Stochastic Calculus for Finance
Solutions to Exercises

Chapter 1

Exercise 1.1: Show that for each n, the random variables $K(1), \ldots, K(n)$ are independent.

Solution: Since $K(r)$ have discrete distributions, the independence of $K(1), \ldots, K(n)$ means that for each sequence $V_1, \ldots, V_n$, $V_i \in \{U, D\}$ we have

$$P(K(1) = V_1, K(2) = V_2, \ldots, K(n) = V_n)$$
$$= P(K(1) = V_1) \cdot \ldots \cdot P(K(n) = V_n).$$

Fix a sequence $V_1, \ldots, V_n$. Start by splitting the interval $[0, 1]$ into two intervals I_U, I_D of length $\frac{1}{2}$, $I_U = \{\omega : K(1) = U\}$, $I_D = \{\omega : K(1) = D\}$. Repeat the splitting for each interval at each stage. At stage two we have $I_U = I_{UU} \cup I_{UD}$, $I_D = I_{DU} \cup I_{DD}$ and the variable $K(2)$ is constant on each $I_{\alpha\beta}$. For example, $I_{UD} = \{\omega : K(1) = U, K(2) = D\}$. Using this notation we have

$$\{K(1) = V_1\} = I_{V_1}, \{K(1) = V_1, K(2) = V_2\} = I_{V_1,V_2},$$
$$\ldots\{K(1) = V_1, \ldots, K(n) = V_n\} = I_{V_1,\ldots,V_n}.$$

The Lebesgue measure of $I_{V_1,\ldots,V_n}$ is $\frac{1}{2^n}$, so that

$$P(K(1) = V_1, \ldots, K(n) = V_n) = \frac{1}{2^n}$$

. From the definition of $K(r)$ follows directly $P(K(1) = V_1) \cdot \ldots P(K(n) = V_n) = \frac{1}{2^n}$.

Exercise 1.2: Redesign the random variables $K(n)$ so that $P(K(n) = U) = p \in (0, 1)$, arbitrary

Solution: Given the probability space $(\Omega, \mathcal{F}, P) = ([0, 1], \mathcal{B}([0, 1]), m)$, where m denotes the Lebesgue measure, we will define a sequence of random variables $K(n)$, $n = 1, 2, \ldots$.on Ω.

First split $[0, 1]$ into two subintervals: $[0, 1] = I_U \cup I_D$, where I_U, I_D are disjoint intervals with lengths $|I_U| = p$, $|I_D| = q$, $p + q = 1$, with I_U to the left on I_D.. Now set

$$K(1,\omega) = \left\{ \begin{array}{c} U \text{ if } \omega \in I_U \\ D \text{ if } \omega \in I_D \end{array} \right\}.$$

Clearly $P(K(1) = U) = p$, $P(K(1) = D) = q$. Repeat the procedure separately on I_U and I_D, splitting each into two subintervals in the proportion p to q. Then $I_U = I_{UU} \cup I_{UD}$, $I_D = I_{DU} \cup I_{DD}$, $|I_{UU}| = p^2$, $|I_{UD}| = pq$, $|I_{DU}| = qp$, $|I_{DD}| = q^2$. Repeating this recursive construction n times we obtain intervals of the form $I_{\alpha_1,\ldots,\alpha_r}$, with α_i either U or D, and with length $p^l q^{r-l}$, where $l = \#\{\alpha_i : \alpha_i = U\}$.
Again set

$$K(r,\omega) = \left\{ \begin{array}{c} U \text{ if } \omega \in I_{\alpha_1,\ldots,\alpha_{r-1},U} \\ D \text{ if } \omega \in I_{\alpha_1,\ldots,\alpha_{r-1},D} \end{array} \right\}.$$

If the value U appears l times in a sequence $\alpha_1,\ldots,\alpha_{r-1}$, then $|I_{\alpha_1,\ldots,\alpha_{r-1},U}| = pp^l q^{r-1-l}$. There are $\binom{r-1}{l}$ different sequences $\alpha_1,\ldots,\alpha_{r-1}$ having U exactly at l places. Then for $A_r = \{K(r) = U\}$ we find

$$\begin{aligned} P(A_r) = P(K(r) = U) &= p \sum_{l=0}^{r-1} \binom{r-1}{l} p^l q^{r-1-l} \\ &= p(p+q)^{r-1} = p \end{aligned}$$

As a consequence is $P(K(r) = D) = q$. The proof that the variables $K(1),\ldots,K(n)$ are independent follows as in Ex. 1.1.

Exercise 1.3: Find the filtration in $\Omega = [0,1]$ generated by the process $X(n,\omega) = 2\omega \mathbf{1}_{[0,1-\frac{1}{n}]}(\omega)$.

Solution: Since $X(1)(\omega) = 0$ for all $\omega \in [0,1]$, we have $\mathcal{F}_{X(1)} = \{\emptyset, [0,1]\}$. For any $B \subset \mathbb{R}$ and $\alpha \in \mathbb{R}$, let $\alpha B = \{\alpha\omega : \omega \in B\}$.
Now for $k > 1$, $B \in \mathcal{B}(\mathbb{R})$,

$$X(k)^{-1}(B) = \left\{ \begin{array}{c} (\frac{1}{2}B) \cap [0, 1-\frac{1}{k}] \text{ if } 0 \notin B \\ (\frac{1}{2}B) \cap [0, 1-\frac{1}{k}] \cup (1-\frac{1}{k}, 1] \text{ if } 0 \in B. \end{array} \right\}$$

Then Hence $\mathcal{F}_{X(k)} = \{A \cup E : A \in \mathcal{B}((0, 1-\frac{1}{k}]) \}, E \in \{\emptyset, \{0\} \cup (1-\frac{1}{k}, 1]\}\}$. Suppose $1 \le k \le n$. If $C \in \mathcal{F}_{X(k)}$ and $C \in \mathcal{B}((0, 1-\frac{1}{k}])$ then $C \in \mathcal{B}((0, 1-\frac{1}{n}]) \subset \mathcal{F}_{X(n)}$. If $C = A \cup \{0\} \cup (1-\frac{1}{k}, 1]$, $A \in \mathcal{B}((0, 1-\frac{1}{k}])$, then $C = (A \cup (1-\frac{1}{k}, 1-\frac{1}{n}]) \cup \{0\} \cup (1-\frac{1}{n}, 1] \in \mathcal{F}_{X(n)}$ because $A \cup (1-\frac{1}{k}, 1-\frac{1}{n}] \in \mathcal{B}((0, 1-\frac{1}{n}])$. In consequence $\mathcal{F}_{X(k)} \subset \mathcal{F}_{X(n)}$ for all k. This implies $\mathcal{F}_n^X = \mathcal{F}_{X(n)}$.

Exercise 1.4: Working on $\Omega = [0,1]$ find (by means of concrete formulae and sketching the graphs) the martingale $\mathbb{E}(Y|\mathcal{F}_n)$ where $Y(\omega) = \omega^2$ and $\mathcal{F}_n$ is generated by $X(n,\omega) = 2\omega\mathbf{1}_{[0,1-\frac{1}{n})}(\omega)$ (see Exercise 1.3).

Solution: According to Exercise 1.3 the natural filtration $\mathcal{F}_n$ of X has the form$\mathcal{F}_n = \mathcal{F}_n^X = \mathcal{F}_{X(n)}$, so

$$\mathcal{F}_n = \{A \cup E : A \in \mathcal{B}((0, 1 - \frac{1}{n}]), E \in \{\emptyset, \{0\} \cup (1 - \frac{1}{n}, 1]\}\}.$$

Hence the restriction of $\mathbb{E}(Y|\mathcal{F}_n)$ to the interval $(0, 1-\frac{1}{n}]$ must be a $\mathcal{B}((0, 1-\frac{1}{n}])$-measurable variable and $\mathbb{E}(Y|\mathcal{F}_n) = Y$ on $(0, 1 - \frac{1}{n}]$ satisfies Def. 1.9 for $A \subset (0, 1 - \frac{1}{n}]$. The restriction of $\mathbb{E}(Y|\mathcal{F}_n)$ to the set $\{0\} \cup (1 - \frac{1}{n}, 1]$ must be measurable with respect to the σ-field $\{\emptyset, \{0\} \cup (1 - \frac{1}{n}, 1]\}$. Thus $\mathbb{E}(Y|\mathcal{F}_n)$ has to be a constant function: $\mathbb{E}(Y|\mathcal{F}_n) = c$, on $\{0\} \cup (1 - \frac{1}{n}, 1]$. Condition 2 of Def. 1.9 gives $\int_{(1-\frac{1}{n},1]} c\, dP = \int_{(1-\frac{1}{n},1]} \omega^2 dP$. It follows that $\mathbb{E}(Y|\mathcal{F}_n)(\omega) = c = 1 - \frac{1}{n} + \frac{1}{3n^2}$ for $\omega \in (1 - \frac{1}{n}, 1]$.

Exercise 1.5: Show that the expectation of a martingale is constant in time. Find an example showing that constant expectation does not imply the martingale property.

Solution: Let ζ be the trivial σ-algebra, consisting of P-null sets and their complements. For every integrable random variable X, $\mathbb{E}(X|\zeta) = \mathbb{E}(X)$. If M is a martingale, then $\mathbb{E}(M(n + 1)|\mathcal{F}_n) = M(n)$ for all $n \geq 0$. Using the tower property we obtain

$$\begin{aligned}\mathbb{E}(M(n)) &= \mathbb{E}(M(n)|\zeta) = \mathbb{E}(M(n + 1)|\mathcal{F}_n)|\zeta) \\ &= \mathbb{E}(M(n + 1)|\zeta) = \mathbb{E}(M(n + 1)).\end{aligned}$$

If $X(n)$, $n \geq 0$ is any sequence of integrable random variables, then for the sequence $\tilde{X}(n) = X(n) - \mathbb{E}(X(n))$ the property $\mathbb{E}(\tilde{X}(n)) = \mathbb{E}(X(n) - \mathbb{E}(X(n))) = \mathbb{E}(X(n)) - \mathbb{E}(X(n)) = 0$ holds for all n.

Exercise 1.6: Show that martingale property is preserved under linear combinations with constant coefficients and adding a constant.

Solution: Let X, Y be martingales with respect to the filtration $\mathcal{F}_n$ and fix $\alpha \in \mathbb{R}$. Define $Z = X + Z$, $W = \alpha X$, $U = X + \alpha$. Then $\mathbb{E}(|Z(n)|) = \mathbb{E}(|X(n) + Y(n)|) \leq \mathbb{E}(|X(n)|) + \mathbb{E}(|Y(n)|) < +\infty$, $\mathbb{E}(|W(n)|) = \mathbb{E}(|\alpha X(n)|) = |\alpha|\mathbb{E}(|X(n)|) < +\infty$. It implies that $Z(n)$ and $W(n)$ are $\mathcal{F}_n$-measurable and they have finite expectation. Finally the linearity of conditional expectation gives $\mathbb{E}(Z(n + 1)|\mathcal{F}_n) = \mathbb{E}(X(n + 1) + Y(n + 1)|\mathcal{F}_n) = \mathbb{E}(X(n + 1)|\mathcal{F}_n) + \mathbb{E}(Y(n + 1)|\mathcal{F}_n) = X(n) + Y(n) = Z(n)$, $\mathbb{E}(W(n + 1)|\mathcal{F}_n) + \mathbb{E}(\alpha X(n + 1)|\mathcal{F}_n) = \alpha\mathbb{E}(X(n + 1)|\mathcal{F}_n) = \alpha X(n) = W(n)$. The process U is the special case of this Z when $Y(n) = \alpha$ for all n.

Exercise 1.7: Prove that if M is a martingale, then for $m < n$,

$$M(m) = \mathbb{E}(M(n)|\mathcal{F}_m).$$

Solution: Using the tower property $n - m - 1$ times we obtain

$$\begin{aligned} M(m) &= \mathbb{E}(M(m+1)|\mathcal{F}_m) = \mathbb{E}(\mathbb{E}(M(m+2)|\mathcal{F}_{m+1})|\mathcal{F}_m) \\ &= \mathbb{E}(M(m+2)|\mathcal{F}_m) = \ldots = \mathbb{E}(M(m)|\mathcal{F}_m). \end{aligned}$$

Exercise 1.8: Let M be a martingale with respect to the filtration generated by $L(n)$ (as defined for random walk), and assume for simplicity $M(0) = 0$. Show that there exists a predictable process H such that $M(n) = \sum_{i=1}^{n} H(i)L(i)$ (i.e. $M(n) = \sum_{i=1}^{n} H(i)[Z(i)-Z(i-1)]$, where $Z(i) = \sum_{j=1}^{i} L(j)$. (We are justified in calling this result a representation theorem: each martingale is a discrete stochastic integral).

Solution: Here the crucial point is that the random variables $L(n)$ have discrete distributions and the process $(M(n))_{n\geq 0}$ is adapted to the filtration $\mathcal{F}_n^L$, which means that $M(n)$, $n \geq 0$ also have discrete distributions and $M(n)$ is constant on the sets of the partition $\mathcal{P}(L_1, \ldots, L_n)$. From the formula $M(n) = \sum_{i=1}^{n} H(i)L(i)$ we obtain $M(n+1) - M(n) = H(n+1)L(n+1)$. Since $L^2(k) = \mathbf{1}_\Omega$ a.s. for all $k \geq 1$, we define $H(n+1) = [M(n+1) - M(n)]L(n$ for $n \geq 0$. To prove that $(H(n+1))_{n\geq 0}$ is a predictable process we have to verify $M(n+1)$ is $\mathcal{F}_n^L$-measurable. This is equivalent to the condition $H(n+1)$ is constant on the sets of the partition $\mathcal{P}(L_1, \ldots, L_n)$. Write $A_{\alpha_1,\ldots,\alpha_k} = \{\omega \in \Omega : L_1(\omega) = \alpha_1, \ldots, L_k(\omega) = \alpha_k; \alpha_j \in \{-1, 1\}\}$. Then $\mathcal{P}(L_1, \ldots, L_k) = \{A_{\alpha_1,\ldots,\alpha_k}; \alpha_j \in \{-1, 1\}\}$ and $A_{\alpha_1,\ldots,\alpha_k,-1} \cup A_{\alpha_1,\ldots,\alpha_k,1} = A_{\alpha_1,\ldots,\alpha_k}$. Moreover, $P(A_{\alpha_1,\ldots,\alpha_k}) = \frac{1}{2^k}$, because the L_j are i.i.d. random variables. Fix n and a set $A_{\alpha_1,\ldots,\alpha_k}$. Next, let $\tilde{\alpha}_0 = M(n)(\omega)$ for $\omega \in A_{\alpha_1,\ldots,\alpha_n}$, $\tilde{\alpha}_{-1} = M(n+1)(\omega)$ for $\omega \in A_{\alpha_1,\ldots,\alpha_n-1}$, $\tilde{\alpha}_1 = M(n+1)(\omega)$ for $\omega \in A_{\alpha_1,\ldots,\alpha_n,1}$. Since M is a martingale

$$\int_{A_{\alpha_1,\ldots,\alpha_n}} M(n)dP = \int_{A_{\alpha_1,\ldots,\alpha_n}} M(n+1)dP$$

and therefore $\tilde{\alpha}_0 P(A_{\alpha_1,\ldots,\alpha_n}) = \tilde{\alpha}_{-1} P(A_{\alpha_1,\ldots,\alpha_n,-1}) + \tilde{\alpha}_1 P(A_{\alpha_1,\ldots,\alpha_n,1})$. From this and the relation $2P(A_{\alpha_1,\ldots,\alpha_n}) = P(A_{\alpha_1,\ldots,\alpha_n,-1}) = P(A_{\alpha_1,\ldots,\alpha_n,1})$ it follows that $2\tilde{\alpha}_0 = \tilde{\alpha}_{-1} + \tilde{\alpha}_1$ or, equivalently, $-(\tilde{\alpha}_{-1} - \tilde{\alpha}_0) = \tilde{\alpha}_1 - \tilde{\alpha}_0$. Using this equality we verify finally that

$$\begin{aligned} &H(n+1)\mathbf{1}_{A_{\alpha_1,\ldots,\alpha_n}} = [M(n+1) - M(n)]L(n+1)\mathbf{1}_{A_{\alpha_1,\ldots,\alpha_n}} \\ &= (-1)(\tilde{\alpha}_{-1} - \tilde{\alpha}_0)\mathbf{1}_{A_{\alpha_1,\ldots,\alpha_n,-1}} + (\tilde{\alpha}_1 - \tilde{\alpha}_0)\mathbf{1}_{A_{\alpha_1,\ldots,\alpha_n,1}} \\ &= (\tilde{\alpha}_1 - \tilde{\alpha}_{-1})\mathbf{1}_{A_{\alpha_1,\ldots,\alpha_n}} \end{aligned}$$

so that $H(n+1)$ is constant on $A_{\alpha_1,\ldots,\alpha_k}$.

Exercise 1.9: Show that the process $Z^2(n)$, the square of a random walk, is not a martingale, by checking that $\mathbb{E}(Z^2(n+1)|\mathcal{F}_n) = Z^2(n)+1$.

Solution: Assume, as before, that $L(k)$, $k = 1, \ldots$ is a symmetric random walk, $L(k) \in \{-1, 1\}$ and set $L(0) = 0$. The variables $(L(k))_{k\geq 0}$ are independent and $Z(k+1) = Z(k)+L(k)$, $k \geq 0$, $\mathcal{F}_k = \mathcal{F}_k^L$. Then $\mathbb{E}(L(k)) = 0$, $\mathbb{E}(L^2(k)) = 1$ for $k \geq 1$ and the variables $Z(k), Z(k^2)$ are $\mathcal{F}_k$-measurable and the variables $L(k+1), L^2(k+1)$ are independent of $\mathcal{F}_k$. Using the properties of conditional expectation we have

$$\begin{aligned}
&\mathbb{E}(Z^2(n+1)|\mathcal{F}_n) = \mathbb{E}((Z(n)+L(n+1))^2|\mathcal{F}_n) \\
&= \mathbb{E}(Z^2(n)|\mathcal{F}_n) + 2Z(n)\mathbb{E}(L(n+1)|\mathcal{F}_n) + \mathbb{E}(L^2(n+1)|\mathcal{F}_n) \\
&\quad \text{(linearity, measurability)} \\
&= Z^2(n) + 2Z(n)\mathbb{E}(L(n+1)) + \mathbb{E}(L^2(n+1)) \\
&\quad \text{(measurability,independence)} \\
&= Z^2(n) + 1 \text{ for } n \geq 0.
\end{aligned}$$

Exercise 1.10: Show that if X is a submartingale, then its expectations increase with n:

$$\mathbb{E}(X(0)) \leq \mathbb{E}(X(1)) \leq \mathbb{E}(X(2)) \leq \cdots,$$

and if X is a supermartingale, then its expectations decrease as n increases:

$$\mathbb{E}(X(0)) \geq \mathbb{E}(X(1)) \geq \mathbb{E}(X(2)) \geq \cdots.$$

Solution: Since X is a submartingale, $X(n) \leq E(X(n+1)|\mathcal{F}_n)$ for all n. Taking expectations on both sides of this inequality we obtain

$$\mathbb{E}(X(n)) \leq \mathbb{E}(\mathbb{E}(X(n+1)|\mathcal{F}_n)) = \mathbb{E}(X(n+1)) \text{ for all n.}$$

For a supermartingale proceed similarly.

Exercise 1.11: Let $X(n)$ be a martingale (submartingale, supermartingale). For a fixed m consider the sequence $X'(k) = X(m+k) - X(m)$, $k \geq 0$ Show that X' is a martingale (submartingale, supermartingale) relative to the filtration $\mathcal{F}'_k = \mathcal{F}_{m+k}$.

Solution: Let X be a martingale (submartingale, supermartingale). Then $X(m)$ is $\mathcal{F}_{m+k}$ measurable variable for all m, k. We have $\mathbb{E}(X'(k+1)|\mathcal{F}'_k) = \mathbb{E}(X(m+k+1) - X(m)|\mathcal{F}_{m+k}) = \mathbb{E}(X(m+k+1)|\mathcal{F}_{m+k}) - \mathbb{E}(X(m)|\mathcal{F}_{m+k}) = (\geq, \leq) X(m+k) - X(m) = X'(k)$, for all k.

Exercise 1.12: Prove the Doob decomposition for submartingales from first principles:

If $Y(n)$ is a submartingale with respect to some filtration, then there exist, for the same filtration, a martingale $M(n)$ and a predictable, increasing process $A(n)$ with $M(0) = A(0) = 0$ such that

$$Y(n) = Y(0) + M(n) + A(n).$$

This decomposition is unique.

Solution: The process $Z(n) = Y(n) - Y(0)$, $n \geq 0$, is a submartingale with $Z(0) = 0$. Therefore we may assume $Y(0) = 0$ without loss of generality. We prove the theorem with the use the principle of induction. For $n = 1$, the decomposition formula would imply the relation

$$\mathbb{E}(Y(1)|\mathcal{F}_0) = \mathbb{E}(M(1)|\mathcal{F}_0) + \mathbb{E}(A(1)|\mathcal{F}_0).$$

If this is to hold with M a martingale and A predictable, we must set

$$A(1) := \mathbb{E}(Y(1)|\mathcal{F}_0) - M(0),$$

which shows that $A(1)$ is $\mathcal{F}_0$-measurable.

To arrive at the composition formula we now define

$$M(1) := Y(1) - A(1).$$

$M(1)$ is $\mathcal{F}_1$-measurable because $Y(1)$ and $A(1)$ are. Moreover,

$$\mathbb{E}(M(1)|\mathcal{F}_0) = \mathbb{E}(Y(1)|\mathcal{F}_0) - \mathbb{E}(A(1)|\mathcal{F}_0) = \mathbb{E}(Y(1)|\mathcal{F}_0) - A(1) = M(0),$$

which completes the initial induction step.

Assume now that we have defined an $\mathcal{F}_k$-adapted martingale $M(k)$ and a predictable, increasing process $A(k)$, $k \leq n$ such that $A(k)$ and $M(k)$ satisfy the decomposition formula for $Y(k)$, for all $k \leq n$. Once again the decomposition formula for $k = n + 1$ gives

$$\mathbb{E}(A(n+1)|\mathcal{F}_n) = \mathbb{E}(Y(n+1)|\mathcal{F}_n) - \mathbb{E}(M(n+1)|\mathcal{F}_n).$$

Hence it is necessary to define

$$A(n+1) := \mathbb{E}(Y(n+1)|\mathcal{F}_n) - M(n). \tag{0.1}$$

Having $A(n + 1)$ to conserve the decomposition formula we set

$$M(n+1) := Y(n+1) - A(n+1). \tag{0.2}$$

Now we verify that $M(k), A(k)$, $k \le n+1$ satisfy the conditions of the theorem. From (0.1) $A(n+1)$ is $\mathcal{F}_n$-measurable, because $M(n)$ is $\mathcal{F}_n$-measurable. Next from (0.1) and the decomposition formula for n we have

$$\begin{aligned} A(n+1) &= \mathbb{E}(Y(n+1)|\mathcal{F}_n) - M(n) \\ &= [\mathbb{E}(Y(n+1)|\mathcal{F}_n) - Y(n)] + A(n) \ge A(n) \end{aligned}$$

because Y is a submartingale. Then $A(k)$ is an increasing, predictable process for $k \le n+1$.

From (0.2), $M(n+1)$ is $\mathcal{F}_{n+1}$-measurable and, since $A(n+1)$ is $\mathcal{F}_n$-measurable,

$$\begin{aligned} \mathbb{E}(M(n+1)|\mathcal{F}_n) &= \mathbb{E}(Y(n+1)|\mathcal{F}_n) - \mathbb{E}(A(n+1)|\mathcal{F}_n) = \mathbb{E}(Y(n+1)|\mathcal{F}_n) - A(n+1) \\ &= M(n), \text{ by (0.1)).} \end{aligned}$$

Thus $M(k)$, $k \le n+1$ is a martingale. By construction, the processes $M(k), A(k)$, $k \le n+1$ satisfy the decomposition formula for $Y(k)$ for all $k \le n+1$. By the principle of induction we may deduce that the processes A and M, given by (0.1), (0.2) for all n, satisfy the conditions of the theorem. The uniqueness is proved in the main text.

Exercise 1.13: Let $Z(n)$ be a random walk (see Example 1.2), $Z(0) = 0, Z(n) = \sum_{j=1}^{n} L(j)$, $L(j) = \pm 1$, and let $\mathcal{F}_n$ be the filtration generated by $L(n)$, $\mathcal{F}_n = \sigma(L(1), \dots, L(n))$. Verify that $Z^2(n)$ is a submartingale and find the increasing process A in its Doob decomposition.

Solution: From relations (0.1), (0.2) we can give explicit formula for the compensator A.

$$\begin{aligned} A(k) &= \mathbb{E}(Y(k)|\mathcal{F}_{k-1}) - M(k-1) \\ &= \mathbb{E}(Y(k)|\mathcal{F}_{k-1}) - Y(k-1) + A(k-1). \end{aligned}$$

Hence $A(k) - A(k-1) = \mathbb{E}(Y(k) - Y(k-1)|\mathcal{F}_{k-1})$. Adding these equalities on both sides we obtain

$$A(n) = \sum_{k=1}^{n} \mathbb{E}(Y(k) - Y(k-1)|\mathcal{F}_{k-1}), \text{ for } n \ge 1. \tag{0.3}$$

By Exercise 1.9, $\mathbb{E}(Z^2(n+1)|\mathcal{F}_n) = Z^2(n) + 1 \ge Z^2(n)$ when $Z(0) = 0$, i.e., Z^2 is a submartingale. Next using the formula (0.3) given in Exercise

1.12 we obtain

$$A(n) = \sum_{k=1}^{n} \mathbb{E}(Z^2(k) - Z^2(k-1)|\mathcal{F}_{k-1})$$
$$= \sum_{k=1}^{n} [(Z^2(k-1)+1) - Z^2(k-1)] = n$$

for $n \geq 1$.

Exercise 1.14: Using the Doob decomposition, show that if Y is a square-integrable submartingale (resp. supermartingale) and H is predictable with bounded non-negative $H(n)$, then the stochastic integral of H with respect to Y is also a submartingale (resp. supermartingale).

Solution: Let Y be a submartingale. Then by the Doob decomposition (Theorem 1.19) there exist unique martingale M and a predictable, increasing process A, $M(0) = A(0) = 0$, such that $Y(k) = Y(0) + M(k) + A(k)$ for $k \geq 0$. Hence $Y(k) - Y(k-1) = [M(k) - M(k-1)] + [A(k) - A(k-1)]$. This relation gives the following representation for the stochastic integral H with respect to Y

$$X(n+1) = \sum_{k=1}^{n+1} H(k)[Y(k) - Y(k-1)]$$
$$= \sum_{k=1}^{n+1} H(k)[M(k) - M(k-1)]$$
$$+ \sum_{k=1}^{n+1} H(k)[A(k) - A(k-1)]$$
$$= Z(n+1) + B(n+1), \ n \geq 0.$$

By the Theorem 1.15 $Z(k)$, $k \geq 1$ is a martingale. For the second term we have

$$\mathbb{E}(B(n+1)|\mathcal{F}_n) = \sum_{k=1}^{n+1} \mathbb{E}(H(k)[A(k) - A(k-1)]|\mathcal{F}_n)$$
$$= \sum_{k=1}^{n+1} H(k)[A(k) - A(k-1)]$$
$$= B(n) + H(n+1)[A(n+1) - A(n)] \geq B(n)$$

because by the predictability of H and A, the random variables $H(k)[A(k) - A(k-1)]$ are $\mathcal{F}_n$-measurable for $k \leq n+1$. Also, $H(k) \geq 0$, and $A(k)$ is an increasing process. Taking together the properties of Z and B we conclude that $\mathbb{E}(X(n+1)|\mathcal{F}_n) = \mathbb{E}(Z(n+1)|\mathcal{F}_n) + \mathbb{E}(B(n+1)|\mathcal{F}_n) \geq Z(n) + B(n) = X(n)$. This proves the claim for a submartingale.

If Y is a supermartingale, $-Y$ is the submartingale, so the above proof applies, which implies that the stochastic integral of a supermartingale is again a supermartingale.

Exercise 1.15: Let τ be a stopping time relative to the filtration $\mathcal{F}_n$. Which of the random variables $\tau + 1$, $\tau - 1$, τ^2 is a stopping time?

Solution: α) $\tau' = \tau + 1$, yes. Because $\{\tau' = n\} = \{\tau = n-1\} \in \mathcal{F}_{n-1} \subset \mathcal{F}_n$ for $n \geq 1$. $\{\tau' = 0\} = \emptyset \in \mathcal{F}_0$.

β) $\tau' = \tau - 1$, no. Because we can only conclude that$\{\tau' = n\} = \{\tau = n+1\} \in \mathcal{F}_{n+1}$ for $n \geq 0$, so this set need not be in $\mathcal{F}_n$.

γ) τ'^2, yes. Because $\{\omega : \tau'^2, k \in \mathbb{N}\} = \{\omega : \tau(\omega) = k\} \in \mathcal{F}_k \subset \mathcal{F}_n$ for $n = k^2$, $k \in \mathbb{N}$. For $n \notin \{k^2; k \in \mathbb{N}\}$, $\{\omega : \tau(\omega) = n\} = \emptyset \in \mathcal{F}_n$.

Exercise 1.16: Show that the constant random variable, $\tau(\omega) = m$ for all ω, is a stopping time relative to any filtration.

Solution: $\{\tau = n\} = \left\{ \begin{array}{ll} \varnothing & \text{if } m \neq n \\ \Omega & \text{if } m = n \end{array} \right\}$ then $\{\tau = n\} \in \mathcal{F}_n$ for all $n \in \mathbb{N}$.

Exercise 1.17: Show that if τ and ν are as in the Proposition, then $\tau \wedge \nu$ is also a stopping time.

Solution: Use the condition (p. 15): $g : \Omega \to \mathbb{N}$, then $\{g = n\} \in \mathcal{F}_n$ for all $n \in \mathbb{N} \Leftrightarrow \{g \leq n\} \in \mathcal{F}_n$ for all $n \in \mathbb{N}$. We have $\{\tau \wedge \nu = n\} \in \mathcal{F}_n$ for all n $\Longleftrightarrow \{\tau \leq n\} \cup \{\nu \leq n\} \in \mathcal{F}_n$ for all n.

Exercise 1.18: Deduce the above theorem from Theorem 1.15 by considering $H(k) = \mathbf{1}_{\{\tau \geq k\}}$. (Let M be a martingale. If τ is a stopping time, then the stopped process M_τ is also a martingale.)

Solution: Let M and τ be a martingale and a stopping time for filtration $(\mathcal{F}_n)_{n \geq 0}$. Take $Y(n) = M(n) - M(0)$ for $n \geq 0$. Then Y is also a martingale and $\mathbb{E}(Y(0)) = 0$. Now write for $n \geq 1$

$$
\begin{aligned}
& Y_\tau(n, \omega) = Y(n \wedge \tau(\omega), \omega) = Y(1, \omega) + (Y(2, \omega) - Y(1, \omega) + \dots \\
& +(Y(n \wedge \tau(\omega), \omega) - Y(n \wedge \tau(\omega) - 1, \omega)) \\
= & \sum_{k=1}^{n} \mathbf{1}_{\{\tau \geq k\}}(\omega)(Y(k) - Y(k-1)).
\end{aligned}
$$

Thus Y_τ can be written in the form $Y_\tau(n) = \sum_{k=1}^{n} H(k)(Y(k) - Y(k-1))$ where

$H(k) = \mathbf{1}_{\{\tau \geq k\}}$. The process H is a bounded predictable process because it is the indicator function of the set $\{\tau \geq k\} = \Omega - \bigcup_{m=1}^{k-1}\{\tau = m\} \in \mathcal{F}_{k-1}$. By Theorem 1.15 Y_τ is a martingale. This gives $\mathbb{E}(Y_\tau(n)) = (Y_\tau(0)) = \mathbb{E}(Y(0)) = 0$. Hence $\mathbb{E}(X_\tau(n)) = \mathbb{E}(X(0))$ for all n.

Exercise 1.19: Using the Doob decomposition show that a stopped submartingale is a submartingale, (and similarly for a supermartingale). Alternatively use the above representation of the stopped process and use the definition to reach the same conclusions.

Solution: Use the form of the stopped process given in the proof of Proposition 1.30. Let M and τ be a submartingale (supermartingale) and a finite stopping time. From the form of M_τ we have

$$M_\tau(n+1) = \sum_{m<n+1} M(m)\mathbf{1}_{\tau=m} + M(n+1)\mathbf{1}_{\tau \geq n+1}.$$

Since each term of the right hand side is integrable variable, $M_\tau(n+1)$ is also integrable variable. Now we can write

$$\mathbb{E}(M_\tau(n+1)|\mathcal{F}_n) = \sum_{m<n+1} \mathbb{E}(M(m)\mathbf{1}_{\tau=m}|\mathcal{F}_n) + \mathbb{E}(M(n+1)\mathbf{1}_{\tau \geq n+1}|\mathcal{F}_n).$$

The processes $M(m)\mathbf{1}_{\tau=m}$, $m < n+1$ and $\mathbf{1}_{\tau \geq n+1} = \mathbf{1}_\Omega - \mathbf{1}_{\tau \leq n}$ are $\mathcal{F}_n$-measurable, then

$$\mathbb{E}(M_\tau(n+1)|\mathcal{F}_n) = \sum_{m<n+1} M(m)\mathbf{1}_{\tau=m} + \mathbf{1}_{\tau \geq n+1}\mathbb{E}(M(n+1)|\mathcal{F}_m)$$

$$\geq (\leq) \sum_{m<n} M(m)\mathbf{1}_{\tau=m} + M(n)\mathbf{1}_{\tau=n} + \mathbf{1}_{\tau \geq n+1} M(n)$$

(M is sub (super) martingale)

$$= \sum_{m<n} M(m)\mathbf{1}_{\tau=m} + M(n)\mathbf{1}_{\tau \geq n} = M_\tau(n) \text{ for all } n \geq 0.$$

Exercise 1.20: Show that $\mathcal{F}_\tau$ is a sub-σ-field of $\mathcal{F}$.

Solution: α) $\Omega \in \mathcal{F}_\tau$ because $\Omega \cap \{\tau = n\} = \{\tau = n\} \in \mathcal{F}_n$ for all $n \geq 0$.
β) Let A belong $\mathcal{F}_\tau$. It is equivalent to the condition $A \cap \{\tau = n\} \in \mathcal{F}_n$ for all n. Then $(\Omega \setminus A) \cap \{\tau = n\} = \{\tau = n\} - (A \cap \{\tau = n\}) \in \mathcal{F}_n$ for all n, because $\mathcal{F}_n$ are σ-fields. The last condition means $\Omega \setminus A \in \mathcal{F}_\tau$.
γ) Let A_k belong $\mathcal{F}_\tau$ for $k = 1, 2, \ldots$. Then $A_k \cap \{\tau = n\} \in \mathcal{F}_n$ for all n. Hence it follows $(\bigcup_{k=1}^\infty A_k) \cap \{\tau = n\} = \bigcup_{k=1}^\infty (A_k \cap \{\tau = n\}) \in \mathcal{F}_n$ for all n. This means $\bigcup_{k=1}^\infty A_k \in \mathcal{F}_\tau$.

Exercise 1.21: Show that if τ, ν are stopping times with $\tau \leq \nu$ then $\mathcal{F}_\tau \subset \mathcal{F}_\nu$.

Solution: Let τ, ν be stopping times such that $\tau \leq \nu$. Let $A \in \mathcal{F}_\tau$. According to the definition of $\mathcal{F}_\tau$, $A \cap \{\tau = m\} \in \mathcal{F}_m$ for $m = 0, 1, \ldots$ Since $\tau \leq \nu$, it holds $(\bigcup_{m=0}^n \{\tau = m\}) \cap \{\nu \leq n\} = \{\nu \leq n\}$. Hence we have $A \cap \{\nu \leq n\} = \bigcup_{m=0}^t (A \cap \{\tau = m\}) \cap \{\nu \leq n\} \in \mathcal{F}_n$ for $\mathcal{F}_m \subset \mathcal{F}_n$ and $\{\nu \leq n\} \in \mathcal{F}_n$. n was arbitrary, according to the condition (p. 15) $A \in \mathcal{F}_\tau$.

Exercise 1.22: Any stopping time τ is $\mathcal{F}_\tau$-measurable.

Solution: We have to prove $\{\tau \leq k\} \in \mathcal{F}_\tau$ for each $k = 0, 1, \ldots$. This is equivalent to the condition $\{\tau < k\} \cap \{\tau = n\} \in \mathcal{F}_n$ for all n, k. We have $B = \{\tau \leq k\} \cap \{\tau = n\} = \left\{ \begin{array}{cc} \emptyset & \text{if } k < n \\ \{\tau = n\} & \text{if } n \leq k \end{array} \right\}$Then $B \in \mathcal{F}_n$. As consequence $\{\tau \leq k\} \in \mathcal{F}_\tau$.

Exercise 1.23: (Theorem 1.35 for supermartingales). If M is a supermartingale and τ, ν are bounded stopping times $\tau \leq \nu$ then

$$\mathbb{E}(M(\nu)|\mathcal{F}_\tau) \leq M(\tau).$$

Solution: It is enough to prove that

$$\int_A \mathbb{E}(M(\nu)|\mathcal{F}_\tau) dP = \int_A M(\nu) dP \leq \int_A M(\tau) dP$$

for all $A \in \mathcal{F}_\tau$. We will prove the equivalent inequality. $\mathbb{E}(\mathbf{1}_A(M(\nu) - M(\tau))) \leq 0$ for arbitrary $A \in \mathcal{F}_\tau$. From the proof of Theorem 1.35 we know that the variable $\mathbf{1}_A(M(\nu) - M(\tau))$ can be written in the form $\mathbf{1}_A(M(\nu) - M(\nu)) = X_\nu(N)$ where the process $X(n)$ is as follows

$$X(n) = \sum_{k=1}^n H(k)(M(k) - M(k-1)),$$

$X(0) = 0$, $H(k) = \mathbf{1}_A \cdot \mathbf{1}_{\{\tau < k\}}$ and N is a constant such that $\nu \leq N$. Additionally H is a bounded and predictable process. Now the assumption M is a supermartingale implies that X is also supermartingale Exercise 1.14). Hence it follows by the results of Exercise 1.19 that the stopped process $X_\nu(n)$ is also supermartingale. But for a supermartingale we have $\mathbb{E}(X_\nu(N)) \leq \mathbb{E}(X_\nu(0)) = \mathbb{E}(X(0)) = 0$, which completes the proof.

Exercise 1.24: Suppose that, with $M(n)$ and λ as in the Theorem, $(M(n))^p$

is integrable for some $p > 1$. Show that we can improve (1.4) to read

$$P(\max_{k\le n} M(k) \ge \lambda) \le \frac{1}{\lambda^p}\int_{\{\max_{k\le n} M(k)\ge\lambda\}} M^p(n)dP \le \frac{1}{\lambda^p}\mathbb{E}(M^p(n)).$$

Solution: If $p \ge 1$ the function x^p, $x \ge 0$ is a convex, nondecreasing function. As $M^p(n)$ is integrable, Jensen's inequality (see p.10) and the fact that M is a submartingale imply

$$\mathbb{E}(M^p(n+1)|\mathcal{F}_n) \ge (\mathbb{E}(M(n+1)|\mathcal{F}_n))^p \ge M^p(n).$$

Applying Doob's maximal inequality (Theorem 1.36) to the event $\{\max_{k\le n} M(k) \ge \lambda\} = \{\max_{k\le n} M^p \ge \lambda^p\}$ we obtain the result.

Exercise 1.25: Extend the above Lemma to L^p for every $p > 1$, to conclude that for non-negative $Y \in L^p$, and with its relation to $X \ge 0$ as stated in the Lemma, we obtain $\|X\|_p \le \frac{p}{p-1}\|Y\|_p$. (Hint: the proof is similar to that given for the case $p = 2$, and utilises the identity $p\int_{\{z\ge x\}} x^{p-1}dx = x^p$.)

Note: The definition of the normed vector space L^p is not given explicitly in the text, but is well-known: one may prove that if $p > 1$ the map $X \longmapsto (\mathbb{E}(|X|^p))^{1/p} = \|X\|_p$ is a norm on the vector space of all p-integrable random variables (i.e. where $\mathbb{E}(|X|^p) < \infty$), again with the proviso that we identify random variables that are a.s. equal. The Schwarz inequality in L^2 then extends to the Hölder inequality: $\mathbb{E}(|XY|) \le \|X\|_p \|Y\|_q$ when $X \in L^p, Y \in L^q, \frac{1}{p} + \frac{1}{q} = 1$.

Proof: The extension of Lemma 1.38 that we require is the following:

Assume that X, Y are non-negative random variables, Y is in $L^p(\Omega)$, $p > 1$. Suppose that for all $x > 0$,

$$xP(X \ge x) \le \int_\Omega \mathbf{1}_{\{X\ge x\}} Y dP.$$

Then X is in $L^p(\Omega)$ and

$$\|\mathbb{E}(X)\|_p = (\mathbb{E}(X^p))^{\frac{1}{p}} \le \frac{p}{p-1}\|\mathbb{E}(Y)\|_p.$$

The proof is similar to that of Lemma 1.38. First consider the case when X is bounded. The following formula is interesting on its own:

$$\mathbb{E}(X^p) = \int_0^\infty px^{p-1}P(X \ge x)dx, \text{ for } p > 0.$$

To prove it, substitute $z = X(\omega)$ in the equality $z^p = p\int_0^\infty \mathbf{1}_{\{x\le z\}}(x)x^{p-1}dx$, and we obtain

$$\mathbb{E}(X^p) = \int_\Omega X^p(\omega)dP(\omega) = \int_\Omega p\left(\int_0^\infty \mathbf{1}_{\{x\le X(\omega)\}}(x)x^{p-1}dx\right)dP(\omega).$$

By Fubini's theorem

$$\mathbb{E}(X^p) = \int_0^\infty x^{p-1}\left(\int_\Omega \mathbf{1}_{\{X\ge x\}}(\omega)dP(\omega)\right)dx = \int_0^\infty px^{p-1}P(X\ge x)dx.$$

Our hypothesis and Fubini's theorem once more give

$$\begin{aligned}
\mathbb{E}(X^p) &\le p\int_0^\infty x^{p-2}\left(\int_\Omega \mathbf{1}_{\{X\ge x\}}(\omega)Y(\omega)dP(\omega)\right)dx \\
&= p\int_\Omega\left(\int_0^\infty \mathbf{1}_{\{x<X(\omega)\}}(x)x^{p-2}dx\right)Y(\omega)dP(\omega) \\
&= p\int_\Omega\left(\int_0^{X(\omega)} x^{p-2}dx\right)Y(\omega)dP(\omega) \\
&= \frac{p}{p-1}\int_\Omega X^{p-1}(\omega)Y(\omega)dP(\omega).
\end{aligned}$$

for $p > 1$.
The Hölder inequality with p and $q = \frac{p}{p-1}$ yields $\int_\Omega X^{p-1}YdP \le (\mathbb{E}((X^{p-1})^{\frac{p}{p-1}}))^{\frac{p-1}{p}}(\mathbb{E}(Y^p))^{\frac{1}{p}}$. The last two inequalities give $\|X\|_p^p = \mathbb{E}(X^p) \le \frac{p}{p-1}\|X\|_p^{p-1}\|Y\|_p$. This is equivalent to our claim for X bounded.

If X is not bounded we can take $X_n = X\wedge n$. The inclusion $\{X_n \le x\} \supset \{X \le x\}$ implies $P(X_n \ge x) \le P(X \ge x)$ and from the assumptions of the theorem we obtain the inequalities

$$xP(X_n\ge x) \le xP(X\ge x) \le \int \mathbf{1}_{\{X\ge x\}}YdP \le \int \mathbf{1}_{\{X_n\ge x\}}YdP.$$

As X_n is bounded this gives $\mathbb{E}(X_n^p) \le (\frac{p}{p-1})^p\mathbb{E}(Y^p)$ for all $n \ge 1$. The sequence X_n^p increases to X^p a.s., the monotone convergence theorem implies $\mathbb{E}(X^p) \le (\frac{p}{p-1})^p\mathbb{E}(Y^p)$ and also as a consequence $X^p \in L^p(\Omega)$.

Exercise 1.26: Find the transition probabilities for the binomial tree. Is it homogeneous?

Solution: From the definition of the binomial tree the behaviour of stock prices is described by a sequence of random variables $S(n) = S(n-1)(1+K(n))$, where $K(n,\omega) = U\mathbf{1}_{A_n}(\omega) + D\mathbf{1}_{[0,1]\setminus A_n}(\omega)$, $S(0)$ given, deterministic.

As in the Exercise 1.2 we have $P(K(n) = U) = 1 - P(K(n) = D) = p$, $p \in (0,1)$ for $n \geq 1$ and the variables $K(n)$ are independent random variables. From the definition of $S(n)$, $S(n) = S(0)\prod_{i=1}^{n}(1 + K(i))$. Then $\mathcal{F}_n^S = \sigma(S(1), \ldots, S(n)) = \mathcal{F}_n^K = \sigma(K(1), \ldots, K(n))$ and for any Borel function $f : \mathbb{R} \to \mathbb{R}$ is

$$\begin{aligned}\mathbb{E}(f(S(n+1))|\mathcal{F}_n^S) &= \mathbb{E}(f(S(n)(1 + K(n+1)))|\mathcal{F}_n^K) \\ &= \mathbb{E}(F(f)(S(n), K(n+1))|\mathcal{F}_n^K)\end{aligned}$$

where $F(f)(x,y) = f(x(1+y))$. The variable $K(n+1)$ is independent of $\mathcal{F}_n^K$ and $S(n)$ is $\mathcal{F}_n^K$ measurable, by the Lemma 1.43 we have

$$\mathbb{E}(f(S(n+1)|\mathcal{F}_n^S) = G(f)(S(n))$$

where

$$\begin{aligned}G(f)(x) &= \mathbb{E}(F(f)(x, K(n+1))) \\ &= pf(x(1+U)) + (1-p)f(x(1+D)).\end{aligned}$$

Since $G(f)$ is a Borel function, the penultimate formula implies, by definition of conditional expectation, that $\mathbb{E}(f(S(n+1))|\mathcal{F}_n^S) = \mathbb{E}(f(S(n+1))|\mathcal{F}_{S(n)})$. So the process $(S(n))_{n\geq 0}$ has the Markov property. Assuming $f = \mathbf{1}_B$, $\mu_n(x, B) = G(\mathbf{1}_B)(x)$ for Borel sets B we see that, for every fixed B, $\mu_n(x, B) = p\mathbf{1}_B(x(1+U))\,(1-p)\mathbf{1}_B(x(1+B))$ is a measurable function and for every fixed $x \in \mathbb{R}$, $\mu_n(x, \cdot)$ is a probability measure on $\mathcal{B}(\mathbb{R})$. We also have

$$P(S(n+1) \in B|\mathcal{F}_{S(n)}) = \mathbb{E}(\mathbf{1}_B(S(n+1))|\mathcal{F}_{S(n)}) = \mu_n(S(n), B).$$

Thus the μ_n are transition probabilities of the Markov process $(S(n))_{n\geq 0}$. This is a homogeneous Markov process, as the μ_n do not depend on n.

Exercise 1.27: Show that symmetric random walk is homogeneous.

Solution: According to its definition, a symmetric random walk is defined by taking $Z(0)$ and defining $Z(n) = Z(n-1) + L(n)$, where the random variables $Z(0), L(1), \ldots, L(n)$ are independent for every $n \geq 1$. Moreover, $P(L(n) = 1) = P(L(n) = -1) = \frac{1}{2}$ (see Examples 1.4 and 1.46). Since $Z(n) = Z(0) + \sum_{i=1}^{n} L(i)$, we have $\mathcal{F}_n^Z = \sigma(Z(0), L(1), \ldots, L(n))$.

For any bounded Borel function $f : \mathbb{R} \to \mathbb{R}$ we have $f(Z(n+1)) = f(Z(n) + L(n+1)) = F(f)(Z(n), L(n+1))$ where $F(f)(x,y) = f(x+y)$. The variable $L(n+1)$ is independent of $\mathcal{F}_n^Z$ and $Z(n)$ is $\mathcal{F}_{Zn}$-measurable, so by Lemma 1.43 we obtain

$$\mathbb{E}(f(Z(n+1))|\mathcal{F}_n^Z) = \mathbb{E}(F(f)(Z(n), L(n+1))) = G(f)(Z(n)),$$

where

$$G(f)(x) = \mathbb{E}(F(f)(x, L(n+1))) = \mathbb{E}(f(x + L(n+1))$$
$$= \frac{1}{2}(f(x+1) + f(x-1)).$$

There last two relations testify that $G(f)$ is a Borel function and that the equality $\mathbb{E}(f(Z(n+1))|\mathcal{F}_n^Z) = \mathbb{E}(f(Z(n+1))|\mathcal{F}_{Z(n)})$ holds.

Thus $(Z(n))_{n\geq 0}$ is a Markov process. Assuming $f = \mathbf{1}_B$, $\mu(x, B) = \frac{1}{2}\mathbf{1}_B(x+1) + \mathbf{1}_B(x-1) = \delta_{x+1}(B) + \delta_{x-1}(B)$, where B is a Borel set, $x \in \mathbb{R}$ we obtain

$$P(Z(n+1) \in B|\mathcal{F}_{Z(n)}) = G(\mathbf{1}_B)(Z(n)) = \mu(Z(n), B).$$

Again, $\mu(\cdot, B)$ is a measurable function for each Borel set B and $\mu(x, \cdot)$ is a probability measurable for every $x \in \mathbb{R}$, so we conclude that μ is a transition probability for the Markov process $(Z(n))_{n\geq 0}$. Since μ does not depend on n, this process is homogeneous.

Exercise 1.28: Let $(Y(n))_{n\geq 0}$, be a sequence of independent integrable random variables on $(\Omega, \mathcal{F}, P)$. Show that the sequence $Z(n) = \sum_{i=0}^{n} Y(i)$ is a Markov process and calculate the transition probabilities dependent on n. Find a condition for Z to be homogeneous.

Solution: From the definition $Z(n) = \sum_{i=1}^{n} Y(i)$ follow the relations $\mathcal{F}_n^Z = \sigma(Z(0), \ldots, Z(n)) = \sigma(Y(0), \ldots, Y(n)) = \mathcal{F}_n^Y$ and $Z(n+1) = Z(n) + Y(n+1)$. For any bounded Borel function $f : \mathbb{R} \to \mathbb{R}$ we have

$$f(Z(n+1)) = f(Z(n) + Y(n+1)) = F(f)(Z(n), Y(n+1))$$

where $F(f)(x, y) = f(x+y)$. The variable $Z(n)$ is $\mathcal{F}_n^Z$ measurable and by our assumption $(Y(i))_{i\geq 0}$ is a sequence of independent variables. Thus $Y(n+1)$ is independent of $\mathcal{F}_n^Z$. Now using Lemma 1.43 we obtain

$$\mathbb{E}(f(Z(n+1))|\mathcal{F}_n^Z) = \mathbb{E}(F(f)(Z(n), Y(n+1))|\mathcal{F}_n^Z)$$
$$= G_n(f)(Z(n))$$

where

$$G_n(f)(x) = \mathbb{E}(F(f)(x, Y(n+1))) = \mathbb{E}(f(x + Y(n+1)))$$
$$= \int_{\mathbb{R}} f(x+y) P_{Y(n+1)}(dy) \text{ for } n \geq 0.$$

$P_{Y(n+1)}$ is the distribution of the random variable $Y(n+1)$. From the form $G_n(f)(x) = \int_{\mathbb{R}} f(x+y) P_{Y(n+1)}(dy)$ and the Fubini theorem, $G_n(f)$ is a measurable function. The equality $\mathbb{E}(f(Z(n+1))|\mathcal{F}_n^Z) = G_n(f)(Z(n))$ implies that $\mathbb{E}(f(Z(n+1))|\mathcal{F}_n^Z)$ is $\mathcal{F}_{Z(n)}$ measurable function. Then from the definition of conditional expectation we have $\mathbb{E}(f(Z(n+1))|\mathcal{F}_{Z(n)}) = \mathbb{E}(f(Z(n+1))|\mathcal{F}_{Z(n)})$ a.e. . So the process $(Z(n))_{n\geq 0}$ is a Markov process.

Putting $\mu_n(x, B) = G_n(\mathbf{1}_B)(x)$, $n \geq 0$, we see that for every Borel set B, $\mu_n(\cdot, B)$ is a measurable function. Next, denote $S_x(y) = x + y$. Of course S_x is a Borel function for every x. From the definition of μ_n we have relations

$$\mu_n(x, B) = \int_{\mathbb{R}} \mathbf{1}_B(S_x(y))P_{Y(n+1)}(dy)$$
$$= \int_{\mathbb{R}} \mathbf{1}_{S_x^{-1}(B)}(y)P_{Y(n+1)}(dy) = P_{Y(n+1)}(S_x^{-1}(B)).$$

This shows that for every $x \in \mathbb{R}$, $\mu_n(x, \cdot)$ are probability measures. Finally

$$P(Z(n+1) \in B|\mathcal{F}_{Z(n)}) = \mathbb{E}(\mathbf{1}_B(Z(n+1))|\mathcal{F}_{Z(n)})$$
$$+ G(\mathbf{1}_B)(Z(n)) = \mu_n(Z(n), B)$$

$n \geq 0$. Collecting together all these properties we conclude that the measures μ_n, $n \geq 0$, are the transition probabilities of the Markov process $(Z(n))_{n\geq 0}$. From the definition of μ_n we see that if the distribution functions $P_{Y(n)}$ of variables $Y(n)$ are different, then μ_n are different and the process $Z(n)$ is not homogeneous. If for all n variables $Y(n)$ have the same distribution function that is $P_{Y(n)} = P_{Y(0)}$ for all n, then $\mu_n = \mu_0$ for all n and the process $Z(n)$ is homogeneous.

Exercise 1.29: A Markov chain is homogeneous if and only if for every pair $i, j \in \mathbb{S}$

$$P(X(n+1) = j|X(n) = i) = P(X(1) = j|X(0) = i) = p_{ij} \tag{0.4}$$

for every $n \geq 0$.

Solution: A Markov chain $X(n)$, $n \geq 0$, is homogeneous if for every Borel set B and $n \geq 0$ the equation $\mathbb{E}(\mathbf{1}_B(X(n+1))|\mathcal{F}_{X(n)}) = \mu(X(n), B)$ is satisfied, where μ is a fixed transition probability, not depending on n. In the discrete case, the variables $X(n)$, $n \geq 0$ take values in a finite set $\{0, \ldots, N\}$. The relation $\mathbf{1}_B(X(n+1)) = \sum_{j\in B} \mathbf{1}_{\{X(n+1)=j\}}$ and additivity of the conditional expectation allows us to restrict attention to sets $B = \{i\}$, $i \in S$. Since here the conditional expectations are simple functions, constant on the sets $A_i^n = \{X(n) = i\}$, the condition that the process $X(n)$ be homogeneous is equivalent to

$$\mathbb{E}(\mathbf{1}_{\{X(n+1)=j\}}|\mathcal{F}_{X(n)}) \cdot \mathbf{1}_{A_i^{(n)}} = \mu(i, \{j\})\mathbf{1}_{A_i^{(n)}}$$

for every $i, j \in S$, $n \geq 0$. Denoting $\mu(i, \{j\}) = p_{ij}$ we obtain that the last equalities are equivalent to the formulae $P(X(n+1) = j|X(n) = i) = p_{ij}$ for all $i, j \in S$ and $n \geq 0$.

Comming back to the financial example (see p.32) based on credit ratings this means that the process of rating of a country is a homogeneous Markov process if the rating of a country at time $n+1$ depends only on its rating at time n (it close not depend on its previous ratings)and the probabilities p_{ij} of rating changes are the same for all times.

Exercise 1.30: Prove that the transition probabilities of a homogeneous Markov chain satisfy the so-called **Chapman-Kolmogorov equation**

$$p_{ij}(k+l) = \sum_{r \in \mathbb{S}} p_{ir}(k) p_{rj}(l).$$

Proof: Denote by $\tilde{P}^k$ the matrix with entries $p_{ij}(k)$ and denote the entries of the k-th power of the transition matrix P^k by $p_{ij}^{(k)}$, $i, j \in \mathbb{S}$. We now claim that $P^k = \tilde{P}^k$ for all $k \geq 0$, or equivalently $p_{ik}^{(k)} = p_{ij}(k)$ for all i, j, k.
To prove our claim we use the induction principle.

Step1. If $k = 1$, then $p_{ij}^{(1)} = p_{ij} = p_{ij}(1)$, so $\tilde{P} = P$.

Step2. The induction hypothesis. Assume that for all $l \leq m$, $\tilde{P}^l = P^l$.

Step3. The inductive step. We will prove that $\tilde{P}^{m+1} = P^{m+1}$. We have

$$\begin{aligned} & p_{ij}(m+1) = P(X(m+1) = j|X(0) = i) \\ &= \sum_{r \in \mathbb{S}} P(X(m) = r|X(0) = i) P(X(m+1) = j|X(m) = r, X(0) = i) \\ &= \sum_{r \in \mathbb{S}} p_{ir}(m) \cdot P(X(m+1) = j|X(m) = r, X(0) = i). \end{aligned}$$

The following relations hold on the set $\{X(m) = r\}$

$$\begin{aligned} & P(X(m+1) = j|X(m) = r, X(0) = i) \\ &= \mathbb{E}(\mathbf{1}_{\{j\}}(X(m+1))|\mathcal{F}_{X(0),X(m)}) \\ &= \mathbb{E}(\mathbb{E}(\mathbf{1}_{\{j\}}(X(m+1))|\mathcal{F}_m^X|\mathcal{F}_{X(0),X(m)}) \text{ (tower property)} \\ &= \mathbb{E}(\mathbb{E}(\mathbf{1}_{\{j\}}(X(m+1))|\mathcal{F}_{X(m)}|\mathcal{F}_{X(0),X(m)}) \text{ (Markov property)} \\ &= \mathbb{E}(\mathbf{1}_{\{j\}}(X(m+1))|\mathcal{F}_{X(m)}) \text{ (tower property)} \\ &= P(X(m+1) = j|X(m) = r) = p_{rj} \end{aligned}$$

(X is Markov, homogeneous, Exercise 1.29). Utilizing this result we obtain $p_{ij}(m+1) = \sum_{r \in \mathbb{S}} p_{ir}(m) p_{rj}$ for all i, j. This equality means that $\tilde{P}^{m+1} =$

$\tilde{P}^m P$. Hence by the induction assumption $\tilde{P}^{m+1} = P^m$. Now by the Induction Principle $\tilde{P}^k = P^k$ for all $k \geq 1$ and the claim is true.

Now our exercise is trivial. From the equality $P^{k+l} = P^k P^l$ it follows that $\tilde{P}^{k+l} = \tilde{P}^k \tilde{P}^l$. Writing out the last equation for the entries completes the proof.

Chapter 2

Exercise 2.1: Show that scalings other than by the square root lead nowhere by proving that $X(n) = h^{\alpha}L(n)$, $\alpha \in (0, \frac{1}{2})$, implies $\sum_{n=1}^{N} X(n) \to 0$ in L^2 while for $\alpha > \frac{1}{2}$ this sequence goes to infinity in this space.

Solution: Since $L(n)$ has mean 0 and variance 1 for each n, we have, by independence and since $h = \frac{1}{N}$,

$$\begin{aligned}\left\|\sum_{n=1}^{N} X(n)\right\|_2^2 &= Var(h^{\alpha}\sum_{n=1}^{N} L(n)) \\ &= h^{2\alpha}\sum_{n=1}^{N} Var(L(n)) \\ &= h^{2\alpha}N = h^{2\alpha-1}.\end{aligned}$$

When $h \to 0$, this goes to 0 if $\alpha < \frac{1}{2}$ and to $+\infty$ if $\alpha < \frac{1}{2}$.

Exercise 2.2: Show that $\mathrm{Cov}(W(s), W(t)) = \min(s, t)$.

Solution: Since $\mathbb{E}(W(t)) = 0$ and $\mathbb{E}(W(t) - W(s))^2 = t - s$ for all $t \geq s \geq 0$, we have $\mathrm{Cov}(W(s), W(t)) = \mathbb{E}(W(s)W(t))$ and $t - s = \mathbb{E}(W(t) - W(s))^2 = \mathbb{E}(W^2(t)) - 2\mathbb{E}(W(s)W(t)) + \mathbb{E}(W^2(s)) = t - 2\mathbb{E}(W(s)W(t)) + s$. This equality implies the formula $\mathbb{E}(W(s)W(t)) = s = \min(s, t)$.

Exercise 2.3 Consider $B(t) = W(t) - tW(1)$ for $t \in [0, 1]$ (this process is called the Brownian bridge, since $B(0) = B(1) = 0$). Compute $\mathrm{Cov}(B(s), B(t))$.

Solution: $\mathbb{E}(B(r)) = \mathbb{E}(W(r)) - r\mathbb{E}(W(s)) = 0$ for all $r \geq 0$ for $\mathbb{E}(W(s)) = 0$ for all r. Then

$$\begin{aligned}Cov(B(s), B(t)) &= Cov(B(s), B(t)) = E(W(s) - sW(1))(W(t) - tW(1)) \\ &= \mathbb{E}(W(s)W(t)) - s\mathbb{E}(W(1)W(t)) - t\mathbb{E}(W(s)W(1)) + st\mathbb{E}(W^2(1)) \\ &= Cov(W(s), W(t)) - sCov(W(1), W(t)) - tCov(W(s), W(1)) + st\mathbb{E}(W^2(1)) \\ &= \min(s, t) - s\min(t, 1) - t(s, 1) + st \\ &= \left\{\begin{array}{ll} s(1-t) & \text{if } s \leq t \leq 1 \\ t(1-s) & \text{if } t \leq s \leq 1 \end{array}\right\}.\end{aligned}$$

Exercise 2.4: Show directly from the definition that if W is a Wiener process, then so are the processes given by $-W(t)$ and $\frac{1}{c}W(c^2t)$ for any $c > 0$.

Solution: The process $-W(t)$ obviously satisfies the Definition 2.4. We consider the process $Y(t) = \frac{1}{c}W(c^2t)$, $c > 0$, $t \geq 0$. It is known (see [PF]) that if, for two given random variables U, V and every continuous bounded function $f : \mathbb{R} \to \mathbb{R}$ we have $\mathbb{E}(f(U)) = \mathbb{E}(f(V))$, then the distributions P_U and P_V of U and V are the same. First note that $P_{Y(t)} = P_{W(t)}$ for all $t \geq 0$, since

$$\mathbb{E}(f(Y(t))) = \int_\Omega f\left(\frac{1}{c}W(c^2t)\right)dP$$
$$= \int_{\mathbb{R}} f\left(\frac{1}{c}x\right)\frac{1}{\sqrt{2\pi c^2 t}}e^{-\frac{x^2}{2c^2t}}dx \text{ (}W\text{ has normal distribution)}$$
$$= \int_{\mathbb{R}} f(y)\frac{1}{\sqrt{2\pi t}}e^{-\frac{y^2}{2t}}dy \text{ (change of variable x=cy)}$$
$$= \mathbb{E}(f(W(t))).$$

(i) We verify the conditions of Definition 2.4. Condition 1 is obvious. For Condition 2 2 take $0 \leq s < t$, $B \in \mathcal{B}(\mathbb{R})$. Then

$$P((Y(t) - Y(s)) \in B) = P\left(\frac{1}{c}(W(c^2t) - W(c^2s)) \in B\right)$$
$$= P((W(c^2t) - W(c^2s)) \in g_c^{-1}(B)) \text{ (where } g_c(x) = \frac{1}{c}x)$$
$$= P((W(c^2t - c^2s)) \in g_c^{-1}(B)) \text{ (Condition 2 for } W\text{, the same increments)}$$
$$= P\left(\left(\frac{1}{c}W(c^2(t-s))\right) \in B\right) = P(Y(t-s) \in B) = P(W(t-s) \in B)$$
$$= P((W(t) - W(s)) \in B).$$

Thus $Y(t) - Y(s)$ and $W(t) - W(s)$ have the same distribution. For the third condition set $0 \leq t_1 < \ldots < t_m$. Then $0 \leq c^2t_1 < \ldots < c^2t_m$ and the increments $W(c^2t_1) - W(c^2t_0), \ldots, W(c^2t_m) - W(c^2t_{m-1})$ are independent by independence of the increments of $W(t)$. Hence the process $Y(t)$ has independent increments. The paths of Y are continuous for almost all ω because this holds for W.

Exercise 2.5: Apply the above Proposition to solve Exercise 2.4. In other words, use the following result to give alternative proofs of Exercise 2.4: If a Gaussian process X has $X(0) = 0$, constant expectations, a.s. continuous paths and $Cov(X(s), X(t) = \min(s, t)$, then it is a Wiener process.

Solution: The proof that $-W$ is again a Wiener process is clear, as it is Gaussian,, a.s. continuous and has the right covariances. For the second part we prove two auxiliary claims:

1.If $X(t)$, $t \geq 0$ is a Gaussian process and $b > 0$, then $Y(t) = X(bt)$, $t \geq 0$ is a Gaussian process.
2. If $Y(t)$, $t \geq 0$ is a Gaussian process, $c > 0$, then $Z(t) = \frac{1}{c}Y(t)$, $t \geq 0$ is a Gaussian process.
Proof of 1: Fix $0 \leq t_0 < t_1 < \ldots < t_n$. Then the distribution vector of increments $(Y(t_1) - Y(t_0), \ldots, Y(t_n) - Y(t_{n-1}))$ is the same as the distribution vector of the increments $(X(s_1) - X(s_0), \ldots, X(s_n) - X(s_{n-1}))$ where $s_i = bt_i$, $i = 0, \ldots, n$. But the last vector is Gaussian because X is a Gaussian process. According to Def. 2.11 Y is a Gaussian process.
For 2. we prove the following more general claim:
If U, $U^T = (U_1, \ldots, U_n)^T$ is a Gaussian random vector with the mean vector μ_U and the covariance matrix Σ_U and A is a nonsingular $n \times n$ (real) matrix, then $V = AU$ is a Gaussian vector with mean vector $\mu_V = A\mu_U$ and covariance matrix $\Sigma_V = A\Sigma_U A^T$.
To prove this consider the mapping $A(u) = Au$ for $u \in \mathbb{R}^n$. Then for every Borel set $B \in \mathcal{B}(\mathbb{R}^n)$ we have

$$P(V \in B) = P(A \circ U \in B) = P(U \in A^{-1}(B)) = \int_{A^{-1}(B)} f_U(u)du$$

where f_U is the density distribution function for U, $du = (du_1, \ldots, du_n)$. Changing the variables, $u = A^{-1}v$, we obtain $P(V \in B) = \int_B f_U(A^{-1}v)\det(A^{-1})dv$. From Definition 2.12 we conclude that

$$P(V \in B) = \int_B (2\pi)^{-\frac{n}{2}}(\det\Sigma_U)^{-\frac{1}{2}} \exp\{(A^{-1}(v - A\mu))^T \Sigma_U^{-1}(A^{-1}(v - A\mu))\}(\det A)^{-1}dv$$

$$= \int_B (2\pi)^{-\frac{n}{2}} det(A\Sigma_U A^T)^{-\frac{1}{2}} \exp\{(v - A\mu)^T (A\Sigma_U A^T)^{-1}(v - A\mu)\}dv.$$

This formula shows that V is a Gaussian vector with mean vector $\mu_V = A\mu_U$ and covariance matrix $\Sigma_V = A\Sigma_U A^T$.

Returning to Point 2 above let V be the vector of increments of the process Z, $V^T = (Z(t_1) - Z(t_0), \ldots, Z(t_n) - Z(t_{n-1}))$ and U the vector of increments of Y, $Y^T = (Y(t_1) - Y(t_0), ..., Y(t_n) - Y(t_{n-1}))$. Denote $A = \text{diag}(\frac{1}{c}, \ldots, \frac{1}{c})$, where diag means diagonal matrix. Then $V = AU$ and A is a non-singular matrix ($c > 0$). Since Y was Gaussian, we know that V is also a Gaussian vector. The proof of 2. is completed.

Now to solve our Exercise we verify the assumptions of Proposition

2.13. Our claims 1 and 2 show that the process $\tilde{W}(t) = \frac{1}{c}W(c^2t)$, $t \geq 0$ is a Gaussian process because W has this property. Next $\mathbb{E}(\tilde{W}(t)) = 0$ for all t because $\mathbb{E}(W(t)) = 0$ for each t. $\tilde{W}$ has continuous paths because W has continuous paths. For the last condition

$$\mathrm{Cov}(\tilde{W}(s), \tilde{W}(t)) = \mathrm{Cov}\left(\frac{1}{c}W(c^2s), \frac{1}{c}W(c^2t)\right) = \frac{1}{c^2}(c^2s \wedge c^2t) = s \wedge t.$$

By Proposition 2.13 $\tilde{W}$ is a Wiener process.

Exercise 2.6: Show that the shifted Wiener process is again a Wiener process and that the inverted Wiener process satisfies conditions 2,3 of the definition.

Solution: 1. Shifted process.
Verify the conditions of Definition 2.4 for Wiener process.
1. $W^u(0) = W(u) - W(u) = 0$.
2. For $0 \leq s < t$, $W^u(t) - W^u(s) = W(u+t) - W(u+s)$. Hence $W^u(t) - W^u(s)$ has normal distribution with mean value 0 and standard deviation $\sqrt{t-s}$.
3. For all m and $0 \leq t_1 < \ldots < t_m$ the increments $W^u(t_{n+1}) - W^u(t_n) = W(u + t_{n+1}) - W(u + t_n)$, $n = 1, \ldots, m-1$, are independent because the increments of Winer process $W(u + t_{n+1}) - W(u + t_n)$, $n = 1, \ldots, m-1$ are independent.
4. For almost all ω the paths of W are continuous functions, then also the paths of W^u are continuous.

2. Inverted process. Consider the process $Y(t) = tW(\frac{1}{t})$ for $t > 0$, $Y(0) = 0$. Since $Y(t) = \frac{1}{c}W(c^2t)$ for $t > 0$, $c = \frac{1}{t}$, by the previous Exercise $Y(t)$ have normal distributions $\mathbb{E}(Y(t)) = 0$, $\mathrm{Var}Y(t) = t$ for $t > 0$. To verify condition 2 of Def. 2.4, choose $0 < s < t$. Then $0 < \frac{1}{t} < \frac{1}{s}$ and $Y(t) - Y(s) = (-s)(W(\frac{1}{s}) - W(\frac{1}{t})) + (t-s)W(\frac{1}{t})$. Since the increments $W(\frac{1}{t})$, $W(\frac{1}{s}) - W(\frac{1}{t})$ are independent, Gaussian variables, the variables $(t-s)W(\frac{1}{t}$ and $(-s)(W(\frac{1}{s}) - W(\frac{1}{t})$ are also independent and Gaussian. Hence their sum $Y(t) - Y(s)$ also has a Gaussian distribution. Now $\mathbb{E}(W(r)) = 0$ for all $r \geq 0$ implies $\mathbb{E}(Y(t) - Y(s)) = 0$. This lets us calculate the standard deviation σ of $Y(t) - Y(s)$ as follows $\sigma^2 = \mathrm{Var}(Y(t) - Y(s)) = \mathrm{Var}((-s)(W(\frac{1}{s}) - W(\frac{1}{t})) + (t - s)W(\frac{1}{t})) = s^2\mathrm{Var}(W(\frac{1}{s}) - W(\frac{1}{t})) + (t-s)^2\mathrm{Var}(W(\frac{1}{t})) = s^2(\frac{1}{s} - \frac{1}{t}) + (t-s)^2\frac{1}{t} = t - s$.

To verify condition 3 of Def. 2.4 take $0 < t_1 < \ldots < t_m$. It is necessary to prove that the components of the vector ΔY_m, $(\Delta Y_m)^T = (Y(t_1), Y(t_2) - Y(t_1), \ldots, Y(t_m) - Y(t_{m-1}))^T$ are independent random variables. To obtain this property we prove that ΔY_m has a Gaussian distribution and $\mathrm{Cov}(Y(s), Y(t)) =$

$\min(t, s)$. These facts give the independence of components of ΔY_m (see proof of Proposition 2.13). It is $0 < \frac{1}{t_m} < \frac{1}{t_{m-1}} < \ldots < \frac{1}{t_1}$. Hence the components of the vector $(\Delta\overline{Z}_m)^T = (W(\frac{1}{t_m}), W(\frac{1}{t_{m-1}}) - W(\frac{1}{t_m}), \ldots, W(\frac{1}{t_1}) - W(\frac{1}{t_2}))^T$ are independent and have normal distributions as increments of a Wiener process. Then the vector $\Delta\overline{Z}_m$ has a Gaussian distribution. Now it is easy to calculate the relation $\Delta\overline{Y}_m = BA\Delta\overline{Z}_m$ where the matrices A and B have the forms

$$A = \begin{pmatrix} t_1 & \cdot & \cdot & t_1 & t_1 \\ t_2 & \cdot & \cdot & t_2 & 0 \\ \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & 0 & 0 & 0 \\ t_m & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & 0 & & & 0 \\ -1 & 1 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ 0 & & & -1 & 1 \end{pmatrix}.$$

(i)

Since $\det A = t_1 \cdot \ldots \cdot t_m \neq 0$, $\det B = 1 \neq 0$, we know by Exercise 2.5, that $\Delta\overline{Y}_m$ is Gaussian vector. Since the sequence (t_i) was arbitrary, $Y(t)$, $t \geq 0$ is a Gaussian process.

The last condition we have to verify $\mathrm{Cov}(Y(t), Y(s)) = \min(t, s)$. Let $0 < s \leq t$. Then $\mathrm{Cov}(Y(t), Y(s)) = \mathbb{E}(tW(\frac{1}{t})sW(\frac{1}{s})) = ts\min(\frac{1}{t}, \frac{1}{s}) = ts\frac{1}{t} = \min(s, t)$. From the proof of Proposition 2.13 the increments of the process $Y(t)$, $t \geq 0$ are independent.

Exercise 2.7: Show that $X(t) = \sqrt{t}Z$ does not satisfy conditions 2,3 of the definition of the Wiener process.

Solution: Assume $0 \leq s < t$. Then we have $X(t) - X(s) = (\sqrt{t} - \sqrt{s})Z$. Hence $\mathbb{E}(X(t) - X(s)) = 0$ and $\mathrm{Var}(X(t) - X(s)) = \mathbb{E}(X(t) - X(s))^2 = (t + s - 2\sqrt{ts})$. The last equality contradicts Condition 2 in Definition 2.4 of the Wiener process.

To check condition 3 of Def. 2.4, consider the increments $X(t_{k+1}) - X(t_k)$, $k = 1, \ldots, m-1$, where $t_{k+1} = (\sqrt{t_k} + 1)^2$, $t_1 \geq 0$. Then $\sqrt{t_{k+1}} - \sqrt{t_k} = 1$ and

$$P(X(t_2) - X(t_1) \leq 0, \ldots, X(t_m) - X(t_{m-1}) \leq 0) = P(Z \leq 0) = \frac{1}{2}, \text{while}$$

$$\prod_{k=1}^{m-1} P(X(t_{k+1}) - X(t_k) \leq 0) = P(Z \leq 0)^{m-1} = (\frac{1}{2})^{m-1}.$$

So Condition 3 of Def. 2.4 is not satisfied.

Exercise 2.8: Prove the last claim **-** i.e.that if X, Y have continuous paths and Y is a modification of X, then these processes are indistinguishable.

Solution: Suppose Y is a modification of X and X and Y have continuous paths. Let $T_0 = \{t_k : k = 1, 2, \ldots\}$ be a dense, countable subset of the time set $\mathbb{T}$. We know that the sets $A_k = \{\omega; X(t_k, \omega) = Y(t_k, \omega)\}, k = 1, 2, \ldots$ have $P(A_k) = 1$ or, equivalently, $P(\Omega \setminus A_k) = 0$. Now take the set $A = \bigcap_{k=1}^{\infty} A_k$. Since

$$P(\Omega \setminus A) = P(\Omega \setminus \bigcap_{k=1}^{\infty} A_k) = P(\bigcup_{k=1}^{\infty} (\Omega \setminus A_k)) \leq \sum_{k=1}^{\infty} P(\Omega \setminus A_k) = 0,$$

we have $P(A) = 1$. If $\omega_0 \in A$, then $\omega_0 \in A_k$ for all $k = 1, 2, \ldots$. This means that $X(t, \omega_0) = Y(t, \omega_0)$ for all $t \in T_0$. Since $X(\cdot, \omega_0)$ and $Y(\cdot, \omega_0)$ are continuous functions and T_0 is a dense subset of T, it follows that $X(t, \omega_0) = Y(t, \omega_0)$ for all $t \in T$. But ω_0 was an arbitrary element of A and $P(A) = 1$, so the processes X and Y are indistinguishable.

Exercise 2.9: Prove that If $M(t)$ is a martingale with respect to $\mathcal{F}_t$, then

$$\mathbb{E}(M^2(t) - M^2(s)|\mathcal{F}_s) = \mathbb{E}([M(t) - M(s)]^2|\mathcal{F}_s).$$

and in particular

$$\mathbb{E}(M^2(t) - M^2(s)) = \mathbb{E}([M(t) - M(s)]^2).$$

Solution: The first equality follows from the relations

$$\begin{aligned}
&\mathbb{E}([M(t) - M(s)]^2|\mathcal{F}_s) \\
&= \mathbb{E}(M^2(t) + M^2(s)|\mathcal{F}_s) - 2\mathbb{E}(M(s)M(t)|\mathcal{F}_s) \text{ (linearity)} \\
&= \mathbb{E}(M^2(t) + M^2(s)|\mathcal{F}_s) - 2M(s)\mathbb{E}(M(t)|\mathcal{F}_s) \text{ } (M(s) \text{ is } \mathcal{F}_s \text{ measurable}) \\
&= \mathbb{E}(M^2(t) + M^2(s)|\mathcal{F}_s) - 2M^2(s) \text{ } (M \text{ is a martingale}) \\
&= \mathbb{E}(M^2(t) - M^2(s)|\mathcal{F}_s).
\end{aligned}$$

The second equality follows from the first by the tower property.

Exercise 2.10: Consider a process X on $\Omega = [0,1]$ with Lebesgue measure, given by $X(0,\omega) = 0$, and $X(t,\omega) = \mathbf{1}_{[0,\frac{1}{t}]}(\omega)$ for $t > 0$. Find the natural filtration $\mathcal{F}_t^X$ for X.

Solution: The definitions of the probability space $([0,1], \mathcal{B}([0,1]), m)$, m-Lebesgue measure, and the process X yield

$$\mathcal{F}_{X(s)} = \begin{cases} \{\emptyset, [0,1]\} & \text{, for } 0 \le s \le 1 \\ \{\emptyset, [0,1], [0,\frac{1}{s}], (\frac{1}{s}, 1]\} & \text{, for } s > 1. \end{cases}$$

This implies $\mathcal{F}_t^X = \sigma(\bigcup_{t \ge s \ge 0} \mathcal{F}_{X(s)}) = \{0, [0,1]\}$ for $0 \le t \le 1$. In the case $t > 1$ all intervals $(\frac{1}{s_1}, \frac{1}{s_2}] = (\frac{1}{s_1}, 1] \cap [0, \frac{1}{s_2}]$, where $0 < s_2 \le s_1 \le t$, also belong to $\mathcal{F}_t^X$. Hence we must have $B((\frac{1}{t}, 1] \subset \mathcal{F}_t^X$ and then $[0, \frac{1}{t}] \in \mathcal{F}_t^X$. These conditions give $\mathcal{F}_t^X = \mathcal{B}((\frac{1}{t}, 1]) \cup B'$, where $B' = \{[0,1] \setminus A : A \in \mathcal{B}((\frac{1}{t}, 1])\}$, because $\mathcal{B}((\frac{1}{t}, 1]) \cup B'$ is a σ-field.

Exercise 2.11: Find $M(t) = \mathbb{E}(Z|\mathcal{F}_t^X)$ where $\mathcal{F}_t^X$ is constructed in the previous exercise.

Solution: Let Z be a random function on the probability space $([0,1], \mathcal{B}([0,1]), m)$ such that $\int_0^1 |Z| dm$ exists. We will calculate the conditional mean values of Z with respect to the filtration $(\mathcal{F}_t^X)_{t \ge 0}$ defined in the Exercise 2.10.

From Exercise 2.10 we know that in the case $t > 1$ every set $A \in \mathcal{F}_t^X$ either belongs to $\mathcal{B}((\frac{1}{t}, 1])$ or it is of the form $A = [0, \frac{1}{t}] \cup C$ where $C \in \mathcal{B}((\frac{1}{t}, 1])$. Hence every $\mathcal{F}_t^X$-measurable variable, including $\mathbb{E}(Z|\mathcal{F}_t^X)$, must be a constant function when restricted to the interval $[0, \frac{1}{t}]$, while restricted to $(\frac{1}{t}, 1]$ it is an $\mathcal{F}((\frac{1}{t}, 1])$-measurable function. Then from the definition of conditional mean value $\int_{[0,\frac{1}{t}]} Z dm = \int_{[0,\frac{1}{t}]} \mathbb{E}(Z|\mathcal{F}_t^X) dm = c\frac{1}{t}$ and for every $A \in \mathcal{B}((\frac{1}{t}, 1])$ we have $\int_A Z dm = \int_A \mathbb{E}(Z|\mathcal{F}_t^X) dm$. The last equality implies $\mathbb{E}(Z|\mathcal{F}_t^X) = Z$ on $(\frac{1}{t}, 1]$ Finally

$$\mathbb{E}(Z|\mathcal{F}_t^X)(\omega) = \begin{cases} t \int_0^{\frac{1}{t}} Z dm & \text{, for } \omega \in [0, \frac{1}{t}] \\ Z(\omega) & \text{, for } \omega \in (\frac{1}{t}, 1] \end{cases}$$

a.e. in the case $t > 1$. In the case $t < 1$ $\mathbb{E}(Z|\mathcal{F}_t^X) = \mathbb{E}(Z)$ a.e.

Exercise 2.12: Is $Y(t,\omega) = t\omega - \frac{1}{2}t$ a martingale ($\mathcal{F}_t^X$ as above)? Compute $\mathbb{E}(Y(t))$.

Solution: It costs little to compute the expectation: $\mathbb{E}(Y(t)) = \int_0^1 (t\omega - \frac{1}{2}t) d\omega = 0$. If the expectation were not constant, we would conclude that the process is not a martingale, however, constant expectation is just a necessary condition, so we have to investigate further. The martingale condition reads $\mathbb{E}(Y(t)|\mathcal{F}_s^X) = Y(s)$. Consider $1 < s < t$. The random variable on the left is $\mathcal{F}_s^X$-measurable, so since $[0, \frac{1}{s}]$ is an atom of the σ-field, it has to be constant on this event. However, $Y(s)$ is not constant (being a linear function of ω), so Y is not a martingale for this filtration.

Exercise 2.13: Prove that for almost all paths of the Wiener process W we have $\sup_{t\geq 0} W(t) = +\infty$ and $\inf_{t\geq 0} W(t) = -\infty$.

Solution

Set $Z = \sup_{t\geq 0} W(t)$. Exercise 2.4 shows that for every $c > 0$ the process $cW(\frac{t}{c^2})$ is also a Wiener process. Hence cZ and Z have the same distribution for all $c > 0$, which implies that $P(0 < Z < \infty) = 0$, and so the distribution of Z is concentrated on $\{0, +\infty\}$. It therefore suffices to show that $P(Z = 0) = 0$. Now we have

$$\begin{aligned} P(Z = 0) &\leq P(\{W(1) \leq 0\} \cap \bigcap_{u\geq 1} \{W(u) \leq 0\}) \\ &= P(\{W(1) \leq 0\} \cap \{\sup_{t\geq 0}(W(1+t) - W(1)) = 0\}) \end{aligned}$$

since the process $Y(t) = W(1 + t) - W(1)$ is also a Wiener process, so that its supremum is almost surely 0 or $+\infty$. But $(Y(t))_{t\geq 0}$ and $(W(t))_{t\in[0,1]}$ are independent, so

$$\begin{aligned} P(Z = 0) &\leq P(W(1) \leq 0)P(\sup_t Y(t) = 0) \\ &= P(W(1) \leq 0)P(Z = 0), \end{aligned}$$

(as Y is a Wiener process, $\sup_{t\geq 0} Y(t)$ has the same distribution as Z) and so $P(Z = 0) = 0$. The second claim is now immediate, since $-W$ is also a Wiener process.

Exercise 2.14: Use Proposition 2.35 to complete the proof that the inversion of a Wiener process is a Wiener process, by verifying path-continuity at $t = 0$.

Solution: We have to verify that the process

$$Y(t) = \begin{cases} tW(\frac{1}{t}) & \text{, for } t > 0 \\ 0 & \text{, for } t = 0 \end{cases}$$

has almost all paths continuous at 0. This follows from Proposition 2.35, since

$$tW\left(\frac{1}{t}\right) = \frac{W(\frac{1}{t})}{\frac{1}{t}} \to 0 \text{ a.s. if } t \to \infty.$$

Exercise 2.15: Let $(\tau_n)_{n\geq 1}$ be a sequence of stopping times. Show that $\sup_n \tau_n$ and $\inf_n \tau_n$ are stopping times.

Erratum: The claim for the infimum as stated in the text is false in general. It requires right-continuity of the filtration, as shown in the proof below.

Solution: $\sup \tau_n$ is a stopping time because for all $t \geq 0$, $\{\sup \tau_n \leq t\} = \bigcap_n \{\tau_n \leq t\} \in \mathcal{F}_t$ as an intersection of sets in σ-field $\mathcal{F}_t$.

The case of $\inf_n \tau_n$ needs an additional assumption.

Definition. A filtration $(\mathcal{F}_t)_{t\leq T}$ is called right continuous if $\mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s = \mathcal{F}_t$.

We now prove the following auxiliary result (see also lemma 2.48 in the text).

Claim. If a filtration $(\mathcal{F}_t)_{t\leq T}$ is right continuous, τ is a stopping time for $(\mathcal{F}_t)_{t\leq T}$ if and only if for every t, $\{\tau < t\} \in \mathcal{F}_t$.

Proof. If τ is stopping time, then for every t and $n = 1, \ldots$, $\{\tau \leq t - \frac{1}{n}\} \in \mathcal{F}_{t-\frac{1}{n}} \subset \mathcal{F}_t$. Hence $\{\tau < t\} = (\bigcup_{n=1}^{\infty} \{\tau \leq t - \frac{1}{n}\}) \in \mathcal{F}_t$. If $\{\tau < t\} \in \mathcal{F}_t$ for all t, then $\{\tau \leq t\} = (\bigcap_{n=1}^{\infty} \{\tau < t + \frac{1}{n}\}) \in \mathcal{F}_{t+} = \mathcal{F}$.

This allows us to prove the desired result: if $(\tau_n)_{n\geq 1}$ is a sequence of stopping times for a right continuous filtration $(\mathcal{F}_t)_{t\in T}$, then $\inf_n \tau_n$ is a stopping time for $(\mathcal{F}_t)_{t\in T}$

Proof. According to the claim τ_n are stopping times imply that for every t and n, $\{\tau_n < t\} \in \mathcal{F}_t$. Hence $\{\inf_n \tau_n < t\} = \bigcup_n \{\tau_n < t\} \in \mathcal{F}_t$ for all t, which, again by virtue of claim, filtration is continuous, testifies, $\inf \tau_n$ is a stopping times.

Exercise 2.16: Verify that $\mathcal{F}_\tau$ is a σ-field when τ is a stopping time.

Solution 2.16:1. Because $\mathcal{F}_t$ is a σ-field , $\emptyset \cap \{\tau \leq t\} = \emptyset \in \mathcal{F}_t$ for all t. Then by the definition of $\mathcal{F}_\tau$, $\emptyset \in \mathcal{F}_\tau$

2.If $A \in \mathcal{F}_\tau$, then $A \cap \{\tau \leq t\} \in \mathcal{F}_t$ for all t_k. Hence$(\Omega \setminus A) \cap \{\tau \leq t\} = \{\tau \leq t\} \setminus (A \cap \{\tau \leq t\}) \in \mathcal{F}_t$ (both sets are in $\mathcal{F}_t$). Since t was arbitrary, $\Omega \setminus A \in \mathcal{F}_\tau$.

3. If A_k, $k = 1, 2, \ldots$ belong to $\mathcal{F}_\tau$, then $A_k \cap \{\tau \leq t\} \in \mathcal{F}_t$ for all t. Now

$(\bigcup_{k=1}^{\infty} A_k) \cap \{\tau \le t\} = \bigcup_{k=1}^{\infty}(A_k \cap \{\tau \le t\}) \in \mathcal{F}_t$ for $\mathcal{F}_t$ is a σ-field. Since t was arbitrary, $\bigcup_{k=1}^{\infty} A_k \in \mathcal{F}_\tau$. 1.,2.,3. imply $\mathcal{F}_\tau$ is a σ-field.

Exercise 2.17: Show that if $\nu \le \tau$ then $\mathcal{F}_\nu \subset \mathcal{F}_\tau$, and that $\mathcal{F}_{\nu\wedge\tau} = \mathcal{F}_\nu \cap \mathcal{F}_\tau$.

Solution: If $A \in \mathcal{F}_\nu$ then $A \cap \{\nu \le \tau\} \in \mathcal{F}_t$ for all t. From the assumption $\nu \le \tau$ it follows that $\{\nu \le t\} \supset \{\tau \le t\}$ and hence $\{\tau \le t\} = \{\nu \le t\} \cap \{\tau \le t\}$ for all t. Now $A \cap \{\tau \le t\} = (A \cap \{\nu \le t\}) \cap \{\tau \le t\} \in \mathcal{F}_t$, as τ is a stopping time and $\mathcal{F}_t$ is a σ-field. Thus $A \in \mathcal{F}_\tau$. For the equality $\mathcal{F}_{\nu\wedge\tau} = \mathcal{F}_\nu \cap \mathcal{F}_\tau$, note that by the previous result the relations $\nu\wedge\tau \le \nu$, $\nu\wedge\tau \le \tau$ imply $\mathcal{F}_{\nu\wedge\tau} \subset \mathcal{F}_\nu \cap \mathcal{F}_\tau$. For the reverse inclusion take $A \in \mathcal{F}_\nu \cap \mathcal{F}_\tau$. hence $A \cap \{\nu \le t\} \in \mathcal{F}_t$ and $A \cap \{\tau \le t\} \in \mathcal{F}_t$ for all t. Since $\{\nu \wedge \tau \le t\} = \{\nu \le t\} \cup \{\tau \le t\}$, we have $A \cap \{\nu \wedge \tau \le t\} = (A \cap \{\nu \le t\}) \cup (A \cap \{\tau \le t\}) \in \mathcal{F}_t$ for all t because $\mathcal{F}_t$ is a σ-field. A was an arbitrary set, so $\mathcal{F}_\nu \cap \mathcal{F}_\tau \subset \mathcal{F}_{\nu\wedge\tau}$ and hence the result follows.

Exercise 2.18: Let W be a Wiener process. Show that the natural filtration is left-continuous: for each $t \ge 0$ we have $\mathcal{F}_t = \sigma(\bigcup_{s<t} \mathcal{F}_s)$. Deduce that if $\nu_n \nearrow \nu$, where ν_n, ν are $\mathcal{F}_t^W$-stopping times, then $\sigma(\bigcup_{n\ge1} \mathcal{F}_{\nu_n}^W) = \mathcal{F}_{\nu.}^W$.

Solution: Proof of the first statement:: For any $s > 0$ the σ-field $\mathcal{F}_s^W$ is generated by sets of the form $A = \{(W(u_1), W(u_2), ..., W(u_n)) \in B\}$ where $B \in \mathcal{B}(\mathbb{R}^n)$ and (u_i) is a partition of $[0, s]$. Now fix $t > 0$. By left-continuity of the paths of W we know that $W(t) = \lim_{s_m \uparrow t} W(s_m)$ a.s., the set A belongs to $\sigma(\bigcup_{m=1}^{\infty} \mathcal{F}_{s_m}^W) \subset \sigma(\bigcup_{s<t} \mathcal{F}_s^W)$. So this σ-field contains the generators of $\mathcal{F}_t^W$, hence contains $\mathcal{F}_t^W$. The opposite conclusion is true for any filtration $(\mathcal{F}_t)_{t\ge0}$, since $\mathcal{F}_s \subset \mathcal{F}_t$ for $s < t$ gives $(\bigcup_{s<t} \mathcal{F}_s) \subset \mathcal{F}_t$.

Erratum: The second statement should be deleted. The claim holds for only for quasi-left-continuous filtrations, which involves concepts well beyond the scope of this text. (See Dellacherie-Meyer, Probabilities and Potential, Vol2, Theorem 83, p.217.)

Exercise 2.19: Show that if $X(t)$ is Markov then for any $0 \le t_0 < t_1 < \cdots < t_N \le T$ the sequence $(X(t_n))_{n=0,\ldots,N}$ is a discrete time Markov process.

Solution: We have to verify that the discrete process $(X(t_n))$, $n = 1, \ldots, N$ is a discrete Markov process with respect to the filtration $(\mathcal{F}_{t_n})$, $n = 0, 1, \ldots, N$. Let f be a bounded Borel function $f : \mathbb{R} \to \mathbb{R}$. Since X is a Markov process, it follows that $\mathbb{E}(f(X(t_{n+1}))|\mathcal{F}_{t_n}) = \mathbb{E}(f(X(_{t_n}))|\mathcal{F}_{X(t_n)})$ for all n. But this means that $(X(t_n))_n$ is a Markov chain (a discrete-parameter Markov process).

Exercise 2.20: Let W be a Wiener process. Show that for $x \in \mathbb{R}$, $t \ge 0$,

$M(t) = \exp\{ixW(t) + \frac{1}{2}x^2t\}$ defines a martingale with respect to the natural filtration of W. (Recall from [PF] that expectations of complex-valued random variables are defined via taking the expectations of their real and imaginary parts separately.)

Solution: Recall that $Z : \Omega \to \mathbb{C}$, where $\mathbb{C}$ is the set of complex numbers, is a complex-valued random variable if $Z = X_1 + iX_2$ and X_1, X_2 are real-valued random variables. Z has mean value $\mathbb{E}Z$ if X_1, X_2 have mean values and $\mathbb{E}Z = \mathbb{E}X_1 + i\mathbb{E}X_2$. If $\mathcal{G}$ is a σ-subfield of $\mathcal{F}$ we also have $\mathbb{E}(Z|\mathcal{G}) = \mathbb{E}(X_1|\mathcal{G}) + i\mathbb{E}(X_2|\mathcal{G})$ when $(Z(t))_t$ is a complex valued process (martingale) if X_1, X_2 are real processes (are real martingales for for the same filtration) and $Z(t) = X_1(t) + iX_2(t)$.

Now for Exercise 2.20 take $0 \leq s < t$ and denote $F(u, v) = e^{ix(u+v)+\frac{1}{2}x^2t}$ where $u, v \in \mathbb{R}$. Let $ReF(u, v) = F_1(u, v)$ and $ImF(u, v) = F_2(u, v)$ be the real and imaginary parts of $F(u, v)$. Write $Y = W(t) - W(s)$, $X = W(s)$, $\mathcal{F}_s^W = \mathcal{G}$. We prove that $((M(t))_t$ is a martingale for $(\mathcal{F})_t$. Using our notation we can write

$$\begin{aligned} \mathbb{E}(M(t)|\mathcal{F}_s^W) &= \mathbb{E}(F(W(s), W(t) - W(s))|\mathcal{F}_s^W) \\ &= \mathbb{E}(F_1(X, Y)|\mathcal{G}) + i\mathbb{E}(F_2(X, Y)|\mathcal{G}). \end{aligned}$$

The variable Y is independent of the σ-field $\mathcal{G}$, X is $\mathcal{G}$-measurable and the mappings F_1, F_2 are measurable (continuous) and bounded. So we have: $\mathbb{E}(F_1(X, Y)|\mathcal{G}) = G_1(X)$, $\mathbb{E}(F_2(X, Y)|\mathcal{G}) = G_2(X)$ where $G_1(u) = \mathbb{E}(F_1(u, Y))$, $G_2(u) = \mathbb{E}(F_2(u, Y))$. Setting $G = G_1 + iG_2$ we have the formula $\mathbb{E}(M(t)|\mathcal{F}_s^W) = G(X)$ where $G(u) = \mathbb{E}(F(u, Y)) = \mathbb{E}(e^{ix(u+Y)+\frac{1}{2}x^2t}) = e^{ixu+\frac{1}{2}x^2t}\mathbb{E}(e^{ixY})$. Since the distribution function of Y is the same as that of $W(t - s)$ and $\mathbb{E}(e^{ixW(t-s)})$ is nothing other than the value at x of the characteristic function of an $N(0, t - s)$-distributed random variable, we obtain $\mathbb{E}(e^{ixY}) = e^{-\frac{1}{2}(t-s)x^2}$ [PF]. Hence $G(u) = e^{ixu+sx^2}$. Finally, $\mathbb{E}(M(t)|\mathcal{F}_s^W) = e^{ixW(s)+sx^2} = M(s)$. Since $0 \leq s < t$ were arbitrary $(M(t))_t$ is a martingale.

Chapter 3

Exercise 3.1: Prove that W and W^2 are in $\mathcal{M}^2$.

Solution: W, W^2 are measurable (continuous, adapted) processes. Since $\mathbb{E}(W^2(t)) = t$, $\mathbb{E}(W^4(t)) = 3t^2$ (see[PF]), by the Fubini theorem we obtain

$$\mathbb{E}\left(\int_0^T W^2(t)dt\right) = \int_0^T \mathbb{E}(W^2(t))dt = \int_0^T tdt = \frac{T^2}{2},$$
$$\mathbb{E}\left(\int_0^T W^4(t)dt\right) = \int_0^T \mathbb{E}(W^4(t))dt = \int_0^T 3t^2dt = T^3.$$

Exercise 3.2: Prove that in general $I(f)$ does not depend on a particular representation of f.

Solution: A sequence $0 = t_0 < t_1 < \ldots < t_n = T$ is called a partition of the interval $[0, T]$. We will denote this partition by $\uparrow (t_i)$. A partition $\uparrow (u_k)$ of $[0, T]$ is a refinement of the partition $\uparrow (t_i)$ if the inclusion $\{t_i\} \subset \{u_k\}$ holds. Let f be a simple process on $[0, T]$, $f \in S^2$ and let $\uparrow (t_i)$ be a partition of $[0, T]$ compatible with f. The latter means that f can be written in the form

$$f(t, \omega) = \xi_0(\omega)\mathbf{1}_{\{0\}}(t) + \sum_{i=1}^{n-1} \xi_i(\omega)\mathbf{1}_{(t_i,t_{i+1}]}(t),$$

where ξ_i is $\mathcal{F}_{t_i}$ measurable and $\xi_i \in L^2(\Omega)$. Note that $f(t_{i+1}) = \xi_i$ for $i \geq 0$. To emphasize the presence of the partition in the definition of the integral of f we also write $I(f) = I_{\uparrow(t_i)}(f)$. If a partition $\uparrow (u_k)$ is a refinement of a partition $\uparrow (t_i)$ and $\uparrow (t_i)$ is compatible with f, then $\uparrow (u_k)$ is also compatible with f. This is because for each i there exists $k(i)$ such that $t_i = u_{k(i)}$. Since $f(t) = f(t_{i+1}) = \xi_i$ for $t_i < t \leq t_{i+1}$, it follows that $f(u_k) = f(t_{i+1}) = \xi_i$ and $f(u_k) \in \mathcal{F}_{t_i} \subset \mathcal{F}_{u_k}$ for all k such that $t_i = u_{k(i)} < u_k \leq u_{k(i+1)} = t_{i+1}$. Additionally,

$$\begin{aligned}&\xi_0\mathbf{1}_{\{0\}}(t) + \sum_{k=0}^{p-1} f(u_{k+1})\mathbf{1}_{(u_k,u_{k+1}]}(t)\\ &= \xi_0\mathbf{1}_{\{0\}}(t) + \sum_{i=0}^{n-1}\left(\sum_{k(i)<k\leq k(i+1)} \xi_i\mathbf{1}_{(u_k,u_{k+1}]}(t)\right) = f(t).\end{aligned}$$

Now for the integral of f we have

$$\begin{aligned}
I_{\uparrow(u_k)}(f) &= \sum_{k=0}^{p-1} f(u_{k+1})(W(u_{k+1}) - W(u_k)) \\
&= \sum_{i=0}^{n-1} \left(\sum_{k(i)<k\leq k(i+1)} \xi_i (W(u_{k+1}) - W(u_k)) \right) \\
&= \sum_{i=0}^{n-1} \xi_i (W(t_{i+1}) - W(t_i)) = I_{\uparrow(t_i)}(f).
\end{aligned}$$

Returning to our exercise let $\uparrow (t_i)$ and $\uparrow (s_j)$ be partitions on $[0,T]$ compatible with f. We can construct the partition $\uparrow (v_k)$, where $\{v_k\} = \{t_i\} \cup \{s_j\}$ and elements v_k are ordered as real numbers. Then the partition $\uparrow (v_k)$ is a refinement of both partitions $\uparrow (t_i)$ and $\uparrow (s_j)$. By the previous results we have $I_{\uparrow(t_i)}(f) = I_{\uparrow(v_k)}(f) = I_{\uparrow(s_j)}(f)$. Thus the Itô integral of a simple process is independent of the representation of that process.

Exercise 3.3: Give a proof for the general case (i.e., linearity of the integral for simple functions).

Solution: We prove two implications.
1. If $f \in S^2$, $\alpha \in \mathbb{R}$, then $\alpha f \in S^2$, $I(\alpha f) = \alpha I(f)$.
2. If $f, g \in S^2$, then $f + g \in S^2$, $I(f+g) = I(f) + I(g)$.
Proof 1. If $f(t) = \xi_0 \mathbf{1}_{\{0\}}(t) + \sum_{i=0}^{n-1} \xi_i \mathbf{1}_{(t_i,t_{i+1}]}(t)$, where $\xi_i \in \mathcal{F}_{t_i}$, the $\alpha\xi_i \in \mathcal{F}_{t_i}$ and $I(\alpha f) = \sum_{i=0}^{n-1} \alpha\xi_i (W(t_{i+1}) - W(t_i)) = \alpha \sum_{i=1}^{n-1} \xi_i (W(t_{i+1}) - W(t_i)) = \alpha I(f)$.

Proof 2. We use the notation and the results of Exercise 3.2. Let $\uparrow (t_i)$ and $\uparrow (s_j)$ be partitions (of the interval $[0,T]$) compatible with processes f and g respectively. We can construct a partition $\uparrow (v_k)$ which is a refinement of both $\uparrow (t_i)$ and $\uparrow (s_j)$ (as was shown in Exercise 3.2). Then $f + g \in S^2$ and $I(t) + I(g) = I_{\uparrow(t_i)}(f) + I_{\uparrow(s_j)}(g) = I_{\uparrow(v_k)}(f) + I_{\uparrow(v_k)}(g) = I_{\uparrow(v_k)}(f+g) = I(f+g)$.

Exercise 3.4: Prove that for $\int_a^b f(t)dW(t)$, $[a,b] \subset [0,T]$ we have

$$\mathbb{E}[\int_a^b f(t)dW(t)] = 0, \ \mathbb{E}[(\int_a^b f(t)dW(t))^2] = \mathbb{E}[\int_a^b f^2(t)dt].$$

Solution: Since for $f \in S^2$ the process $\mathbf{1}_{[}a,b]\cdot f$ belongs to S^2, it follows that $\mathbb{E}(\int_a^b f\,dW) = \mathbb{E}(\int_0^T \mathbf{1}_{[a,b]} f\,dW)$ (definition) $= 0$ (Theorem 3.9). Similarly $\mathbb{E}(\int_a^b f\,dW)^2 = \mathbb{E}(\int_0^T \mathbf{1}_{[a,b]} f\,dW)^2$ (definition)$= \mathbb{E}(\int_0^T \mathbf{1}_{[a,b]} f^2 dt)$(Theorem 3.10) $= \mathbb{E}(\int_a^b f^2 dt)$.

Exercise 3.5: Prove that the stochastic integral does not depend on the choice of the sequence f_n approximating f.

Solution: Assume $f \in M^2$ and let (f_n) and (g_n) be two sequences approximating f. That is $f_n, g_n \in S^2$ for all n and $\mathbb{E}(\int_0^T (f_n - f)^2 dt) \xrightarrow[n\to\infty]{} 0$, $\mathbb{E}(\int_0^T (f - g_n)^2 dt) \xrightarrow[n\to\infty]{} 0$. The last two relations imply, by the inequality $(a+b)^2 \le 2a^2 + 2b^2$,

$$\mathbb{E}\left(\int_0^T (f_n - g_n)^2 dt\right) \le 2\mathbb{E}\left(\int_0^T (f_n - f)^2 dt\right) \\ +2\mathbb{E}\left(\int_0^T (f - g_n)^2 dt\right) \xrightarrow[n\to\infty]{} 0.$$

By assumption, the integrals $I(g_n), I(f_n)$ exist and there exist $\lim_{n\to\infty} I(f_n)$ and $\lim_{n\to\infty} I(g_n)$ in $L^2(\Omega)$.

We want to prove that $\lim_{n\to\infty} I(f_n) = \lim_{n\to\infty} I(g_n)$. Now

$$\mathbb{E}((I(f_n) - I(g_n))^2) = \mathbb{E}\left(\left(\int_0^T (f_n(t) - g_n(t)t) dW(t)\right)^2\right)$$

$$\text{(linearity in } S^2)$$

$$= \mathbb{E}\left(\int_0^T (f_n(t) - g_n(t))^2 dt\right) \text{(isometry in } S^2) \to 0 \text{ as } n \to \infty.$$

That last convergence was shown above. Thus $\lim_{n\to\infty} I(f_n) = \lim_{n\to\infty} I(g_n)$ in $L^2(\Omega)$-norm.

Exercise 3.6: Show that

$$\int_0^t s dW(s) = tW(t) - \int_0^t W(s) ds.$$

Solution: We have to choose an approximating sequence for our integrals and next to calculate the limit of Itô integrals for the approximating sequence. Denote $f(s) = s$ and take $f_n(s) = \sum_{i=0}^{n-1} \mathbf{1}_{(\frac{it}{n}, \frac{(i+1)t}{n}]}(s)\frac{it}{n}$ for $0 < s \le t$, $f_n(0) = 0$. Then $f_n$are simple functions and $f_n \in S^2$ (they do not depend on ω). (f_n) is an approximating sequence for f because $|f_n(s) - f(s)| \le \frac{t}{n}$ for all $0 \le s \le t$. This inequality gives $\mathbb{E}(\int_0^t (f(s) - f_n(s))^2 ds) \le \frac{t^2}{n^2} t \to 0$ as $n \to \infty$. Now we calculate $\lim_{n\to\infty} I(f_n)$. According to the definition of the integral of a simple function

$$I(f_n) = \sum_{i=0}^{n-1} \frac{it}{n}\left(W\left(\frac{(i+1)t}{n}\right) - W\left(\frac{it}{n}\right)\right)$$
$$= tW(t) - \sum_{i=1}^{n-1} \frac{t}{n} W\left(\frac{it}{n}\right) \to tW(t) + \int_0^t W(s)ds \ \ a.e.$$

This holds because for almost all ω $W(\cdot, \omega)$ is continuous function and $\sum_{i=1}^{n-1} \frac{t}{n} W(\frac{it}{n})$ is a Riemann approximating sum for the integral of $W(\cdot, \omega)$. Convergence with probability one is not sufficient. We need the convergence in $L^2(\Omega)$ norm. We verify the Cauchy condition to prove $L^2(\Omega)$-norm convergence of $(I(f_n))$:

$$\mathbb{E}((I(f_n) - I(f_m))^2) = \mathbb{E}((I(f_n - f_m))^2) \text{ (linearity in } S^2)$$
$$= \mathbb{E}\left(\left(\int_0^t (f_n(s) - f_m(s))dW(s)\right)^2\right) = \mathbb{E}\left(\int_0^t (f_n(s) - f_m(s))^2 ds\right)$$
$$\text{(Itô isometry)} .$$

Since $f_n \to f$ in $L^2([0, T] \times \Omega)$ norm, it satisfies the Cauchy condition. The Itô isometry guarantees that the sequence of integrals also satisfies the Cauchy condition in $L^2(\Omega)$. Then $(I(f_n))_n$ converges in $L^2(\Omega)$-norm. But the limits of a sequence convergent with probability one and at the same time convergent in $L^2(\Omega)$-norm must be the same. Thus

$$I(f)(s) = (\lim_{n\to\infty} I(f_n))(s) = tW(t) - \int_0^t W(s)ds.$$

Exercise 3.7: Compute the variance of the random variable $\int_0^T (W(t) - t)dW(t)$.

Solution: In order to calculate the variance of a random variable we need its mean value. Denote by f_n an approximating sequence for the process $f(t) = W(t) - t$. From the definition of the Itô integral $I(f_n) \to I(f)$ in $L^2(\Omega)$-norm. We have the inequalities

$$|\mathbb{E}(I(f)) - \mathbb{E}(I(f_n))| \le \mathbb{E}(|I(f) - I(f_n)|) \le \text{ (Schwarz inequality)}$$
$$\mathbb{E}((I(f) - I(f_n))^2) \to 0 \text{ as } n \to \infty.$$

Since $\mathbb{E}(I(f_n)) = 0$ for all n (Theorem 3.9), it follows that $\mathbb{E}(I(f)) = 0$. Now we calculate

$$Var(I(f)) = \mathbb{E}((I(f))^2)\ (\mathbb{E}(I(f)) = 0) = \mathbb{E}\left(\int_0^T f^2(t)dt\right)$$

$$(\text{Itô isometry}) = \int_0^T \mathbb{E}(W(t) - t)^2 dt = \int_0^T (\mathbb{E}(W^2(t)) + t^2)dt$$

$$= \int_0^T (t + t^2)dt = \frac{T^2}{2} + \frac{T^3}{3}.$$

Exercise 3.8: Prove that if $f, g \in \mathcal{M}^2$ and α, β are real numbers, then

$$I(\alpha f + \beta g) = \alpha I(f) + \beta I(g).$$

Solution: Let (f_n) and (g_n), $f_n, g_n \in S^2$ be approximating sequences for f and g, respectively and fix $\alpha, \beta \in \mathbb{R}$. This gives $\alpha f_n \to \alpha f$, $\beta g_n \to \beta g$ in $L^2([0,T] \times \Omega)$-norm. Hence the sum of these sequences $(\alpha f_n + \beta g_n)$ converges in $L^2([0,T] \times \Omega)$-norm to $\alpha f + \beta g$ and of course $\alpha f_n + \beta g_n \in S^2$. So the sequence $(\alpha f_n + \beta g_n)$ is an approximating sequence for the process $\alpha f + \beta g$. From the definition of integral follow the relations

$$I(\alpha f_n + \beta g_n) \to I(\alpha f + \beta g)$$

and

$$I(\alpha f_n + \beta g_n) = \ (\text{linearity in } S^2)$$
$$\alpha I(f_n) + \beta I(g_n) \to \alpha I(f) + \beta I(g)$$

in $L^2(\Omega)$-norm. Since the limit of a sequence in a normed vector space is determined explicitly, it follows that

$$\alpha I(f) + \beta I(g) = I(\alpha f + \beta g).$$

Exercise 3.9: Prove that for $f \in \mathcal{M}^2$, $a < c < b$,

$$\int_a^c f(s)dW(s) = \int_a^b f(s)dW(s) + \int_b^c f(s)dW(s).$$

Solution: Let $a < c < b$. Then

$$\int_a^c f(s)dW(s) + \int_c^b f(s)dW(s) = \int_0^T f(s)\mathbf{1}_{[a,c]}(s)dW(s)$$
$$+ \int_0^T f(s)\mathbf{1}_{[c,b]}(s)dW(s) = \int_0^T (f(s)\mathbf{1}_{[a,c]}(s) + f(s)\mathbf{1}_{[c,b]}(s))dW(s)$$
$$(\text{linearity}) = \int_0^T (f(s)\mathbf{1}_{[a,c]}(s) + f(s)\mathbf{1}_{(c,b]}(s))dW(s)$$
$$= \int_0^T f(s)\mathbf{1}_{[a,b]}(s)dW(s) = \int_a^b f(s)dW(s).$$

Note that a change of value of an integrand at one point has no influence on the value of the integral. For example, let (f_n) be an approximating sequence for $f \in M^2$ and let $0 = t_0^{(n)} < t_1^{(n)} < \ldots < t_n^{(n)} = T$ be the partition for f_n. Then

$$\mathbb{E}((I(f_n) - I(f_n\mathbf{1}_{(0,T]})^2) = \mathbb{E}((f(0)(W(t_1^{(n)}) - W(0)))^2)$$
$$= \mathbb{E}((f^2(0)\mathbb{E}(W^2(t_1^{(n)})|\mathcal{F}_0)) = \mathbb{E}(f^2(0))\mathbb{E}(W^2(t_1))$$
$$(W^2(t_1) \text{ independent of } \mathcal{F}_0) = \mathbb{E}(f^2(0))t_1^{(n)} \xrightarrow[n\to\infty]{} 0.$$

Exercise 3.10: Show that the process $M(t) = \int_0^t \sin(W(t))dW(t)$ is a martingale.

Solution: Let be $0 < s < t$. We have to prove the equality

$$\mathbb{E}\left(\int_0^t \sin(W(u))dW(u)|\mathcal{F}_s\right) = \int_0^s \sin(W(u))dW(u).$$

Beginning with the equality

$$\int_0^t \sin(W(u))dW(u)) = \int_0^s \sin(W(u))dW(u)$$
$$+ \int_s^t \sin(W(u))dW(u) = \eta + \xi$$

We see that to solve the problem it is enough to show that $\mathbb{E}(\eta|\mathcal{F}_s) = \eta$ and $\mathbb{E}(\xi|\mathcal{F}_s) = 0$. The first equality means that η should be an $\mathcal{F}_s$ measurable random variable. To prove it let $(f_n)_n$, $f_n \in S^2$ be an approximating sequence for the process W on the interval $[0, s]$. Then $(\sin(f_n))_n$ is an approximating sequence for $\sin(W)$ and of course $\sin(f_n) \in S^2$. The integrals $\eta_n = I(\sin(f_n))$ converge in $L^2(\Omega)$ norm to η and they are of the form $\eta_n = \sum_{i=1}^{m^{(n)}-1} \eta_i^{(n)}(W(t_{i+1}^{(n)}) - W(t_i^{(n)}))$ where $t_i^{(n)} \le s$ and $\eta_i^{(n)} \in \mathcal{F}_{t_i^{(n)}} \subset \mathcal{F}_s$. Then η_n are $\mathcal{F}_s$-measurable variables and as a consequence η must be an

$\mathcal{F}_s$ measurable variable. For the equality $\mathbb{E}(\xi|\mathcal{F}_s) = 0$, let $(g_n)_n$, $g_n \in S^2$, $g_n = \mathbf{1}_{[s,t]} g_n$ be an approximating sequence for the process W on the interval $[s,t]$. Then similarly as in the previous case $\xi_n = I(\mathbf{1}_{[s,t]} \sin(g_n))$ are of the form $\xi_n = \sum_{j=1}^{l^{(n)}-1} \xi_j^{(n)}(W(s_{j+1}^{(n)}) - W(s_j^{(n)}))$ where $\xi_j^{(n)} \in \mathcal{F}_{s_j^{(n)}}$ and $s \leq s_j^{(n)} \leq t$ for all j and n. These conditions and the definition of Wiener process W adapted to the filtration $(\mathcal{F}_n)$ give the variable $W(s_{j+1}^{(n)}) - W(s_j^{(n)})$ is independent of the σ-field $\mathcal{F}_{s_j^{(n)}}$ for all $s_j^{(n)}$ and n. This property and the fact that $\xi_j^{(n)} \in \mathcal{F}_{s_j^{(n)}}$ implies that

$$\mathbb{E}(\xi_n|\mathcal{F}_s) = \sum_{j=1}^{l^{(n)}-1} \mathbb{E}(\xi_j^{(n)}(W(s_{j+1}^{(n)}) - W(s_j^{(n)}))|\mathcal{F}_s)$$

and further

$$\begin{aligned}
&\mathbb{E}(\xi_j^{(n)}(W(s_{j+1}^{(n)}) - W(s_j^{(n)}))|\mathcal{F}_s) \\
&= \mathbb{E}(\mathbb{E}(\xi_j^{(n)}(W(s_{j+1}^{(n)}) - W(s_j^{(n)}))|\mathcal{F}_{s_j^{(n)}})|\mathcal{F}_s) \text{ (tower property)} \\
&= \mathbb{E}(\xi_j^{(n)}\mathbb{E}(W(s_{j+1}^{(n)}) - W(s_j^{(n)})|\mathcal{F}_{s_j^{(n)}})|\mathcal{F}_s) \\
&= \mathbb{E}(\xi_j^{(n)}\mathbb{E}(W(s_{j+1}^{(n)}) - W(s_j^{(n)}))|\mathcal{F}_s) \text{ (independence)} \\
&= \mathbb{E}(W(s_{j+1}^{(n)}) - W(s_j^{(n)}))\mathbb{E}(\xi_j^{(n)}|\mathcal{F}_s) = 0
\end{aligned}$$

for all j, n because $\mathbb{E}(W(s_{j+1}^{(n)}) - W(s_j^{(n)})) = 0$. Thus $\mathbb{E}(\xi_n|\mathcal{F}_s) = 0$ for all n. The convergence $\xi_n \to \xi$ in $L^2(\Omega)$-norm implies $\mathbb{E}(\xi_n|\mathcal{F}_s) \to \mathbb{E}(\xi|\mathcal{F}_s)$ in $L^2(\Omega)$ (see [PF]). The last result $\mathbb{E}((\mathbb{E}(\xi|\mathcal{F}_s))^2) = 0$ gives $\mathbb{E}(\xi|\mathcal{F}_s) = 0$ almost everywhere. The proof is completed.

Exercise 3.11: For each t in $[0,T]$ compare the mean and variance of the Itô integral $\int_0^T W(s)dW(s)$ with those of the random variable $\frac{1}{2}(W(T)^2 - T)$.

Solution: We have $\mathbb{E}(\int_0^T W(s)dW(s)) = 0$ (Theorem 3.14). As a consequence

$$\begin{aligned}
&Var\left(\int_0^T W(s)dW(s)\right) = \mathbb{E}\left(\left(\int_0^T W(s)dW(s)\right)^2\right) \\
&\mathbb{E}\left(\int_0^T W^2(s)ds\right) \text{ (isometry) } = \int_0^T \mathbb{E}(W^2(s))ds = \int_0^T s ds = \frac{T^2}{2}.
\end{aligned}$$

For the second random variable we obtain

$$\mathbb{E}\left(\frac{1}{2}(W^2(T) - T)\right) = \frac{1}{2}\left(\mathbb{E}(W^2(T)) - T\right) = \frac{1}{2}(T - T) = 0$$

and so

$$Var\left(\frac{1}{2}(W^2(T)-T)\right)=\frac{1}{4}Var\left(W^2(T)\right)=\frac{1}{4}\left(\mathbb{E}((W^2(T))^2)-(\mathbb{E}(W^2(T)))^2\right)$$
$$=\frac{1}{4}\left(\mathbb{E}(W^4(T))-T^2\right)=\frac{1}{4}\left(3T^2-T^2\right)=\frac{T^2}{2}.$$

Exercise 3.12: Use the identity $2a(b-a)=(b^2-a^2)-(b-a)^2$ and appropriate approximating partitions to show from first principles that $\int_0^T W(s)dW(s)=\frac{1}{2}(W(T)^2-T)$.

Solution: Since the process W belongs to M^2, we know that the integral of W exists and it is enough to calculate the limit of integrals for an approximating W sequence $(f_n)_n$. We take f_n, $n=1,\ldots$ given by the partitions $t_i^{(n)}=\frac{iT}{n}$, $i=0,1,\ldots,n-1$. So we have $f_n(t)=\sum_{i=0}^{n-1}W(t_i^{(n)})\mathbf{1}_{(t_i^{(n)},t_{i+1}^{(n)}]}(t)$, $f_n(0)=0$, $n=1,2,\ldots$. It is easy to verify that $f_n\to W$ in $L^2([0,T]\times\Omega)$-norm. Using our hypothesis about $\lim_{n\to\infty}I(f_n)$ we see that it is necessary to prove that $I(f_n)=\sum_{i=0}^{n-1}W(t_i^{(n)})(W(t_{i+1}^{(n)})-W(t_i^{(n)}))\to\frac{1}{2}(W^2(T)-T)$ in $L^2(\Omega)$-norm. The identity $2a(b-a)=(b^2-a^2)-(b-a)^2$ lets us write the Itô sum of $I(f_n)$ as follows

$$I(f_n)=\frac{1}{2}\sum_{i=0}^{n-1}(W^2(t_{i+1}^{(n)})-W^2(t_i^{(n)}))-\frac{1}{2}\sum_{i=1}^{n-1}(W(t_{i+1}^{(n)})-W(t_i^{(n)}))^2$$
$$=\frac{1}{2}W^2(T)-\frac{1}{2}\eta_n$$

where $\eta_n=\sum_{i=0}^{n-1}(W(t_{i+1}^{(n)})-W(t_i^{(n)}))^2$. Then it is sufficient to show that $\mathbb{E}(\eta_n-T)^2\to 0$. Since $\mathbb{E}((W(t_{i+1}^{(n)})-W(t_i^{(n)}))^2)=\mathbb{E}(W^2(t_{i+1}^{(n)}-t_i^{(n)}))$ (the same distributions) $=\mathbb{E}(W^2(\frac{T}{n}))=\frac{T}{n}$, we obtain $\mathbb{E}(\eta_n)=T$. Hence

$$\mathbb{E}(\eta_n-T)^2=Var(\eta_n)=\sum_{i=0}^{n-1}Var((W(t_{i+1}^{(n)})-W(t_i^{(n)}))^2)\text{ (independence)}$$
$$=\sum_{i=1}^{n-1}Var(W^2(t_{i+1}^{(n)}-t_i^{(n)}))\text{ (the same distributions) }=\sum_{i=0}^{n-1}Var\left(W^2\left(\frac{T}{n}\right)\right)$$
$$=n\left(\mathbb{E}\left(W^4\left(\frac{T}{n}\right)\right)-\left(\mathbb{E}\left(W^2\left(\frac{T}{n}\right)\right)\right)^2\right)=n\left(3\left(\frac{T}{n}\right)^2-\left(\frac{T}{n}\right)^2\right)$$
$$=\frac{2T^2}{n}\to 0$$

as $n\to\infty$. The proof is completed.

Exercise 3.13: Give a direct proof of the conditional Itô isometry (Theorem 3.20) : if $f \in M^2$, $[a,b] \subset [0,T]$, then

$$\mathbb{E}([\int_a^b f(s)dW(s)]^2|\mathcal{F}_a) = \mathbb{E}(\int_a^b f^2(s)ds|\mathcal{F}_a)$$

following the method used for proving the unconditional Itô isometry.

Solution: (Conditional Itô isometry.) The proof has two steps. In step 1. we prove the theorem for $f \in S^2$. In this case, let $a = t_0 < t_1 < \ldots < t_n = b$ be a partition of $[a,b]$ and let f be of the form $f(t) = \xi_0 \mathbf{1}_{\{a\}}(t) + \sum_{k=0}^{n-1} \xi_k \mathbf{1}_{(t_k,t_{k+1}]}(t)$, where, for $k < n$, ξ_k is an $\mathcal{F}_{t_k}$-measurable variable. Then we can calculate similarly as in Theorem 3.10

$$\mathbb{E}\left(\left[\int_a^b f(s)dW(s)\right]^2 |\mathcal{F}_a\right) = \mathbb{E}\left(\left[\sum_{k=0}^{n-1} \xi_k(W(t_{k+1}) - W(t_k))\right]^2 |\mathcal{F}_a\right)$$
$$= \sum_{k=0}^{n-1} \mathbb{E}([\xi_k(W(t_{k+1}) - W(t_k))]^2|\mathcal{F}_a)$$
$$+2\sum_{i<k} \mathbb{E}([\xi_i\xi_k(W(t_{i+1}) - W(t_i))(W(t_{k+1}) - W(t_k))]|\mathcal{F}_a) = A + 2B.$$

Consider A.We have

$$\begin{aligned}
& \mathbb{E}(\xi_k^2(W(t_{k+1}) - W(t_k))^2)|\mathcal{F}_a) \\
&= \mathbb{E}(\mathbb{E}(\xi_k^2(W(t_{k+1}) - W(t_k))^2|\mathcal{F}_{t_k})|\mathcal{F}_a) \text{ (tower property)} \\
&= \mathbb{E}(\xi_k^2(\mathbb{E}[(W(t_{k+1}) - W(t_k))^2|\mathcal{F}_{t_k}]|\mathcal{F}_a) \text{ (ξ_k^2 is $\mathcal{F}_{t_k}$-measurable)} \\
&= \mathbb{E}(\xi_k^2(\mathbb{E}[W(t_{k+1}) - W(t_k)]^2)|\mathcal{F}_a) \text{ (independence)} \\
&= \mathbb{E}((W(t_{k+1} - t_k))^2)\mathbb{E}(\xi_a^2|\mathcal{F}_a) \text{ (linearity, the same distribution)} \\
&= (t_{k+1} - t_k)\mathbb{E}(\xi_k^2|\mathcal{F}_a) = \mathbb{E}(\xi_k^2(t_{k+1} - t_k)|\mathcal{F}_a).
\end{aligned}$$

This proves that

$$A = \sum_{k=0}^{n-1} \mathbb{E}(\xi_k^2(t_{k+1} - t_k)|\mathcal{F}_a) = \mathbb{E}(\sum_{k=0}^{n-1} \xi_k^2(t_{k+1} - t_k)|\mathcal{F}_a)$$
$$= \mathbb{E}\left(\int_a^b f^2(t)dt|\mathcal{F}_a\right).$$

Now for B we have

$$\begin{aligned}&\mathbb{E}(\xi_i\xi_k(W(t_{i+1})-W(t_i))(W(t_{k+1})-W(t_k))|\mathcal{F}_a)\\ =\ &\mathbb{E}(\mathbb{E}[\xi_i\xi_k(W(t_{i+1})-W(t_i))(W(t_{k+1})-W(t_k))|\mathcal{F}_{t_k}]|\mathcal{F}_a)\\ &\text{(tower property)} \ = \mathbb{E}(\xi_i\xi_k(W(t_{i+1})-W(t_i))\\ &\cdot\mathbb{E}(W(t_{k+1})-W(t_k)|\mathcal{F}_{t_k})|\mathcal{F}_a) \text{ (terms for } i<k \text{ are } \mathcal{F}_{t_k}\text{-measurable)} \ = 0\end{aligned}$$

because $W(t_{k+1})-W(t_k)$ is independent of the σ-field $\mathcal{F}_k$, hence $\mathbb{E}(W(t_{k+1})-W(t_k)|\mathcal{F}_{t_k}) = \mathbb{E}(W(t_{k+1})-W(t_k)) = 0$. Hence also $B = 0$.
Step 2. The general case. Let be $f \in M^2$ and let (f_n) be a sequence of approximating processes for f on $[a,b]$. Then $f_n \in S^2$, $n = 1,2,\ldots$ and $\|f - f_n\|_{L^2([a,b]\times\Omega)} \longmapsto 0$ as $n \to \infty$. The last condition implies $\|I(\mathbf{1}_{[a,b]} f_n)\|_{L^2(\Omega)} \to 0$.

Now we will want to utilize the conditional isometry for f_n and to take the limit as $n\to\infty$. This needs the following general observation.
Observation. Let (Z_n) be a sequence of random variables on a probability space $(\Omega',\mathcal{F}',P')$ and let ζ be a sub σ-field of $\mathcal{F}'$. If $Z_n \in L^1(\Omega')$, $n = 1,2,\ldots$ and $Z_n \to Z$ in $L^1(\Omega)$-norm, then $\mathbb{E}(Z_n|\zeta) \to \mathbb{E}(Z|\zeta)$ in $L^1(\Omega)$-norm.

Proof. First note that $\mathbb{E}(Z_n|\zeta)$, $\mathbb{E}(Z|\zeta)$ belong to $L^1(\Omega)$.

Now we have

$$\begin{aligned}&\mathbb{E}(|\mathbb{E}(Z_n|\zeta)-\mathbb{E}(Z|\zeta)|) = \mathbb{E}(|\mathbb{E}(Z_n - Z|\zeta)|)\\ &\le \mathbb{E}(\mathbb{E}(|Z_n - Z||\zeta)) = \mathbb{E}(|Z_n - Z|) \to 0 \text{ as } n\to\infty\ .\end{aligned}$$

To use our Observation we have to verify that $[I(\mathbf{1}_{[a,b]} f_n)]^2 \to [I(\mathbf{1}_{[a,b]} f)]^2$ and $\int_0^T (\mathbf{1}_{[a,b]} f_n)^2 ds \to \int_0^T (\mathbf{1}_{[a,b]} f)^2 ds$ in $L^1(\Omega)$-norm. The following relations hold

$$\begin{aligned}&\mathbb{E}(|[I(\mathbf{1}_{[a,b]} f_n)]^2 - [I(\mathbf{1}_{[a,b]} f)]^2|) = \mathbb{E}(|I(\mathbf{1}_{[a,b]}(f_n - f))||I(\mathbf{1}_{[a,b]}(f_n + f))|)\\ &\le (\mathbb{E}([I(\mathbf{1}_{[a,b]}(f_n - f))]^2))^{\frac12}(\mathbb{E}([I(\mathbf{1}_{[a,b]}(f_n + f))]^2))^{\frac12} \text{ (Schwarz inequality)}\\ &= (\mathbb{E}(\int_0^T \mathbf{1}_{[a,b]}(f_n - f)^2 ds))^{\frac12}(\mathbb{E}(\int_0^T \mathbf{1}_{[a,b]}(f_n + f)^2 ds))^{\frac12} \text{ (isometry)}\\ &= \|f_n - f\|_{L^2([a,b]\times\Omega)}\|f_n + f\|_{L^2([a,b]\times\Omega)} \to 0\end{aligned}$$

as $n \to \infty$ because the second sequence is bounded. For the second sequence we have similarly

$$\mathbb{E}|\int_0^T \mathbf{1}_{[a,b]}(f_n^2 - f^2)ds| \leq \mathbb{E}(\int_0^T |\mathbf{1}_{[a,b]}(f_n - f)||\mathbf{1}_{[a,b]}(f_n + f)|ds)$$
$$\leq (\mathbb{E}(\int_0^T \mathbf{1}_{[a,b]}(f_n - f)^2 ds))^{\frac{1}{2}}(\mathbb{E}(\int_0^T \mathbf{1}_{[a,b]}(f_n + f)^2 ds))^{\frac{1}{2}}$$
$$= \|f_n - f\|_{L^2([a,b]\times\Omega)}\|f_n + f\|_{L^2([a,b]\times\Omega)} \to 0 \text{ as } n \to \infty.$$

Thus we have by our Observation that

$$\mathbb{E}([I(\mathbf{1}_{[a,b]} f_n)]^2|\mathcal{F}_a) \to \mathbb{E}([I(\mathbf{1}_{[a,b]} f)]^2|\mathcal{F}_a)$$

and

$$\mathbb{E}(\int_0^T \mathbf{1}_{[a,b]} f_n^2 ds|\mathcal{F}_a) \to \mathbb{E}(\int_0^T \mathbf{1}_{[a,b]} f^2 ds|\mathcal{F}_a)$$

in $L^1(\Omega)$ norm. Hence and from the equality

$$\mathbb{E}([I(\mathbf{1}_{[a,b]} f_n)]^2|\mathcal{F}_a) = \mathbb{E}(\int_0^T \mathbf{1}_{[a,b]} f_n^2 ds|\mathcal{F}_a)$$

valid for all $f_n \in S^2$, we can obtain the final result.

Exercise 3.14: Show $\int_0^t g(s)ds = 0$ for all $t \in [0, T]$ implies $g = 0$ almost surely on $[0, T]$.

Solution: Denote by g^+ and g^- the positive and the negative parts of g. Then $g^+ \geq 0$, $g^- \geq 0$ and $g^+ - g^- = g$. The assumption about g implies $\int_a^b g^+(s) = \int_a^b g^-(s) = 0$ for all intervals $[a, b] \subset [0, T]$. Write $\nu^+(A) = \int_A g^+(s)ds$, $\nu^-(A) = \int_A g^-(s)ds$ for $A \in \mathcal{B}([0, T])$. Of course ν^- and ν^+ are measures on $\mathcal{B}([0, T])$ and the properties of g^+ and g^- give $\nu^+([a, b]) = \nu^-([a, b])$ for every interval $[a, b]$. Since intervals generate the σ-field $\mathcal{B}([0, T])$, we must have $\nu^+(A) = \nu^-(A)$ for all $A \in \mathcal{B}([0, T])$. Suppose now that the Lebesbue measure of the set $\{x : g(x) \neq 0\} = \{x : g^+(x) \neq g^-(x)\}$ is positive. Then the measure of the set $B = \{x : g^+(x) > g^-(x)\}$ (or the set $\{x : g^+(x) < g^-(x)\}$) must be also positive. But this conjecture leads to the conclusion

$$\nu(B) = \nu^+(B) - \nu^-(B) = \int_B (g^+ - g^-)ds > 0$$

which contradicts the assumption $\nu(A) = 0$ for all $A \in \mathcal{B}([0, T])$.

Chapter 4

Exercise 4.1: Show that for cross-terms all we need is the fact that $W^2(t)-t$ is a martingale.

Solution: We need to calculate $H = \mathbb{E}(F''(W(t_i))F''(W(t_j))X_iX_j)$ for $i < j$, where $X_k = [W(t_{k+1}) - W(t_k]^2 - [t_{k+1} - t_k]$. As in the proof of Theorem 4.5, (Hurdle 1-Cross terms) we have $H = \mathbb{E}(F''(W(t_i))F''(W(t_j))X_i\mathbb{E}(X_j|\mathcal{F}_{t_j}))$. So we calculate $\mathbb{E}(X_j|\mathcal{F}_{t_j})$. It is possible to write it in the form

$$\begin{aligned} \mathbb{E}(X_j|\mathcal{F}_j) &= \mathbb{E}([W^2(t_{j+1}) - t_{j+1}]|\mathcal{F}_{t_j}) - 2W(t_j)\mathbb{E}(W(t_{j+1})|\mathcal{F}_{t_j}) \\ &\quad + W^2(t_j) + t_j \ (W(t_j) \text{ is } \mathcal{F}_{t_j}\text{-measurable}) \\ &= W^2(t_j) - t_j - 2W^2(t_j) + W^2(t_j) + t_j = 0 \\ &\quad (W^2(t) - t, W(t) \text{ are martingales}). \end{aligned}$$

Hence also $H = 0$.

Exercise 4.2: Verify the convergence claimed in (4.3), using the fact that quadratic variation of W is t.

Solution: Actually we have to repeat the calculus done for the quadratic variation of W. As in the previous exercise write $X_i = (W(t_{i+1}) - W(t_i))^2 - (t_{i+1} - t_i)$, $i = 1, \ldots, n-1$. Since $\mathbb{E}(X_i) = 0$ and hence $\mathbb{E}(\sum_{i=0}^{n-1} X_i) = 0$, and since X_i, $i = 1, \ldots, n-1$ are independent random variables, we have

$$\begin{aligned} &\sum_{i=0}^{n-1} \mathbb{E}(\{(W(t_{i+1}) - W(t_i))^2 - (t_{i+1} - t_i)\}^2) = \sum_{i=0}^{n-1} \mathbb{E}X_i^2 \\ &= \sum_{i=0}^{n-1} Var(X_i) \ (\mathbb{E}(X_i) = 0) \ = Var(\sum_{i=0}^{n-1} X_i) \ (X_i \text{ are independent}) \\ &= \mathbb{E}((\sum_{i=0}^{n-1} X_i)^2) \ (\mathbb{E}(\sum_{i=0}^{n-1} X_i) = 0) = \mathbb{E}[((\sum_{i=0}^{n-1}(W(t_{i+1}) - W(t_i))^2) - t)] \\ &= \mathbb{E}((V^2_{[0,t]}(n) - t)^2) \to \mathbb{E}(([W, W](t) - t)^2) = 0 \end{aligned}$$

in $L^2(\Omega)$-norm, independently of the sequence of partitions with mesh going to 0 as $n \to \infty$. (See Proposition 2.2.)

Exercise 4.3: Prove that

$$\tau_M = \inf\{t : \int_0^t |f(s)|ds \geq M\}$$

is a stopping time.

Solution: The process $t \mapsto \int_0^t |f(s)|ds$ has continuous paths and we can apply the argument given at the beginning of the section provided it is adapted. For this we have to assume that the process $f(t)$ is adapted and notice that the integral $\int_0^t |f(s)|ds$, computed pathwise, is the limit of approximating sums. These sums are $\mathcal{F}_t$-measurable and measurability is preserved in the limit.

Exercise 4.4: Find a process that is in $\mathcal{P}^2$ but not in $\mathcal{M}^2$.

Solution: If a process has continuous paths, it is in $\mathcal{P}^2$ since the integral over any finite time interval of a continuous function is finite. We need an example for which the expectation $\mathbb{E}\int_0^T f^2(s)ds$ is infinite. Fubini's theorem implies that it is sufficient to find f such that $\int_0^T \mathbb{E}(f^2(s))ds$ is infinite. Going for a simple example, let $\Omega = [0,1]$ and $T = 1$. The goal will be achieved if $\mathbb{E}(f^2(s)) = \frac{1}{s}$. Now $\mathbb{E}(f^2(s)) = \int_0^1 f^2(s,\omega)d\omega$ so we need a random variable, i.e. a Borel function $X : [0,1] \to \mathbb{R}$ such that $\int_0^1 X(\omega)d\omega = \frac{1}{s}$. Clearly, $X(\omega) = \frac{1}{s^2}\mathbf{1}_{[0,s]}(\omega)$ does the trick, so $f(s,\omega) = \frac{1}{s}\mathbf{1}_{[0,s]}(\omega)$ is the example we are looking for.

Exercise 4.5: Show that the Itô process $dX(t) = a(t)dt + b(t)dW(t)$ has quadratic variation $[X,X](t) = \int_0^t b^2(s)ds$.

Solution: Under the additional assumption that $\int_0^T b(s)dW(s)$ is bounded the result is given by Theorem 3.26. For the general case let $\tau_n = \min\{t : \int_0^t b(s)dW(s) \geq n\}$, so that, writing $M(t) = \int_0^t b(s)dW(s)$, the stopped process $M^{\tau_n}(t)$ is bounded (by n). Since $M^{\tau_n}(t) = \int_0^T \mathbf{1}_{[0,\tau_n]}b(s)dW(s)$, $[X^{t_n}, X^{\tau_n}](t) = \int_0^t \mathbf{1}_{[0,\tau_n]}b^2(s)ds \to \int_0^t b^2(s)ds$ almost surely, because τ_n is localising.

Exercise 4.6: Show that the characteristics of an Itô process are uniquely defined by the process, i.e. prove that $X = Y$ implies $a_X = a_Y$, $b_X = b_Y$, by applying the Itô formula to find the form of $(X(t) - Y(t))^2$.

Solution: Let $Z(t) = X(t) - Y(t)$ and by the Itô formula $dZ^2(t) = 2Z(t)a_Z(t)dt + 2Z(t)b_Z(t)dW(t) + b_Z^2(t)dt$ with $a_Z = a_X - a_Y$, $b_Z = b_X - b_Y$. But $Z(t) = 0$ hence $\int_0^t b_Z^2(s)ds = 0$, all t, so $b_Z = 0$. This implies $\int_0^t a_Z(s)ds = 0$ for all t hence $a_Z(t) = 0$ as well.

Exercise 4.7: Suppose that the Itô process $dX(t) = a(t)dt + b(t)dW(t)$ is positive for all t and find the characteristic of the processes $Y(t) = 1/X(t)$, $Z(t) = \ln X(t)$.

Solution: $Y(t) = \frac{1}{X(t)} = F(X(t))$, $F(x) = \frac{1}{x}$, $F'(x) = -\frac{1}{x^2}$, $F''(x) = 2\frac{1}{x^3}$

$$dY = -\frac{1}{X^2}adt - \frac{1}{X^2}bdW(t) + \frac{1}{2}\frac{1}{X^3}b^2dt,$$

$Z(t) = \ln X(t)$, so $Z(t) = F(X(t))$ with $F(x) = \ln x$, $F'(x) = \frac{1}{x}$, $F''(x) =$

$-\frac{1}{x^2}$ and

$$dZ = \frac{1}{X}adt + \frac{1}{X}bdW(t) - \frac{1}{2}\frac{1}{X^2}b^2dt.$$

Exercise 4.8: Find the characteristics of $\exp\{at + X(t)\}$, given the form of the Itô process X.

Solution: Let $F(t,x) = \exp\{at + x\}$ so $F_t = aF$, $F_x = F$, $F_{xx} = F$ and with $Z(t) = \exp\{at + X(t)\}$, $dX(t) = a_X(t)dt + b_X(t)dW(t)$ we have

$$dZ = aZdt + a_xZdt + b_XZdW(t) + \frac{1}{2}b_X^2Zdt.$$

Exercise 4.9: Find a version of Corollary 4.32 for the case where σ is a deterministic function of time.

Solution: Let $M(t) = \exp\{\int_0^t \sigma(s)dW(s) - \frac{1}{2}\int_0^t \sigma^2(s)ds\} = \exp\{X(t)\}$ where $X(t)$ is Itô with $a_X = -\frac{1}{2}\sigma^2$, $b_X = \sigma$. Since $X(t)$ has normal distribution (σ is deterministic), $\sigma M \in \mathcal{M}^2$ can be show in the same way as in the proof of the corollary and (4.16) is clearly satisfied.

Exercise 4.10: Find the characteristics of the process $e^{-rt}X(t)$.

Solution: Let $Y(t) = e^{-rt}$, $a_Y(t) = -re^{-rt}$, $b_Y(t) = 0$, $dY(t) = -re^{-rt}dt$ so integration by parts (Itô product rule, in other words) gives

$$d[e^{-rt}X(t)] = -re^{-rt}X(t)dt + e^{-rt}dX(t).$$

Exercise 4.11: Find the form of the process X/Y using Exercise 4.7.

Solution: Write $dX(t) = a_X(t)dt + b_X(t)dW(t)$, $d(\frac{1}{Y(t)}) = a_{1/Y}(t)dt + b_{1/Y}(t)dW(t)$ with the characteristics of Y given by Exercise 4.7:

$$a_{1/Y} = -\frac{1}{Y^2}a_Y + \frac{1}{2}\frac{1}{Y^3}b_Y^2,$$
$$b_{1/Y} = -\frac{1}{Y^2}b_Y.$$

All that is left is to plug these into the claim of Theorem 4.36

$$d(X\frac{1}{Y}) = Xd(\frac{1}{Y}) + \frac{1}{Y}dX + b_Xb_{1/Y}dt.$$

Chapter 5

Exercise 5.1: Find an equation satisfied by $X(t) = S(0)\exp\{\mu_X t + \sigma W(t)\}$.

Solution: Write the process in (5.3) in the form $S(t) = S(0)\exp\{\mu_S t + \sigma W(t)\}$ with $\mu_S = \mu_X - \frac{1}{2}\sigma^2$ and (5.2) takes the form $dS(t) = (\mu_S + \frac{1}{2}\sigma^2)S(t)dt + \sigma S(t)dW(t)$ so immediately

$$dX(t) = (\mu_X + \frac{1}{2}\sigma^2)X(t)dt + \sigma X(t)dW(t)$$

Exercise 5.2: Find the equations for the functions $t \mapsto \mathbb{E}(S(t))$, $t \mapsto \mathrm{Var}(S(t))$.

Solution: We have $\mathbb{E}(S(t)) = S(0)\exp\{\mu t\} = m(t)$, say, so $m'(t) = \mu m(t)$ with $m(0) = S(0)$. Next,l $\mathrm{Var}(S(t)) = \mathbb{E}(S(t) - S(0)e^{\mu t})^2 = S^2(0)e^{2\mu t}(e^{\sigma^2 t} - 1) = v(t)$, say and

$$\begin{aligned} v'(t) &= 2\mu S^2(0)e^{2\mu t}(e^{\sigma^2 t} - 1) + \sigma^2 S^2(0)e^{2\mu t}e^{\sigma^2 t} \\ &= [2\mu + \sigma^2]v(t) + \sigma^2 m^2(t). \end{aligned}$$

Exercise 5.3: Show that the linear equation

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW(t)$$

with continuous deterministic functions $\mu(t)$ and $\sigma(t)$ has a unique solution

$$S(t) = S(0)\exp\{\int_0^t \left(\mu(s) - \frac{1}{2}\sigma^2(s)\right)ds + \int_0^t \sigma(s)dW(s)\}.$$

Solution: For uniqueness we can repeat the proof of Proposition 5.3 (or notice that the coefficients of the equation satisfy the conditions of Theorem 5.8). To see that the process solves the equation, take

$$F(t, x) = S(0)\exp\{\int_0^t \left(\mu(s) - \frac{1}{2}\sigma^2(s)\right)ds + x\},$$

and $S(t) = F(t, X(t))$ with $X(t) = \int_0^t \sigma(s)dW(s)$. Now

$$\begin{aligned} F_t(t, x) &= (\mu(t) - \frac{1}{2}\sigma^2(t))S(0)F(t, x), \\ F_x(t, x) &= F_{xx}(t, x) = S(0)F(t, x), \\ dX(t) &= \sigma(t)dW(t), \end{aligned}$$

so by the Itô formula we get the result:

$$
\begin{aligned}
dS(t) = {} & (\mu - \frac{1}{2}\sigma^2)S(0)\exp\{\int_0^t \left(\mu(s) - \frac{1}{2}\sigma^2(s)\right) ds + X(t)\}dt \\
& +\sigma S(0)\exp\{\int_0^t \left(\mu(s) - \frac{1}{2}\sigma^2(s)\right) ds + X(t)\}dW(t) \\
& +\frac{1}{2}\sigma^2 S(0)\exp\{\int_0^t \left(\mu(s) - \frac{1}{2}\sigma^2(s)\right) ds + X(t)\}dt \\
= {} & \mu S(t)dt + \sigma S(t)dW(t)
\end{aligned}
$$

(We have essentially repeated the proof of Theorem 5.2.)

Exercise 5.4: Find the equation solved by the process $\sin W(t) = X(t)$, say.

Solution: Take $F(x) = \sin(x)$, $F'(x) = \cos(x)$, $F''(x) = -\sin(x)$ and the simplest version of Itô formula gives

$$dX(t) = \cos(W(t))dW(t) - \frac{1}{2}\sin(W(t))dt = \sqrt{1 - X^2(t)}dW(t) - \frac{1}{2}X(t)dt$$

Exercise 5.5: Find a solution to the equation $dX = -\sqrt{1 - X^2}dW + \frac{1}{2}Xdt$ with $X(0) = 1$.

Solution: Comparing with Exercise 5.4 we can guess $X(t) = \cos(W(t))$ and check that with $F(x) = \cos x$ the Itô formula gives the result.

Exercise 5.6: Find a solution to the equation

$$dX(t) = 3X^2(t)dt - X^{3/2}(t)dW(t)$$

bearing in mind the above derivation of $dX(t) = X^3(t)dt + X^2(t)dW(t)$.

Solution: An educated guess (the 'educated' part is to solve $F' = -F^{3/2}$ so that the stochastic term agrees, the 'guess' is to use F of some special form $(1+ax)^{-b}$, then keep fingers crossed that the dt term will be as needed) gives $F(x) = (1 + \frac{1}{2}x)^{-2}$ with $F'(x) = -(1 + \frac{1}{2}x)^{-3} = -[F(x)]^{\frac{3}{2}}$, $F''(x) = 3(1 + \frac{1}{2}x)^{-4} = 3F^2(x)$ so $X(t) = F(W(t))$ satisfies the equation.

Exercise 5.7: Solve the following Vasicek equation $dX(t) = (a-bX(t))dt + \sigma dW(t)$.

Solution: Observe that $d\left[e^{bt}X(t)\right] = ae^{bt}dt + \sigma e^{bt}dW(t)$ (Exercise 4.10) hence

$$
\begin{aligned}
e^{bt}X(t) &= X(0) + a\int_0^t e^{bu}du + \sigma\int_0^t e^{bu}dW(u) \\
&= X(0) + \frac{a}{b}\left(e^{bt} - 1\right) + \sigma\int_0^t e^{bu}dW(u),
\end{aligned}
$$

so that

$$X(t) = e^{-bt}X(0) + \frac{a}{b}\left(1 - e^{-bt}\right) + \sigma e^{-bt}\int_0^t e^{bu}dW(u).$$

Exercise 5.8: Find the equation solved by the process X^2 where X is the Ornstein-Uhlenbeck process.

Solution: Recall $dX(t) = \mu X(t)dt + \sigma dW(t)$, so by the Itô formula, $dX^2(t) = 2X(t)dX(t) + \sigma^2 dt = 2\mu X^2(t)dt + 2\sigma X(t)dW(t) + \sigma^2 dt$.

Exercise 5.9: Prove uniqueness using the method of Proposition 5.3 for a general equation with Lipschitz coefficients (take any two solutions and estimate the square of their difference to show that it is zero).

Solution: Suppose

$$X_i(t) = X_0 + \int_0^t a(s, X_i(s))ds + \int_0^t b(s, X_i(s))dW(s), \quad i = 1, 2.$$

Then

$$\begin{aligned} X_1(t) - X_2(t) = &\int_0^t [a(u, X_1(u)) - a(u, X_2(u))]du \\ &+ \int_0^t [b(u, X_1(u)) - b(u, X_2(u))]dW(u) \end{aligned}$$

and using $(a + b)^2 \leq 2a^2 + 2b^2$ and taking expectation we get

$$\begin{aligned} &f(t) := \mathbb{E}(X_1(t) - X_2(t))^2 \\ &\leq 2\mathbb{E}\left(\int_0^t [a(u, X_1(u)) - a(u, X_2(u))]du\right)^2 \\ &\quad +2\mathbb{E}\left(\int_0^t [b(u, X_1(u)) - b(u, X_2(u))]dW(u)\right)^2. \end{aligned}$$

Using the Lipschitz condition for a, the first term on the right is estimated by $2\mathbb{E}\left(\int_0^t K[X_1(u) - X_2(u)]du\right)^2$ and we can continue from here as in the proof of Proposition 5.3.

Itô isometry and the Lipschitz condition for b allow us to estimate the second term by

$$\begin{aligned} &2\mathbb{E}\left(\int_0^t [b(u, X_1(u)) - b(u, X_2(u))]dW(u)\right)^2 \\ &= 2\int_0^t \mathbb{E}[b(u, X_1(u)) - b(u, X_2(u))]^2 du \\ &\leq 2\int_0^t K^2\mathbb{E}[X_1(u)) - X_2(u)]^2 du \end{aligned}$$

Putting these together we obtain $f(t) \leq 2K^2(1+T)\int_0^t f(u)du$ and the Gronwall lemma implies $f(t) = 0$, i.e. $X_1(t) = X_2(t)$.

Exercise 5.10: Prove that the solution depends continuously on the initial value in the L^2 norm, namely show that if X, Y are solutions of (5.4) with initial conditions X_0, Y_0, respectively, then for all t we have $\mathbb{E}(X(t) - Y(t))^2 \leq c\mathbb{E}(X_0 - Y_0)^2$. Find the form of the constant c.

Solution: We proceed as in Exercise 5.9 but the first step is

$$X(t) - Y(t) = X(0) - Y(0) + \int_0^t [a(u, X(u)) - a(u, Y(u))]du + \int_0^t [b(u, X(u)) - b(u, Y(u))]dW(u).$$

After taking squares, expectation and following the same estimations we will end up with

$$f(t) \leq 2\mathbb{E}(X_0 - Y_0)^2 + 2K^2(1+T)\int_0^t f(u)du$$

so after Gronwall

$$\mathbb{E}(X_1(t) - X_2(t))^2 \leq 2\exp\{2K^2(1+T)T\}\mathbb{E}(X_0 - Y_0)^2.$$