a pair (of different rank), such as three queens and a pair of 7's, or three 4's and a pair of aces. In five-card stud, where just five cards are dealt face down at random from a well-shuffled deck., what is the probability of obtaining a full house?

Solution: The total number of five-card hands is $C(52,5)=\frac{52!}{47!5!}=2{,}598{,}960$. To count the number of full houses, we must select two ranks from 13 (such as threes and jacks, for example). However, since one rank if for a pair and the other is for the three of a kind, the order of selection matters (say the first rank selected is for the three of a kind, and the second for the pair.) So, there are $P(13,2)$ ways to do this. Once the ranks are selected, then we must select the right number of cards from the four suits of that rank, and for this part of the problem, the order of selection does not matter. Therefore, the total number of different full houses is (by the multiplication principle):

$$P(13,2)\cdot C(4,3)\cdot C(4,2)=13\cdot 12\cdot 4\cdot 6=3744$$

Thus, the probability of dealing a full house is $\frac{3744}{2{,}598{,}960}=\frac{6}{4165}\approx 0.0014406.$

b) What is the probability of getting a hand which has two pairs? This means two pairs of different rank and fifth card which is a different rank from both of the pairs, like a pair of 3's a pair of jacks, and a 9.

Solution: Like part (a), we begin by selecting two (distinct) ranks. However, since both ranks with be for pairs, the order now does NOT matter (a pair of 3's and a pair of jacks is the same as a pair of jacks and a pair of 3's.) Also, we must select the fifth card from the remaining 44 cards, since the fifth card cannot match either of the previously chosen ranks. Thus, the total number of hands which are two pairs is

$$C(13,2) \cdot C(4,2) \cdot C(4,2) \cdot C(44,1) = 78 \cdot 6 \cdot 6 \cdot 44 = 123{,}552.$$

The probability of dealing a two-pair hand is therefore

$$\frac{123{,}552}{2{,}598{,}960} = \frac{198}{4165} \approx 0.047539$$

c) What is the probability of getting one pair?

Solution: First we select the rank for the pair, then select the pair itself from the four cards of that rank. There are C(13,1)$\cdot$ $C(4,2) = 13 \cdot 6 = 78$ ways to do this. Now we must select 3 more distinct ranks for the remaining three cards, and then select one of the four cards of that rank for each of the three. Thus, the total number of one-pair hands is$78 \cdot C(12,3) \cdot C(4,1)^3 = 78 \cdot 220 \cdot 64 = 1{,}098{,}240$. The probability of dealing a one-pair hand is therefore:

$$\frac{1{,}098{,}240}{2{,}598{,}969} = \frac{352}{833} \approx 0.42257$$

5. A party has 30 attendees. What is the probability that at least 2 people at the party have the same birthday? Assume 365 days in a year, and each day is equally likely as a birthday. [**Hint**: See Example A.37 above. This is the well-known "birthday problem."]

Solution: The hint was intended to suggest finding the probability of the complementary event as we did in Example A.37. The complementary event is that none of the guest share a birthday. That means we are selecting 30 distinct days out of 365 (order matters), while the sample space has 365^{30} elements (again, order matters.) Thus the probability of 30 distinct birthdays is $\frac{P(365,30)}{365^{30}} \approx .294$. Thus, the probability of at least two shared birthdays is approximately $1 - .294 = .706$. (This is surprising – many people would not guess the chances are that high with only 30 people present.)

Section A.5 Conditional Probability and Independent Events

Music References in the text:

Example A.40 – the example with the box of CDs alludes to the British rock band Genesis, the drummer of which (and vocalist after the departure of Peter Gabriel) was Phil Collins. Charisma was one of the labels that carried Genesis. (Website: https://www.genesis-music.com/)

Solutions to exercises and more music references:

1. Suppose A and B are events in a sample space, and suppose $p(A) = .3, p(B) = .6$, and $p(A \cap B) = .1$.

a) Compute $p(A|B)$ and $p(B|A)$.

Solution:

$$p(A|B) = \frac{p(A \cap B)}{p(B)} = \frac{.1}{.6} = \frac{1}{6} \approx 0.16667$$

$$p(B|A) = \frac{p(B \cap A)}{p(A)} = \frac{.1}{.3} = \frac{1}{3} \approx 0.3333$$

b) Compare $p(A^c|B)$ and $p(A^c|B^c)$.

Solution: Observe we can write B as a disjoint union: $B = (A \cap B) \cup (A^c \cap B)$. Thus,

$$p(B) = p(A \cap B) + p(A^c \cap B), \text{ and}$$

$$p(A^c|B) = \frac{p(A^c \cap B)}{p(B)} = \frac{p(B) - p(A \cap B)}{p(B)} = 1 - \frac{p(A \cap B)}{p(B)}$$

$$p(A^c|B) = 1 - p(A|B)$$

This makes sense – it shows that $A^c|B$ and $A|B$ are complementary events. In this example, we have $p(A^c|B) = 1 - p(A|B) = 1 - \frac{1}{6} = \frac{5}{6}$.

Now by the same argument

$$p(A^c|B^c) = 1 - p(A|B^c)$$

Now $p(B^c) = 1 - p(B) = 1 - 0.6 = 0.4$. Also, A is a disjoint union: $A = (A \cap B) \cup (A \cap B^c)$, so $p(A \cap B^c) = p(A) - p(A \cap B) = .3 - .1 = 0.2$. Thus

$$p(A^c|B^c) = 1 - \frac{.2}{.4} = 0.5$$

c) Are A and B independent events? Explain.

Solution: $p(A) = .3 \neq \frac{1}{6} = p(A|B)$, so by definition, A and B are not independent.

2. Suppose $p(E) = \frac{1}{3}$ and $p(F) = \frac{1}{4}$.

a) Find the probability $p(E \cup F)$ if E and F are mutually exclusive.

Solution:

$$\begin{aligned} p(E \cup F) &= p(E) + p(F) - p(E \cap F) \\ &= \frac{1}{3} + \frac{1}{4} - p(E \cap F) \\ &= \frac{7}{12} - p(E \cap F) \end{aligned}$$

In the case when E and F are mutually exclusive, then $E \cap F = \emptyset$, whence $p(E \cap F) = 0$, and so $p(E \cup F) = \frac{7}{12}$.

b) Find the probability $p(E \cup F)$ if E and F are independent.

Solution: As in part (a), we have

$$p(E \cup F) = \frac{7}{12} - p(E \cap F)$$

This time, when E and F are independent, we know $p(E \cap F) = p(E) \cdot p(F) = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$, so

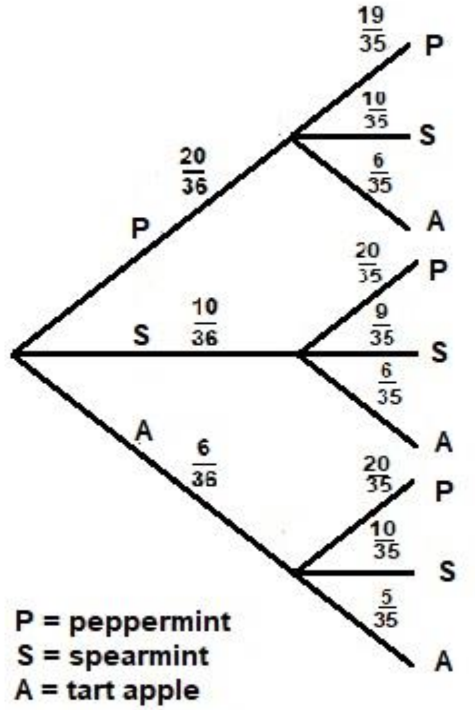
$$p(E \cup F) = \frac{7}{12} - \frac{1}{12} = \frac{1}{2}$$

3. Can two nonempty events be simultaneously independent and mutually exclusive? Explain.

Solution: No, they can't. For independent events, we have $p(A \cap B) = p(A)p(B)$, but if they are also mutually exclusive, then $p(A \cap B) = 0$. It follows that $p(A)p(B) = 0$, which means $p(A) = 0$ or $p(B) = 0$. But in a finite sample space, $p(A) = 0$ if and only if A is the impossible event $A = \emptyset$. Thus, either A or B must be empty, a contradiction since we assumed they were both nonempty.

4. A jar contains 20 peppermint candies, 10 spearmint candies, and 6 tart apple candies, which all look alike. Two candies are drawn randomly in succession (without replacement). Draw a tree diagram and label the branches with appropriate probabilities. Use the diagram and/or the results of this section to answer the following:

Solution: The relevant tree diagram:



a) Find the probability that the first candy drawn is not apple.

Solution: Let's denote by $P1$ the event of drawing Peppermint on the first draw, and $P2$ the event of drawing peppermint on the second draw. We use similar notation for spearmint ($S1$ and $S2$) and tart apple ($A1$ and $A2$). Thus:

$$p(P1 \cup S1) = \frac{20}{36} + \frac{10}{36} = \frac{30}{36} = \frac{5}{6}$$

b) Find the probability that the second candy drawn is apple, given that the first one is spearmint.

Solution: $p(A2|S1) = \frac{6}{35} \approx 0.17143$

c) Find the probability that both candies are peppermint.

Solution:

$$p(P1 \cap P2) = p(P1) \cdot p(P2|P1) = \frac{20}{36} \cdot \frac{19}{35} = \frac{19}{63} \approx 0.30159$$

d) Find the probability that the second candy is spearmint.

Solution: Adding together the three branches ending in $S2$:

$$p(S2) = \frac{20}{36} \cdot \frac{10}{35} + \frac{10}{36} \cdot \frac{9}{35} + \frac{6}{36} \cdot \frac{10}{35} = \frac{5}{18} \approx 0.27778$$

5. In Example A46, part (c), in the text, verify that the probability that the second card is an ace is indeed $\frac{1}{13}$.

Solution: Let $A1$ be the event the first card is an ace, and $B1$ the event the first card is not an ace. Similarly, for $A2$ and $B2$. Then

$$p(A2) = p\big((A2 \cap A1) \cup (A2 \cap B1)\big)$$

$$= p(A2 \cap A1) + p(A2 \cap B1)$$

$$= p(A1) \cdot p(A2|A1) + p(B1) \cdot p(A2|B1)$$

$$= \frac{1}{13} \cdot \frac{3}{51} + \frac{12}{13} \cdot \frac{4}{51} = \frac{51}{13 \cdot 51} = \frac{1}{13}$$

You can also draw a tree diagram to show this.

6. Sheila is taking classes at Leibniz Academy. She estimates the probability that she passes her English class is $.95$, the probability that se passes her History class is $.98$, and the probability that she passes her Music class is $.94$. If these three events are independent, what is the probability that she will not pass at least one class?

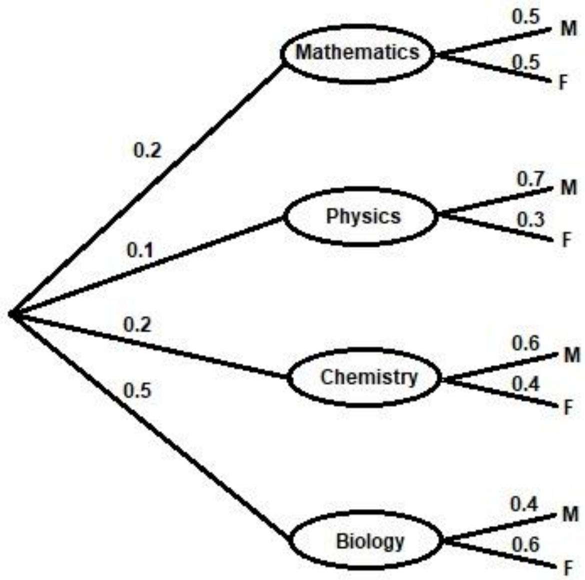
Solution: Denote passing English by E, passing History by H, and passing music by M. The event A of not passing at least one course is the complement of passing all three. Therefore, if these vents are independent,

$$p(A) = 1 - p(E \cap H \cap M) = 1 - p(E) \cdot p(H) \cdot p(M)$$

$$= 1 - (.95)(.98)(.94) = .12486$$

7. The Gender Studies Dept. at Leibniz Academy is conducting a study of the distribution of women students majoring in S.T.E.M. disciplines (Science, Technology, Engineering, and Mathematics). Assume that at Leibniz Academy, there are no double majors, and four STEM disciplines have majors: Mathematics, Physics, Chemistry, and Biology. In the study it is found that if a STEM major at the Academy is chosen at random, the probability that they are a mathematics major is $.2$, the probability that they are a physics major is $.1$, the probability that they are a chemistry major is $.2$, and the probability that they are a biology major is $.5$. Among the math majors at the Academy, 50% are females, among the physic majors, 30% are female, among the chemistry majors, 40% are female, and among the biology majors, 60% are female.

All the questions below can be answered from the tree diagram:



a) If a STEM major is randomly selected, what is the probability that they are male, given that they are a biology major?

Solution: $p(M|Bio) = .04$

b) If a STEM major is randomly selected, what is the probability that they are a female physic major?

Solution: $p(F \cap Physics) = p(Physics) \cdot p(F|Physics) = (0.1)(0.3) = 0.03$

c) If a STEM major is randomly selected, what is the probability that they female? Overall, do you thin that Leibniz Academy is having trouble attracting female STEM majors?

Solution: Adding all the branches that end in F:

$$p(F) = (.2)(.5) + (.1)(.3) + (.2)(.4) + (.5)(.6) = 0.51$$

Since 51% of their STEM majors are female, they are probably not having trouble attracting female STEM majors.

8. A card is selected randomly from a well-shuffled deck. Determine whether or not the following events are independent:

a) E is the event the card is red, and F is the event the card is a face card (Jack, Queen, or King).

Solution: $p(E) = \frac{26}{52} = \frac{1}{2}$, and $p(F) = \frac{12}{52} = \frac{3}{13}$. Also, $p(E \cap F) = \frac{6}{52} = \frac{3}{26}$. Observe that $p(E) \cdot p(F) = \frac{1}{2} \cdot \frac{3}{13} = \frac{3}{26} = p(E \cap F)$. This means the events are independent.

b) E is the event the card is a spade suit, and F is the event the card is a face card.

Solution: $p(E) = \frac{13}{52} = \frac{1}{4}$, and $p(F) = \frac{12}{52} = \frac{3}{13}$. Also, $p(E \cap F) = \frac{3}{52}$. Observe that $p(E) \cdot p(F) = \frac{1}{4} \cdot \frac{3}{13} = \frac{3}{52} = p(E \cap F)$. This means the events are independent.

c) E is the event the card is red, and F is the event the card is a heart suit.

Solution: $p(E) = \frac{26}{52} = \frac{1}{2}$, and $p(F) = \frac{13}{52} = \frac{1}{4}$. Also, $p(E \cap F) = \frac{1}{4}$, because $E \cap F = F$. Observe that $p(E) \cdot p(F) = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8} \neq \frac{1}{4} = p(E \cap F)$. This means the events are NOT independent.

d) E is the event the card is a King, and F is the event the card is a face card.

Solution: $p(E) = \frac{4}{52} = \frac{1}{13}$, $p(F) = \frac{3}{13}$, and p(E$\cap$ $F) = \frac{1}{13}$ because $E \cap F = E$. Observe $p(E) \cdot p(F) = \frac{1}{13} \cdot \frac{3}{13} = \frac{3}{169} \neq \frac{1}{4} = p(E \cap F)$. This means the events are NOT independent.

9. At the *Nilsson Medical Clinic*, patients are tested for Lyme disease by using two different diagnostic tests. The probability that test A correctly detects Lyme disease in a patient who has the disease is $.6$, and the probability that test B correctly detects Lyme disease in a patient who has the disease is $.7$. Suppose that the probability that at least one of the tests will correctly detect the disease is $.85$.

Music allusion to: American singer-songwriter Harry Nilsson. In 1971 he released his well-known song 'Coconut' which contained the lyric 'put the lime in the coconut, drink 'em both up'. (Website: https://harrynilsson.com/)

a) Find the probability that both tests correctly detect the disease.

Solution: Let E be the event that A correctly detects the disease, and F the event that B correctly detects the disease. Then

$$p(E \cup F) = p(E) + p(F) - p(E \cap F)$$

$$.85 = .6 + .7 - p(E \cap F)$$

$$p(E \cap F) = 1.3 - .85 = .45$$

b) Find he probability that Test A correctly detects the disease, given that test B has correctly detected it.

Solution:

$$P(E|F) = \frac{p(E \cap F)}{p(F)} = \frac{.45}{.7} = 0.64286$$

c) Find the probability that Test B correctly detects Lyme disease, given that test A has correctly detected it.

Solution:

$$P(F|E) = \frac{p(E \cap F)}{p(E)} = \frac{.45}{.6} = 0.75$$

d) Are the events "Test A correctly detects Lyme disease" and "Test B correctly detects Lyme disease" independent?

Solution: E and F are not independent, since $p(F) = .7 \neq .75 = p(F|E)$.

Section A.6 Bayes Theorem

Music References in the text:

Example A.49, with the private detective firm, makes several music allusions. First, to English singer-songwriter Elvis Costello, who had a hit in 1977 with the song 'Watching the Detectives'. Second, to American Blue-Eyed soul duo Hall and Oates, who had a 1981 album with title track 'Private Eyes'. Third, to the 1980's new-wave band Missing Persons. (Websites: https://www.elviscostello.com, https://www.hallandoates.com/, https://en.wikipedia.org/wiki/Missing_Persons)

Example A.50, about the medical test, pays homage to electric guitar virtuoso and pioneer Jimi Hendrix, whose full name was James Marshall Hendrix. Hendrix released a song in 1967 entitled 'Manic Depression'. (Website: https://jimihendrix.com/)

Solutions to Exercises (and more music references):

1. In Example A.49, Costello's Private Eyes, write out the events symbolically and their probabilities for the calculation done above.

Solution: Let M denote the event is a missing persons specialist, D the event that the detective is a divorce specialist, and O the event their specialization is Other. Let L be the event they are based in Los Angeles, C that they are based in Chicago, S the event they are based in St. Louis, and B the event they are based in Boston. Then the calculation done above in the text is:

$$p(C|D) = \frac{p(C \cap D)}{p(D)} = \frac{p(C)p(D|C)}{p(L)p(D|L) + p(C)p(D|C) + p(S)p(D|S) + p(B)p(D|B)}$$

$$= \frac{.3(.3)}{.4(.6) + .3(.3) + .2(.5) + .1(.5)} = \frac{3}{16}$$

2. a) In Example A.49, Costello's Private Eyes, find the probability that a detective is based in St. Louis, given that she is a missing persons specialist.

Solution: Using the notation of Exercise 1,

$$p(S|M) = \frac{.2(.25)}{.4(.3) + .3(.2) + .2(.25) + .1(.1)} \approx 0.20833$$

b) Find the probability that a detective is based in Boston, given that she is a missing persons specialist.

Solution:

$$p(B|M) = \frac{.1(.1)}{.4(.3) + .3(.2) + .2(.25) + .1(.1)} \approx 0.041667$$

c) Mr. Hall is a divorce specialist who works in the firm. What is the probability that he is based in Los Angeles?

Solution:

$$p(L|D) = \frac{.4(.6)}{.4(.6) + .3(.3) + .2(.5) + .1(.5)} \approx 0.090909$$

3. In Example A.50, the medical test problem, suppose you take the test and it comes back negative. What is the probability that you really do not have the disease?

Solution:

$$p(D^c|N) = \frac{p(D^c)p(N|D^c)}{p(D)p(N|D) + p(D^c)p(N|D^c)}$$

$$= \frac{.994(.97)}{.006(.07) + .994(.97)} \approx 0.99956$$

4. In the medical test of Example A.50, suppose that the test was improved so that the probability of a false negative was $.05$ and a false positive was $.02$. Now compute $p(D|P)$.

Solution:

$$p(D|P) = \frac{p(D)p(P|D)}{p(D)p(P|D) + p(D^c)p(P|D^c)}$$

$$= \frac{.006(.95)}{.006(.95) + .994(.02)} \approx 0.22283$$

This is improved from the text example, but still far from infallible!

5. An urn contains 4 black balls and 2 red balls. Two balls are drawn in sequence without replacement. If the second ball is read, what is the probability that the first ball was black?

Solution: *Let* $B1$ *(resp.* $B2$) denote the event that the first ball is black (resp., that the second ball is black), and $R1$ *(resp.* $R2$) denote the event that the first ball is red (resp., that the second ball is red.) Then Bayes' theorem implies:

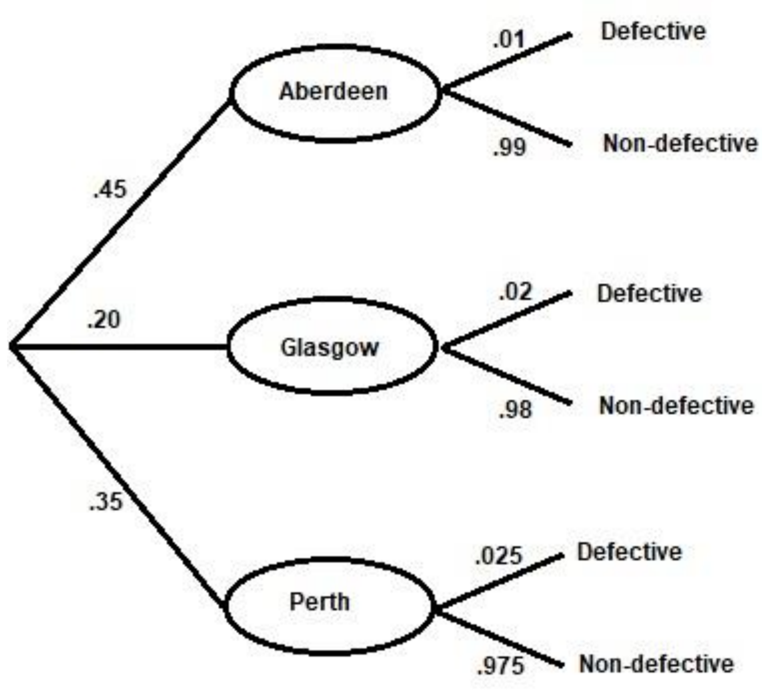
$$p(B1|R2) = \frac{p(B1)p(R2|B1)}{p(B1)p(R2|B1) + p(R1)p(R2|R1)}$$

$$= \frac{\frac{4}{6}\left(\frac{2}{5}\right)}{\frac{4}{6}\left(\frac{2}{5}\right) + \frac{2}{6}\left(\frac{1}{5}\right)} = \frac{4}{5}$$

6) *Leitch Vintage Instruments, Ltd*. Is a company in Scotland that manufactures classic musical instruments such as bagpipes and hurdy-gurdys. They have three factories, one in Aberdeen, one in Glasgow, and one in Perth. While most of the workmanship is high quality, sometimes a defective instrument is produced due to a malfunctioning valve or broken drone string. The probability of a defective instrument from Aberdeen is $.01$, the probability of a defective instrument from Glasgow is $.02$, and the probability of a defective instrument form Perth is $.025$. The plant in Aberdeen produces 45% of the company's output, the plant in Glasgow produces 20%, and the plant in Perth produces 35%. If a Leitch Hurdy-Gurdy chosen at random is found to be defective, what is the probability it came from each of the three cities?

Music homage to: Scottish singer-songwriter Donovan (full name Donovan Phillips Leitch), who had a 1968 hit with the song 'Hurdy-Gurdy Man.' (Website: https://donovan.ie/)

Solution: Here is the tree diagram depicting the production of Leitch Hurdy-Gurdys:



Let D be the event a hurdy-gurdy is defective, and N that it is non-defective. Let A, G, P be the event a hurdy-gurdy is produced in Aberdeen, Glasgow, and Perth, respectively. Then, the probability that a defective instrument is from each of the three cities is given by Bayes' theorem:

For Aberdeen:

$$p(A|D) = \frac{p(A)p(D|A)}{p(A)p(D|A) + p(G)p(D|G) + p(P)p(D|P)}$$

$$= \frac{.45(.01)}{.45(.01) + .20(.02) + .35(.025)} = .26087$$

For Glasgow:

$$p(G|D) = \frac{p(G)p(D|G)}{p(A)p(D|A) + p(G)p(D|G) + p(P)p(D|P)}$$

$$= \frac{.20(.02)}{.45(.01) + .20(.02) + .35(.025)} = .23188$$

For Perth:

$$p(P|D) = \frac{p(P)p(D|P)}{p(A)p(D|A) + p(G)p(D|G) + p(P)p(D|P)}$$

$$= \frac{.35(.025)}{.45(.01) + .20(.02) + .35(.025)} = .50725$$