Black-Scholes Model
Solutions to Exercises

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Chapter 1

Exercise 1.1.
Show that the process

$$S(t) = S(0)\exp\{\mu t - \frac{\sigma^2}{2}t + \sigma W(t)\}.$$

solves

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t).$$

Solution.
Apply the Itô formula (see [SCF]) with $F(t,x) = S(0)\exp\{\mu t - \frac{\sigma^2}{2}t + \sigma x\}$, $X(t) = W(t)$ (so $a(t) = 0$, $b(t) = 1$) to find the stochastic differential of the process $F(t, W(t)) = S(t)$:

$$\begin{aligned} dS(t) &= F_t(t, W(t))dt + F_x(t, W(t))dW(t) + \frac{1}{2}F_{xx}(t, W(t))dt \\ &= \mu F(t, W(t)) + \sigma F(t, W(t)) \end{aligned}$$

since $F_t(t,x) = (\mu - \frac{1}{2}\sigma^2)F(t,x)$, $F_x(t,x) = \sigma F(t,x)$, $F_{xx}(t,x) = \sigma^2 F(t,x)$.

Exercise 1.2.
Find the probability that $S(2t) > 2S(t)$ for some $t > 0$.

Solution.
The inequality is equivalent to

$$\exp\{2\mu t - \sigma^2 t + \sigma W(2t)\} > 2\exp\{\mu t - \frac{\sigma^2}{2}t + \sigma W(t)\}$$

and after rearranging this becomes

$$\exp\{\sigma[W(2t) - W(t)]\} > \exp\{\ln 2 - \mu t + \frac{\sigma^2}{2}t\}$$

which is equivalent to

$$W(2t) - W(t) > \frac{1}{\sigma}[\ln 2 - \mu t + \frac{\sigma^2}{2}t].$$

Writing $W(2t) - W(t) = \sqrt{t}X$, where $X \sim N(0,1)$, we can see that the probability of the above event is

$$1 - N(\frac{1}{\sigma\sqrt{t}}[\ln 2 - \mu t + \frac{\sigma^2}{2}t]),$$

where N is the standard normal cumulative distribution function.

Exercise 1.3.
Find the formula for the variance of the stock price: $\mathrm{Var}(S(t))$.

Solution.

First we find the expectation

$$\mathbb{E}(S(t)) = S(0)\exp\{\mu t\}$$

using the formula $\mathbb{E}(e^X) = e^{\frac{1}{2}\mathrm{Var}(X)}$ where X has normal distribution with zero expectation, and next we compute

$$\begin{aligned}\mathbb{E}(S(t) - S(0)e^{\mu t})^2 &= S^2(0)e^{2\mu t}\mathbb{E}(e^{-\frac{1}{2}\sigma^2 t + \sigma W(t)} - 1)^2 \\ &= S^2(0)e^{2\mu t}\mathbb{E}(e^{-\sigma^2 t + 2\sigma W(t)} - 2e^{-\frac{1}{2}\sigma^2 t + \sigma W(t)} + 1).\end{aligned}$$

Finally,

$$\begin{aligned}\mathbb{E}(e^{-\sigma^2 t + 2\sigma W(t)}) &= e^{-\sigma^2 t}e^{2\sigma^2 t} = e^{\sigma^2 t} \\ \mathbb{E}(e^{-\frac{1}{2}\sigma^2 t + \sigma W(t)}) &= 1\end{aligned}$$

so

$$\mathrm{Var}S(t) = S^2(0)e^{2\mu t}(e^{\sigma^2 t} - 1).$$

Exercise 1.4.

Consider an alternative model where the stock prices follow an Ornstein-Uhlenbeck process: this is a solution of $dS_1(t) = \mu_1 S_1(t)dt + \sigma_1 dW(t)$ (see [SCF]). Find the probability that at a certain time $t_1 > 0$ we will have negative prices: i.e. compute $P(S_1(t_1) < 0)$. Illustrate the result numerically.

Solution.

THe Itô formula gives the form of the solution

$$S_1(t) = S(0)e^{\mu_1 t} + \int_0^t \sigma e^{\mu_1(t-s)}dW(s)$$

and

$$P(S_1(t) < 0) = P(\int_0^t \sigma_1 e^{\mu_1(t-s)}W(s) < -S(0)e^{\mu_1 t}).$$

The random variable $\int_0^t \sigma_1 e^{\mu_1(t-s)}W(s)$ has normal distribution with zero mean and variance

$$\begin{aligned}\int_0^t \sigma_1^2 e^{2\mu_1(t-s)}ds &= \sigma_1^2 e^{2\mu_1 t}\int_0^t e^{-2\mu_1 s}ds \\ &= \frac{\sigma_1^2}{2\mu_1}(e^{2\mu_1 t} - 1)\end{aligned}$$

so

$$P(S_1(t) < 0) = N(-\frac{S(0)e^{\mu_1 t}}{\sqrt{\frac{\sigma_1^2}{2\mu_1}(e^{2\mu_1 t} - 1)}})$$

With $S(0) = 100$, $\mu_1 = 10\%$, $\sigma_1 = 30$, (this parameter is related to prices, not returns), $t_1 = 1$ we obtain 0.000231481.

Exercise 1.5.

Allowing time-dependent but deterministic σ_1 in the Ornstein-Uhlenbeck model, find its shape so that $\mathrm{Var}(S(t)) = \mathrm{Var}(S_1(t))$.

Solution.

$$\begin{aligned}\mathrm{Var}(S_1(t)) &= \mathrm{Var}(\int_0^t \sigma e^{\mu_1(t-s)} dW(s)) = \frac{\sigma_1^2}{2\mu_1}(e^{2\mu_1 t} - 1), \\ \mathrm{Var}S(t) &= S^2(0)e^{2\mu t}(e^{\sigma^2 t} - 1),\end{aligned}$$

so

$$\sigma_1^2 = \frac{2\mu_1 S^2(0)e^{2\mu t}(e^{\sigma^2 t} - 1)}{e^{2\mu_1 t} - 1}$$

Exercise 1.6.

Let L be a random variable representing the loss on some business activity. Value at Risk at confidence level $a\%$ is defined as $\nu = \inf\{x : P(L \leq x) \geq \frac{a}{100}\}$. Compute v for $a = 5\%$, where L is the loss on the investment in a single share of stock purchased at $S(0) = 100$ and sold at $S(T)$ with $\mu = 10\%$, $\sigma = 40\%$, $T = 1$.

Solution.

The loss can be defined in a simplified way, setting $L = S(T) - S(0)$ and neglecting time value of money and lost opportunity in alternative investment, or by taking $L = S(T)e^{-\mu T} - S(0)$, where the discounting uses the average growth rate for stock (the risk-free rate would be inappropriate since the rate should reflect the risk). We use the latter approach. Now

$$\begin{aligned}P(L &\leq x) = P(S(T)e^{-\mu T} - S(0) \leq x) \\ &= P(S(0)\exp\{\frac{\sigma^2}{2}T + \sigma W(T)\} \leq x + S(0)) \\ &= P(W(T) \leq \frac{1}{\sigma}[\ln(\frac{x}{S(0)} + 1) - \frac{\sigma^2}{2}T]) \\ &= N(\frac{1}{\sigma\sqrt{T}}[\ln(\frac{x}{S(0)} + 1) - \frac{\sigma^2}{2}T]).\end{aligned}$$

Due to the monotonicity and continuity of the exponential function, ν is the solution to

$$\frac{a}{100} = N(\frac{1}{\sigma\sqrt{T}}[\ln(\frac{\nu}{S(0)} + 1) - \frac{\sigma^2}{2}T])$$

so

$$\nu = S(0)[\exp\{\sigma\sqrt{T}N^{-1}(\frac{a}{100}) + \frac{\sigma^2}{2}T\} - 1]$$

and we obtain -43.89 for the given data. (Often loss is defined as the opposite difference so that it is positive when we lose the money and negative in case of profit.)

Chapter 2

Exercise 2.1.
Prove that, for $u < t$, $\mathbb{E}(\tilde{S}(t)|\mathcal{F}_u^S) = \tilde{S}(u)\exp\{(\mu - r)t\}$.

Solution.

$$
\begin{aligned}
&\mathbb{E}[\tilde{S}(t)|\mathcal{F}_u^S] \\
&= S(0)\mathbb{E}[\exp\{(\mu - r)t - \frac{1}{2}\sigma^2 t + \sigma W(t)\}|\mathcal{F}_u^S] \\
&= S(0)\exp\{(\mu - r)t - \frac{1}{2}\sigma^2 t\}\mathbb{E}[\exp\{\sigma[W(t) - W(u)]\}\exp\{\sigma W(u)\}|\mathcal{F}_u^S] \\
&= S(0)\exp\{(\mu - r)t - \frac{1}{2}\sigma^2 t\}\exp\{\sigma W(u)\}\mathbb{E}[\exp\{\sigma[W(t) - W(u)]\}|\mathcal{F}_u^S] \\
&\qquad \text{(taking out what is known)} \\
&= S(0)\exp\{(\mu - r)t - \frac{1}{2}\sigma^2 t\}\exp\{\sigma W(u)\}\mathbb{E}[\exp\{\sigma[W(t) - W(u)] \qquad \text{(by independence)} \\
&= S(0)\exp\{(\mu - r)t - \frac{1}{2}\sigma^2 t\}\exp\{\sigma W(u)\}\exp\{\frac{1}{2}\sigma^2(t - u)\} \\
&\qquad \text{(computing the expectation)} \\
&= S(0)\exp\{(\mu - r)u - \frac{1}{2}\sigma^2 u + \sigma W(u)\}\exp\{(\mu - r)(t - u)\} \\
&= \tilde{S}^\delta(u)\exp\{(\mu - r)t\}.
\end{aligned}
$$

Exercise 2.2.
Consider the following strategy: $x(t) = \frac{V(0)}{S(0)}$ for $t \in [0, t_1)$, $x(t) = 2x(0)$ for $t \in [t_1, t_2)$ and $x(t_2) = 0$ with $V(0)$, $S(0)$ known, $0 < t_1 < t_2$ prescribed in advance (so all money is invested in stock at the beginning, then the number of shares is doubled at time t_1 with liquidation of the risky position at time t_2). Choose the process y so that the strategy is self-financing. Within the Black-Scholes model, with given μ, σ, r, what is the probability that $y(t_2) < 0$? Give a numerical example.

Solution.
Recall:

$$
y(t) = \frac{1}{A(t)}\left(\mathrm{e}^{rt}[V(0) + \int_0^t x(u)d\tilde{S}(u)] - x(t)S(t)\right)
$$

so for $t \in [0, t_1)$, we have $y(t) = 0$, since for such t the integral equals $\frac{V(0)}{S(0)}[\widetilde{S}(t) - S(0)]$.

At time t_1 we have to borrow to finance the purchase of additional shares. Before the purchase $V(t_1) = x(0)S(t_1)$ and after $V(t_1) = x(t_1)S(t_1) + y(t_1)A(t_1) = 2x(0)S(t_1) + y(t_1)A(t_1)$ so to maintain the self-financing property we need $y(t_1) = -\frac{1}{A(t_1)}x(0)S(t_1)$.

At time t_2 we have $V(t_2) = 2x(0)S(t_2) - \frac{1}{A(t_1)}x(0)S(t_1)A(t_2)$ before liquidation and $V(t_2) = y(t_2)A(t_2)$ thereafter. Again, by the self-financing property,

$$y(t_2) = \frac{1}{A(t_2)}(2x(0)S(t_2) - \frac{1}{A(t_1)}x(0)S(t_1)A(t_2)).$$

Finally,

$$\begin{aligned}
P(y(t_2) &< 0) = P(2S(t_2)A(t_1) < S(t_1)A(t_2)) \\
&= P(2\exp\{rt_1 + \mu t_2 - \frac{\sigma^2}{2}t_2 + \sigma W(t_2)\} < \exp\{rt_2 + \mu t_1 - \frac{\sigma^2}{2}t_1 + \sigma W(t_1)\}) \\
&= P([W(t_2) - W(t_1)] < \frac{1}{\sigma}[-\ln 2 + r(t_2 - t_1) + \mu(t_1 - t_2) - \frac{\sigma^2}{2}(t_1 - t_2)]) \\
&= N(\frac{1}{\sqrt{t_2 - t_1}\sigma}[-\ln 2 + r(t_2 - t_1) + \mu(t_1 - t_2) - \frac{\sigma^2}{2}(t_1 - t_2)])
\end{aligned}$$

For $t_2 = 2$, $t_1 = 1$, $\mu = 10\%$, $\sigma = 30\%$, $r = 5\%$ we find 0.24254258.

Exercise 2.3.
Design a version of this strategy with positive risk-free rate.

Solution. The strategy in question is that of Example 2.18 (p. 22).
The modifications are as follows: at time t_1 we have to find $x(t_1)$ so that $P(V(t_2) < 2) = p$. The bond position is

$$y(t_1) = \frac{1}{A(t_1)}[V(t_1) - x(t_1)S(t_1)]$$

and

$$P(V(t_2) < 2) = P(x(t_1)S(t_2) + \frac{A(t_2)}{A(t_1)}[V(t_1) - x(t_1)S(t_1)] < 2) = p$$

has to be solved for $x(t_1)$.

Exercise 2.4.
Prove that if the value process of an asset $B(t)$ satisfies the equation $dB(t) = g(t)B(t)dt$, where g is a stochastic process, then $g(t) = r$ a.s. for all $t \geq 0$.

Solution.
The money market account satisfies

$$dA(t) = rA(t)dt,$$
$$A(0) = 1.$$

We can assume without loss of generality that

$$B(0) = 1.$$

Take a strategy consisting of $x(t)$ units of security $B(t)$ and $y(t)$ units of the money market account $A(t)$ such that

$$x(t) = \begin{cases} 1 & \text{if } B(t) > A(t) \\ 0 & \text{if } B(t) = A(t) \\ -1 & \text{if } B(t) < A(t) \end{cases}, \quad y(t) = -x(t)$$

The value of the strategy

$$\begin{aligned} V(t) &= x(t)B(t) + y(t)A(t) \\ &= \begin{cases} B(t) - A(t) & \text{if } B(t) > A(t) \\ 0 & \text{if } B(t) = A(t) \\ A(t) - B(t) & \text{if } B(t) < A(t) \end{cases} \geq 0 \end{aligned}$$

satisfies

$$\frac{dV(t)}{dt} = \frac{d}{dt}\left(x(t)B(t) + y(t)A(t)\right) = x(t)\frac{d}{dt}B(t) + y(t)\frac{d}{dt}A(t)$$

a.e. with respect to $t \geq 0$, that is

$$dV(t) = x(t)dB(t) + y(t)dA(t).$$

Therefore, we have a self-financing strategy such that $V(0) = 0$ and $V(t) \geq 0$ for all $t \geq 0$. By the no-arbitrage principle it follows that $V(t) = 0$ for all $t \geq 0$. As a result, for almost all paths we have

$$B(t) = A(t) = e^{rt},$$

which forces $g(t) = r$ for all $t \geq 0$.

Exercise 2.5: Given a filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and an adapted process X with a.s. continuous paths. Show that the first hitting of a closed set in $\mathbb{R}$ is an $\mathcal{F}_t$-stopping time.

Solution.
Suppose X is a process with a.s. continuous paths and $A \in \mathbb{R}$ is a closed set. Define the first hitting time of A by X as

$$\tau_A(\omega) = \inf\{t \leq T : X(t,\omega) \in A\}.$$

Consider a nested sequence of open neighbourhoods of A defined by

$$O_n = \{x \in \mathbb{R} : \inf(|a - x| : a \in A) < \frac{1}{n}\},$$

and let τ_n define the first hitting time of O_n by A. We check that for any $t \leq T$ we have

$$\{\tau_n < t\} = \bigcup_{r\in\mathbb{Q}, 0\leq r<t} \{X(r) \in O_n\}.$$

If $\{X(r) \in O_n$ for some rational $r < t$, clearly $\inf\{s : X(s) \in O_n\} < t$. Conversely path-continuity of X ensures that if this infimum is less than t, then $X(r) \in O_n$ for some rational $r < t$. So we have shown that $\{\tau_n < t\} \in \mathcal{F}_t$.

The decreasing sequence of stopping times $(\tau_n)_n$ satisfies $\tau_n < \tau_A$ for all n, so $\tau = \lim_n \tau_n \leq \tau_A$. We show that the stopping time τ must equal τ_A.

If $\tau = 0$ there is nothing to prove. On $\{\tau > 0\}$ we can find $k = k(\omega) > 1$ such that $\tau_n = 0$ for $n < k$ and $0 < \tau_n < \tau_{n+1} < \tau$, since $t \to X(t, \omega)$ is continuous for almost all ω, so that, as soon as $\tau_k(\omega) > 0$, the first hitting times of O_n for $n > k$ are a strictly increasing sequence strictly below τ, as the O_n are open and O_{n+1} is strictly contained in O_n. But $A = \bigcap_{n\geq 1} O_n$, and by continuity again, $X_{\tau_A} = \lim_n X_{\tau_n}$. As X_{τ_m} lies in the closure $\overline{O}_m$ of O_m, hence lies in O_n for $n < m$, letting $m \to \infty$ ensures that $X_{\tau_A} \in O_n$, which means that $\tau_A \leq \tau$, hence they are equal.

So we have shown that $\{\tau_A \leq t\} = \bigcap_{n=1}^{\infty}\{\tau_n < t\}$ and the latter set is in $\mathcal{F}_t$, so τ_A is a stopping time.

Exercise 2.6.

Prove that if $f, g \in M^2$ and τ_1, τ_2 are stopping times such that $f(s,\omega) = g(s,\omega)$ whenever $\tau_1(\omega) \leq s < \tau_2(\omega)$, then for any $t_1 \leq t_2$

$$\int_{t_1}^{t_2} f(s)dW(s) = \int_{t_1}^{t_2} g(s)dW(s)$$

for almost all ω satisfying $\tau_1(\omega) \leq t_1 \leq t_2 < \tau_2(\omega)$.

Solution.

Fix $t_1 \leq t_2$. By linearity we need only show that if $f(s,\omega) = 0$ on $\{(s,\omega) : \tau_1(\omega) \leq s \leq \tau_2(\omega)\}$ then $\int_{t_1}^{t_2} f(s)dW(s) = 0$ for $\{\omega : \tau_1(\omega) \leq t < \tau_2(\omega)\} = A_{12}(t)$. Note that $A_{12}(t) \in \mathcal{F}_t$, since $A_{12}(t) = \{\tau_1 \leq t\} \cap [\Omega \setminus \{\tau_2 \leq t\}] \in \mathcal{F}_t$.

Step 1. Suppose first that $f \in \mathcal{M}^2$ is simple

$$f(t,\omega) = \xi_0(\omega)\mathbf{1}_{\{0\}}(t) + \sum_{k=1}^{N} \xi_k(\omega)\mathbf{1}_{(t_k,t_{k+1}]}(t).$$

For any particular ω, $\tau_1(\omega) \in (t_{m_1}, t_{m_1+1}]$, $\tau_2(\omega) \in (t_{m_2}, t_{m_2+1}]$ for some $m_1 \leq m_2$. For f to vanish on $\{(s,\omega) : \tau_1(\omega) \leq s \leq \tau_2(\omega)\}$, the coefficients $\xi_k(\omega)$ must be zero if $k \in [m_1, m_2]$. The stochastic integral is easily computed: let $t_1 \in (t_{n_1}, t_{n_1+1}]$, $t_2 \in (t_{n_2}, t_{n_2+1}]$ and by definition

$$\begin{aligned}
&\left(\int_{t_1}^{t_2} f(s)dW(s)\right)(\omega) \\
= \;& \sum_{k=1}^{n_2-1} \xi_k(\omega)[W(t_{k+1},\omega) - W(t_k,\omega)] + \xi_{n_2}(\omega)[W(t_2,\omega) - W(t_{n_2},\omega)] \\
&- \sum_{k=1}^{n_1-1} \xi_k(\omega)[W(t_{k+1},\omega) - W(t_k,\omega)] + \xi_{n_1}(\omega)[W(t_1,\omega) - W(t_{n_2},\omega)].
\end{aligned}$$

If $\tau_1(\omega) \le t_1 \le t_2 \le \tau_2(\omega)$, $\xi_k(\omega) = 0$ for $n_1 \le k \le n_2$ so the above sum vanishes.

Step 2: Take a bounded f and choose an increasing sequence of simple processes f_n converging to f in $\mathcal{M}^2$. The difficulty in applying the first part of the proof lies in the fact that f_n do not have to vanish for $\tau_1(\omega) \le t \le \tau_2(\omega)$ even if f does. Hence we truncate f_n by forcing it to be zero for those t by writing

$$g_n(t,\omega) = f_n(t,\omega)\mathbf{1}_{A_{12}(t)}(t).$$

The idea is that this should mean no harm as f_n is going to zero anyway in this region, so we are just speeding this up a bit. For any t, the random variable $\mathbf{1}_{A_{12}(t)}(t)$ is 1 on $A_{12}(t)$ which belongs to $\mathcal{F}_t$. So g_n is an adapted simple process and Step 1 applies to give

$$\int_{t_1}^{t_2} g_n(s)dW(s) = 0 \text{ on } \{\tau_1 \le t_1 \le t_2 \le \tau_2\}.$$

The convergence $f_n \to f$ in $\mathcal{M}^2$ implies that $f_n\mathbf{1}_{A_{12}(t)} \to f\mathbf{1}_{A_{12}(t)} = f$ in this space so

$$\int_{t_1}^{t_2} g_n(s)dW(s) \to \int_{t_1}^{t_2} f(s)dW(s) \quad \text{in } L^2(\Omega),$$

thus a subsequence converges almost surely, hence $\int_{t_1}^{t_2} f(s)dW(s) = 0$ if $\{\tau_1 \le t_1 \le t_2 \le \tau_2\}$ holds on a set Ω_t of full probability. Taking rational times q we get

$$\int_{t_1}^{q_k} f(s)dW(s) = 0 \text{ on } \bigcup_{q_k \in \mathbb{Q}, q_k \uparrow t_2} \Omega_{q_k},$$

which by continuity of stochastic integral extends to all $t \in [0,T]$.

Step 3. For an arbitrary $f \in \mathcal{M}^2$ let $f_n(t,\omega) = f(t,\omega)\mathbf{1}_{\{|f(t,\omega)| \le n\}}(\omega)$. Clearly $f_n \to f$ pointwise and by the dominated convergence theorem this convergence is also in the norm on $\mathcal{M}^2$. By the Itô isometry and linearity it follows that $\int_{t_1}^{t_2} f_n(s)dW(s) \to \int_{t_1}^{t_2} f(s)dW(s)$ in $L^2(\Omega)$. But f_n is bounded, it is zero if $\{\tau_1 \le t_1 \le t_2 \le \tau_2(\omega)\}$, so $\int_{t_1}^{t_2} f_n(s)dW(s) = 0$ by Step 2, and consequently $\int_{t_1}^{t_2} f(s)dW(s) = 0$.

Chapter 3

Exercise 3.1.
Find the representation of $M(t) = \left(\int_0^t g dW\right)^2 - \int_0^t g^2 ds$.

Solution.
Write $X(t) = \int_0^t g(s)dW(s)$, then $M(t) = \int_0^t 2g(s)X(s)dW(s)$ (with g deterministic).

Exercise 3.2.
Show that

$$\begin{aligned}
\mathbb{E}_Q(\mathbf{1}_{\{S(T)\geq K\}}|\mathcal{F}_t) &= N(-d(t, S(t)) + \sigma\sqrt{T-t})), \\
\mathbb{E}_Q(S(T)\mathbf{1}_{\{S(T)\geq K\}}|\mathcal{F}_t) &= \mathrm{e}^{r(T-t)}S(t)N(-d(t, S(t))).
\end{aligned}$$

Solution.
By restricting to the set where $\{S(T) \geq K\}$ the call price can be written as

$$\begin{aligned}
C(t) &= \mathbb{E}_Q(\mathrm{e}^{-r(T-t)}(S(T) - K)^+|\mathcal{F}_t) \\
&= \mathbb{E}_Q(\mathrm{e}^{-r(T-t)}(S(T) - K)\mathbf{1}_{\{S(T)\geq K\}}|\mathcal{F}_t) \\
&= \mathbb{E}_Q(\mathrm{e}^{-r(T-t)}S(T)\mathbf{1}_{\{S(T)\geq K\}}|\mathcal{F}_t) - \mathbb{E}_Q(\mathrm{e}^{-r(T-t)}K\mathbf{1}_{\{S(T)\geq K\}}|\mathcal{F}_t)
\end{aligned}$$

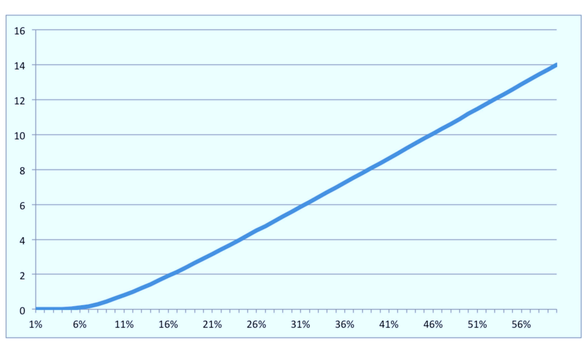
and in our calculation we dealt separately with these two terms to obtain the claimed identities.

Exercise 3.3.
Find the prices for a call and a put, and the probabilities (both risk-neutral and physical) of these options being in the money, if $S(0) = 100$, $K = 110$, $T = 0.5$, $r = 5\%$, $\mu = 8\%$, $\sigma = 35\%$.

Solution.
The call price is 6.97167, while the put price is 14.2558. The risk-neutral probability is 0.4363, and the physical probability is 0.46027.

Exercise 3.4.
Sketch the graph of $\sigma \to C(T, K, r, S(0), \sigma)$

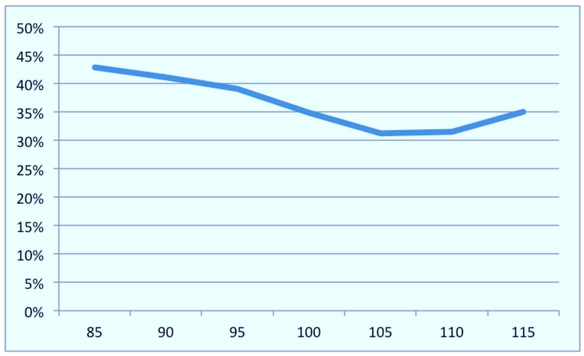
Solution.



Exercise 3.5.
Sketch the graph of the function $k \mapsto \sigma(T, K, r, S(0), C)$ where $S(0) = 100$, $T = 0.5$, $r = 5\%$ and the prices are as below.

K	85	90	95	100	105	110	115
C	21.59	18.3	14.67	10.97	7.74	6.01	5.46

Solution.



Exercise 3.6.
Consider the Bachelier model, i.e. assume $S(t) = S(0)+\mu t+\sigma W(t)$. Assume $r = 0$ and find a formula for the call price.

Solution.
From the Girsanov theorem we have $S(t) = S(0) + \sigma W_Q(t)$,

$$\begin{aligned} C(0) &= \mathbb{E}_Q((S(T) - K)^+) = \mathbb{E}_Q((S(0) + \sigma W_Q(T) - K)^+) \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (S(0) + \sigma y\sqrt{T} - K)^+ e^{-\frac{y^2}{2}} dy. \end{aligned}$$

Next $S(0) + \sigma y\sqrt{T} - K \geq 0$ if $y \geq \frac{1}{\sigma\sqrt{T}}(K - S(0)) =: d$ so

$$\begin{aligned} C(0) &= \frac{1}{\sqrt{2\pi}} \int_d^{\infty} (S(0) + \sigma y\sqrt{T} - K) e^{-\frac{y^2}{2}} dy \\ &= (S(0) - K)\frac{1}{\sqrt{2\pi}} \int_d^{\infty} e^{-\frac{y^2}{2}} dy + \frac{\sigma\sqrt{T}}{\sqrt{2\pi}} \int_d^{\infty} y e^{-\frac{y^2}{2}} dy \\ &= (S(0) - K)N(1 - d) + \frac{\sigma\sqrt{T}}{\sqrt{2\pi}} e^{-\frac{d^2}{2}}. \end{aligned}$$

Exercise 3.7.
Derive the version of the Black-Scholes PDE for the Bachelier model.

Solution.

We have to find u such that $C(t) = u(t, S(t))$, so (with $r = 0$)

$$\begin{aligned} C(t) &= \mathbb{E}_Q((S(T) - K)^+ | \mathcal{F}_t) \\ &= \mathbb{E}_Q((S(t) + \sigma[W_Q(T) - W_Q(t)] - K)^+ | \mathcal{F}_t) \\ &= \psi(S(t)) \end{aligned}$$

where (according to Lemma 3.13)

$$\psi(z) = \mathbb{E}_Q((z + \sigma[W_Q(T) - W_Q(t)] - K)^+).$$

We find the lower limit of integration, d, by solving

$$z + \sigma\sqrt{T-t}y - K \geq 0$$

to get

$$y \geq \frac{1}{\sigma\sqrt{T-t}}(K - z) = d(t, z)$$

so that

$$\begin{aligned} \psi(z) &= \frac{1}{\sqrt{2\pi}} \int_d^\infty (z + \sigma\sqrt{T-t}y - K) e^{-\frac{1}{2}y^2} dy \\ &= (z - K) N(1 - d(t, z)) + \frac{\sigma\sqrt{T-t}}{\sqrt{2\pi}} e^{-\frac{d^2(t,z)}{2}} \\ &= u(t, z) \end{aligned}$$

We compute the partial derivatives to confirm that u satisfies the equation informally derived here along the lines of Chapter 1: $C(t) = u(t, S(t))$ is an Itô process, $dS = \mu dt + \sigma dW_Q$ and

$$\begin{aligned} dC(t) &= (u_t + \mu u_z + \frac{1}{2}\sigma^2 u_{zz}) dt + \sigma u_z dW. \\ dC(t) &= x\mu dt + x\sigma dW \end{aligned}$$

so that $x = u_z$ and the equation has the form

$$u_t + \frac{1}{2}\sigma^2 u_{zz} = 0.$$

Exercise 3.8.

Give a detailed justification of the claims of Remark 3.26.

Remark 3.26 : The above considerations show a deep relationship between solutions to stochastic differential equations and solutions to partial differential equations. The main idea is best seen in a simple case, so consider a version of Lemma 3.23 with $S(t + u) = x + \sigma W(u)$, so that $S(t) = x$, to find that if u is a solution of

$$\begin{aligned} u_t + \frac{1}{2}\sigma^2 u_{zz} &= 0, \quad s < T, \\ u(T, z) &= h(z). \end{aligned}$$

Then $u(t, x + \sigma W(T - t))$ is a martingale and

$$u(t, x) = \mathbb{E}(h(x + \sigma W(T - t))),$$

which is a particular case of the famous Feynman-Kac formula.

Solution.

If $u_t + \frac{1}{2}\sigma^2 u_{zz} = 0$ then $u(s, x + \sigma W(s - t))$ is a martingale $s \in [t, T]$ (this is the correct formulation, rather than $u(t, x + \sigma W(T - t))$ as stated in the Remark).

Write $X(s) = u(s, x + \sigma W(s - t))$ and by the Itô formula

$$dX(s) = u_t ds + \sigma u_z dW(s) + \frac{1}{2}\sigma^2 u_{zz} ds$$

so

$$u(T, x + \sigma W(T - t)) = u(t, x) + \int_t^T \sigma u_z(s, x + \sigma(W(s - t))dW(s)$$

which is a martingale provided $u_x(t, x + \sigma W(T - t))$ is in $\mathcal{M}^2$ – this additional condition is required. Consequently, the expectation of the stochastic integral vanishes which implies

$$u(t, x) = \mathbb{E}(u(T, x + \sigma W(T - t)).$$

Exercise 3.9.

Show that the function defined by $u(t, z) = \mathbb{E}(h(z+\sigma W(T-t)))$ is sufficiently regular and solves $u_t + \frac{1}{2}\sigma^2 u_{zz} = 0$, for $s < T$, with $u(T, z) = h(z)$.

Solution.

The terminal condition is obvious: plug $t = T$ into u. Next

$$\mathbb{E}(h(z + \sigma W(T - t))) = \frac{1}{\sqrt{2\pi}} \int h(z + \sigma\sqrt{T - t}y)e^{-\frac{1}{2}y^2} dy$$

Then change variables: $x = z + \sigma\sqrt{T - t}y$ and since $dx = \sigma\sqrt{T - t}dy$

$$\ldots = \frac{1}{\sqrt{2\pi}\sigma\sqrt{T - t}} \int h(x)e^{-\frac{1}{2z\sqrt{T-t}}(x-z)^2} dx$$

which is smooth by results from elementary calculus (the dependence on t, z is taken out of h, which does not have to be smooth) and differentiation gives the equation.

Exercise 3.10.

Given an European call C and put P, both with strike K and expiry T , show that $\text{delta}_C - \text{delta}_P = 1$. Deduce that $\text{delta}_P = -N(-d_+)$ and that $\text{gamma}_P = \text{gamma}_C$.

Solution.

By call-put parity, $\text{delta}_{C-P} = 1$ and the result follows from the linearity of the delta operator. Since $\text{delta}_C = N(d_+)$, $\text{delta}_P = 1 - N(d_+) = N(-d_+)$. The relation for the gammas can be obtained immediately by differentiating the relation for the deltas.

Exercise 3.11.
Show that analogous calculations, with T replaced by the time to expiry $T - t$, and with $d_\pm(t)$ replacing $d_\pm$, apply to give the Greeks evaluated at time $t < T$.

Solution.
Routine differentiation gives

$$\begin{aligned}
\text{delta}_C(t) &= N(d_+(t)), \\
\text{gamma}_C(t) &= \frac{1}{S\sigma\sqrt{T-t}} n(d_+(t)), \qquad \text{where } n(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}x^2} \\
\text{theta}_C(t) &= -\frac{S\sigma}{2\sqrt{T-t}} n(d_+(t)) - rK\mathrm{e}^{-r(T-t} N(d_-(t)), \\
\text{vega}_C(t) &= S\sqrt{T-t} n(d_+(t)), \\
\text{rho}_C(t) &= (T-t)K\mathrm{e}^{-r(T-t)} N(d_-(t)),
\end{aligned}$$

where

$$d_\pm = \frac{\ln \frac{S(0)}{K} + (r \pm \frac{1}{2}\sigma^2)(T-t}{\sigma\sqrt{T-t}}.$$

Exercise 3.12.
Verify that $\text{theta}_P = \text{theta}_C + rK\mathrm{e}^{-r(T-t)}$, where C and P are a call and a put respectively, with the same strike K and expiry T. Deduce a formula for theta_P.

Solution.
By call-put parity, $\text{theta}_{C-P} = -rK\mathrm{e}^{-r(T-t)}$ the result follows from the linearity of differentiation. Consequently

$$\begin{aligned}
\text{theta}_P &= -\frac{S\sigma}{2\sqrt{T}} n(d_+) - rK\mathrm{e}^{-rT} N(d_-) + rK\mathrm{e}^{-rT} \\
&= -\frac{S\sigma}{2\sqrt{T}} n(d_+) + rK\mathrm{e}^{-rT} N(-d_-).
\end{aligned}$$

Chapter 4

Exercise 4.1.
Derive the equation satisfied by the futures price assuming that the interest rate is constant. Find the version in the risk-neutral world.

Solution.
If the interest rate is constant, the futures and forward prices coincide, the latter is given by $e^{r(T-t)}S(t) = X(t)$ and

$$\begin{aligned} dX(t) &= -re^{r(T-t)}S(t) + e^{r(T-t)}\mu S(t)dt + e^{r(T-t)}\sigma S(t)dW(t) \\ &= (\mu - r)X(t)dt + \sigma X(t)dW(t). \end{aligned}$$

In the risk-neutral world for S this reduces to $dX(t) = \sigma X(t)dW_Q(t)$ so X is a Q-martingale.

Exercise 4.2.
Derive a formula for the price of an option written on futures, assuming that the interest rate is constant.

Solution.
We have $S(t) = e^{-r(T-t)}X(t)$, so that by the Black-Scholes formula the call price is

$$C_S(t) = S(t)N(d_+(t, S(t))) - e^{-r(T-t)}KN(d_-(t, S(t)))$$

where

$$\begin{aligned} d_+(t, z) &= \frac{\ln(\frac{z}{K}) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}; \\ d_-(t, z) &= d_+(t, z) - \sigma\sqrt{T-t}. \end{aligned}$$

Substituting the expression for $X(t)$ the call price on X is then

$$\begin{aligned} C_F(t) &= e^{-r(T-t)}X(t)N(d_+(t, S(t)) - e^{-r(T-t)}KN(d_-(t, S(t)) \\ &= e^{-r(T-t)}[X(t))d_1(t) - KN(d_2(t))]. \end{aligned}$$

where

$$\begin{aligned} d_1(t) &= \frac{\ln(\frac{X(t)}{K}) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} = \frac{\ln(\frac{S(t)e^{r(T-t)}}{K}) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \\ &= \frac{\ln(\frac{S(t)}{K}) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = d_+(t, S(t), \text{ and} \\ d_2(t) &= d_1(t) - \sigma\sqrt{T-t} = d_-(t, S(t). \end{aligned}$$

This is the *Black formula* for a call on futures in a constant interest rate model. A similar argument gives the put price from the BS-put-price for $S(t)$.

Exercise 4.3.

Construct an alternative proof of Proposition 4.2 by observing that the function v, defined by $v(t,z) = \mathrm{e}^{\delta(T-t)}u(t,z)$, satisfies

$$v_t + \rho z v_z + \frac{1}{2}\sigma^2 z^2 v_{zz} = \rho v, \;\; v(T,z) = (K-z)^+.$$

where $\rho = r - \delta$.

Solution.

We write $\rho = r - \delta$ and $v(\rho, t, z) = e^{\delta(T-t)}u(r.t, z)$, where, since $e^{-r(T-t)} = e^{-\delta(T-t)}e^{-\rho(T-t)}$ we obtain

$$v(\rho,t,z) = zN(\frac{\ln(\frac{z}{K}) + (\rho + \frac{1}{2}\sigma^2((T-t)}{\sigma\sqrt{T-t}}) - e^{-\rho(T-t)}KN(\frac{\ln(\frac{z}{K}) + (\rho - \frac{1}{2}\sigma^2((T-t)}{\sigma\sqrt{T-t}}).$$

By the previous chapter the function v satisfies the PDE

$$v_t + \rho z v_z + \frac{1}{2}\sigma^2 z^2 v_{zz} = \rho v$$

and the final condition remains, as before, $v(\rho, T, z) = (K-z)^+ = u(r, T, z)$ since $e^{\delta(T-T)} = 1$.

Since

$$v = e^{\delta(T-t)}u$$

we obtain

$$\begin{aligned} v_t &= e^{\delta(T-t)}u_t - \delta e^{\delta(T-t)}u \\ v_z &= e^{\delta(T-t)}u_z \\ v_{zz} &= e^{\delta(T-t)}u_{zz} \end{aligned}$$

hence

$$e^{\delta(T-t)}u_t - \delta e^{\delta(T-t)}u + (r-\delta)ze^{\delta(T-t)}u_z + \frac{1}{2}\sigma^2 z^2 e^{\delta(T-t)}u_{zz} = (r-\delta)e^{\delta(T-t)}u$$

and so the function u satisfies

$$\begin{aligned} u_t + (r-\delta)zu_z + \frac{1}{2}\sigma^2 z^2 u_{zz} = ru, \quad t < T, z > 0 \\ u(T,z) = (z-K)^+ \end{aligned}$$

Exercise 4.4.

Show that, with N_2 as in the bivariate normal distribution,

$$\begin{aligned} CC(0) &= S(0)N_2(d_+T_1, S^*), d_+(T_2, K_2); \rho) \\ &\quad -K_2\mathrm{e}^{-rT_2}N_2(d_-T_1, S^*), d_-(T_2, K_2); \rho) \\ &\quad -K_1\mathrm{e}^{-rT_1}N_1(d_-(T_1, S^*(T_1)), \end{aligned}$$

where $\rho = \sqrt{\frac{T_1}{T_2}}$ and S^* is the solution of the equation $u(T_1, z) = S(T_1)$.

Solution.
We have found that

$$\begin{aligned} CC(0) &= S(0)\int_{x_1-\sigma\sqrt{T_1}}^{\infty}\int_{x_2-\sigma\sqrt{T_2}}^{\infty} f_{X_1X_2}(x,y)dxdy \\ &\quad -e^{-rT_2}K_2\int_{x_1}^{\infty}\int_{x_2}^{\infty} f_{X_1X_2}(x,y)dxdy \\ &\quad -e^{-rT_1}K_1(1-N(x_1)). \end{aligned}$$

First, recall that $f_{X_1,X_2}(x,y)$ remains unchanged when we replace x by $x' = -x$ and y by $y' = -y$, and therefore for any real a, b we find

$$\begin{aligned} \int_a^{\infty}\int_b^{\infty} f_{X_1X_2}(x,y)dxdy &= \int_{\{x\geq a\}}\int_{\{y\geq b\}} f_{X_1X_2}(x,y)dxdy \\ &= \int_{\{x'\leq -a\}}\int_{\{y'\leq -b\}} f_{X_1X_2}(x',y')dx'dy' \\ &= N_2(-a,-b;\rho). \end{aligned}$$

Apply this with $a = x_1, b = x_2$ to obtain $e^{-rT_2}K_2N_2(-x_1,-x_2;\rho)$ for the second term above, and $a = x_1 - \sigma\sqrt{T_1}, b = x_2 - \sigma\sqrt{T}_2$ yields $S(0)N_2(-x_1 + \sigma\sqrt{T}_1, -x_2 + \sigma\sqrt{T_2};\rho)$ for the first. The third term is simply $e^{-rT_1}K_1N(-x_1)$ by the symmetry of the (univariate) standard normal distribution function N.

Recall that $x_1 = \phi^{-1}(K_1)$, where

$$\begin{aligned} \phi(x) &= u(T_1, S(T_1)) = S(T_1)N(d_+(T_1S(T_1)) \\ &\quad -K_2e^{-r(T_2-T_1)}N(d_-(T_1,S(T_1)). \end{aligned}$$

We write $S^*(T_1)$ for the solution of $u(T_1, S(T_1)) = K_1$, then x_1 solves the equation

$$S(0)\exp((r-\frac{1}{2}\sigma^2)T_1 + \sigma\sqrt{T}_1x) = S^*(T_1),$$

so that

$$x_1 = \frac{\ln(\frac{S^*(T_1)}{S(0)}) - (r-\frac{1}{2}\sigma^2)T_1}{\sigma\sqrt{T}_1},$$

which means that

$$-x_1 = \frac{\ln(\frac{S(0)}{S^*(T_1)}) + (r-\frac{1}{2}\sigma^2)T_1}{\sigma\sqrt{T}_1} = d_-(T_1, S^*(T_1))$$

We also have

$$\begin{aligned} -x_2 &= -\frac{\ln\frac{K_2}{S(0)} - (r-\frac{1}{2}\sigma^2)T_2}{\sigma\sqrt{T_2}} \\ &= \frac{\ln\frac{S((0)}{K_2} + (r-\frac{1}{2}\sigma^2)T_2}{\sigma\sqrt{T_2}} = d_-(T_2, K_2). \end{aligned}$$

Setting $a_2 = -x_1$, $b_2 = -x_2$ and $a_1 = a_2 + \sigma\sqrt{T}_1 = d_+(T_1, S^*(T_1)$ and $b_1 = b_2 + \sigma\sqrt{T}_2 = d_+(T_2, K_2)$.

the formula for the call-on-call can be written, with $\rho = \sqrt{\frac{T_1}{T_2}}$ and $N_1 = N$, as

$$\begin{aligned} CC(0) &= S(0)N_2(a_1, b_1; \rho) - K_2 e^{-rT_2} N_2(a_2, b_2; \rho) - K_1 e^{-rT_1} N_1(a_2) \\ &= S(0)N_2(d_+T_1, S^*(T_1)), d_+(T_2, K_2); \rho) \\ &\quad -K_2 e^{-rT_2} N_2(d_-T_1, S^*(T_1)), d_-(T_2, K_2); \rho) \\ &\quad -K_1 e^{-rT_1} N_1(d_-(T_1, S^*(T_1)). \end{aligned}$$

Exercise 4.5.

Prove that the unique solution of

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW(t),$$

where the coefficients defined on $[0, T]$ are bounded and measurable, is of the form

$$S(t) = S(0)\exp\{\int_0^t [\mu(s) - \frac{1}{2}\sigma^2(s)]ds + \int_0^t \sigma(s)dW(s)\}.$$

Solution.

A routine application of the Itô formula shows that $S(t)$ solves the equation. Uniqueness does not follow from the general uniqueness theorem proved for SDEs in section 5.2 of [SCF], despite the fact that the Lipschitz condition holds for linear equations, since random coefficients are not covered. We follow the proof given on pp 165-168 of [SCF] for constant coefficients.

Proof: Suppose S_1, S_2 are solutions, then

$$S_1(t) - S_2(t) = \int_0^t \mu(t)[S_1(u) - S_2(u)]du + \int_0^t \sigma(t)[S_1(u) - S_2(u)]dW(u)$$

and, since $(a+b)^2 \le 2a^2 + 2b^2$,

$$\begin{aligned} &(S_1(t) - S_2(t))^2 \\ &= \left(\int_0^t \mu(t)[S_1(u) - S_2(u)]du + \int_0^t \sigma(t)[S_1(u) - S_2(u)]dW(u)\right)^2 \\ &\le 2\left(\int_0^t \mu(t)[S_1(u) - S_2(u)]du\right)^2 + 2\left(\int_0^t \sigma(t)[S_1(u) - S_2(u)]dW(u)\right)^2. \end{aligned}$$

Take the expectation on both sides

$$\begin{aligned} \mathbb{E}(S_1(t) - S_2(t))^2 &\le 2\mathbb{E}\left(\int_0^t \mu(t)[S_1(u) - S_2(u)]du\right)^2 \\ &\quad +2\mathbb{E}\left(\int_0^t \sigma(t)[S_1(u) - S_2(u)]dW(u)\right)^2 \end{aligned}$$

The Itô isometry gives

$$\mathbb{E}\left(\int_0^t \sigma(t)(S_1(u)-S_2(u))dW(u)\right)^2 = \mathbb{E}\int_0^t \sigma^2(t)(S_1(u)-S_2(u))^2 du.$$

Next, exchange the order of integration in the integral on the right, which is legitimate, since we are working with a class of processes where Fubini's theorem applies. Thus if we set

$$f(t) = \mathbb{E}(S_1(t)-S_2(t))^2$$

the inequality in question takes the form

$$\begin{aligned} f(t) &\leq 2\sup_{t\in[0,T]}\{\mu(t)\}\mathbb{E}\left(\int_0^t [S_1(u)-S_2(u)]du\right)^2 + 2\sup_{t\in[0,T]}\{\sigma^2(t)\}\int_0^t f(u)du \\ &= 2\mu\mathbb{E}\left(\int_0^t [S_1(u)-S_2(u)]du\right)^2 + 2\sigma^2\int_0^t f(u)du \end{aligned}$$

say, so we can follow the rest of the proof (using the Gronwall Lemma (Lemma 5.4 in [SCF]) without change.

Exercise 4.6.
Show that

$$M(t) = \exp\{-\frac{1}{2}\int_0^t b^2(s)ds - \int_0^t b(s)dW(s)\}.$$

is a martingale.

Solution.
This process is called the exponential martingale. By the Itô formula with $F(x)=e^x$, $X(t) = -\frac{1}{2}\int_0^t b^2(s)ds - \int_0^t b(s)dW(s)$, so that $dX(t) = -\frac{1}{2}b^2(t)dt - b(t)dW(t)$ we have

$$\begin{aligned} dM(t) &= -F_x(X(t))\frac{1}{2}b^2(t)dt - F_x(X(t))b(t)dW(t) + \frac{1}{2}F_{xx}(X(t)b^2(t)dt \\ &= -M(t)b(t)dW(t). \end{aligned}$$

The problem boils down to showing that $M \in \mathcal{M}^2$, since b is bounded and the same is true for the product. Since $\mu(t) - r = \sigma(t)b(t)$, b is deterministic (μ and σ are assumed deterministic) so $\int_0^T b(s)dW(s)$ is Gaussian and $\mathbb{E}(\exp\{\int_0^t b(s)dW(s)\}) = \exp\{\frac{1}{2}\int_0^t b^2(s)ds\}$ which is square-integrable over $[0,T]$.

Exercise 4.7.
Prove that the discounted stock prices follow a martingale and

$$S(t) = S(0)\exp\{rt - \int_0^t \frac{1}{2}\sigma^2(s)ds + \int_0^t \sigma(s)dW_Q(s)\}$$

where $W_Q(t) = W(t) + \int_0^t b(s)ds$, $\mu(t) - r = \sigma(t)b(t)$.

Solution.
The formula for the discounted prices is

$$\tilde{S}(t) = S(0)\exp\{-\int_0^t \frac{1}{2}\sigma^2(s)ds + \int_0^t \sigma(s)dW_Q(s)\}$$

which is the familiar exponential martingale provided σ is sufficiently smooth. For instance, boundedness is sufficient (σ is assumed deterministic in this section). An alternative is the *Noviko*v condition $\mathbb{E}[\exp(\int_0^T \sigma^2(s)ds)] < \infty$, the proof of which is beyond the scope of this text.

Exercise 4.8.
Find a PDE for the function $u(t, z)$ generating the option pricess by the formula $H(t) = u(t, S(t))$ for $H = h(S(T))$ for time dependent volatility.

Solution.
This is a straightforward generalisation of the Black-Scholes setting, and we omit the details. The equation is

$$u_t(t, z) = -\frac{1}{2}\sigma^2(t)z^2 u_{zz}(t, z) - r(t)zu_x(t, z) + r(t)u(t, z) \quad \text{for } 0 < t < T,\ z \in \mathbb{R}.$$

Exercise 4.9.
Show that benchmarked pricing of plain vanilla options gives the well-known Black-Scholes formula.

Solution.
This is immediate since the argument deriving benchmarked prices was based on risk-neutral valuation, which leads to Black-Scholes formula. A dIrect argument is also possible. For $t = 0$, for call

$$\begin{aligned} C(0) &= \mathbb{E}(\exp(-aT - bW(T))(S(T) - K)^+) \\ &= \exp\{-rT - \frac{1}{2\sigma^2}(\mu - r)^2 T\}\frac{1}{\sqrt{2\pi}} \\ &\quad \int_d^\infty \exp\{-\frac{\mu - r}{\sigma}y\sqrt{T}\}(S(0)\exp\{\mu T + \sigma y\sqrt{T}\} - K)\exp\{-\frac{1}{2}y^2\}dy \end{aligned}$$

and some elementary, though somewhat tedious, calculus gives the result.

Chapter 5

Exercise 5.1.
Examine the case where $P_{\text{UO}}(0) = P(0)$ (ordinary put).
Solution.
The ordinary put payoff is the limit of

$$P_{\text{UO}}(T) = (K - S(T))^{+}\mathbf{1}_{\{\max_{t\in[0,T]} S(t)\leq L\}}$$

as $L \to \infty$. The ingredients of the pricing formula also converge

$$d_2 = \frac{\ln \frac{S(0)K}{L^2} - \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \to -\infty,$$

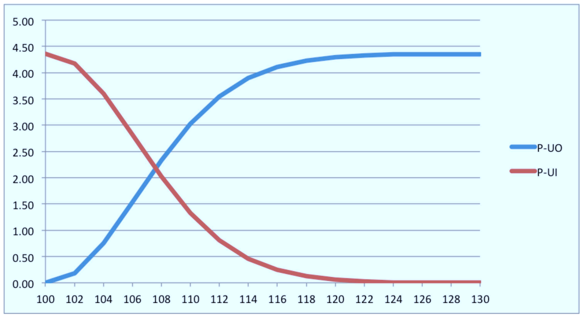
and

$$d_4 = \frac{\ln \frac{S(0)K}{L^2} - \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \to -\infty$$

as $L \to \infty$, hence $N(d_2) \to 0$, $N(d_4) \to 0$. The factors $\left(\frac{L}{S(0)}\right)^{2\frac{r}{\sigma^2}-1}$ and $\left(\frac{L}{S(0)}\right)^{2\frac{r}{\sigma^2}+1}$ tend to infinity but slower that $N(d_2)$, $N(d_{4)}go$ to zero so in the limit the second and the forth term in the formula for $P_{\text{UO}}(0)$ disappear and we end up with the ingredients of the Black-Scholes formula for $P(0)$.

Exercise 5.2.
Consider $S(0) = 100$, $K = 100$, $T = 0.25$, $r = 5\%$, $\sigma = 25\%$. Is it possible to find L so that $P_{\text{UO}}(0) = \frac{1}{2}P(0)$?
Solution.
Since $P_{\text{UO}}(0) = 0$ if $L = 100$ and converges to $P(0)$ as $L \to \infty$, this must be posible. For the given data we find $L = 107.59026$.

Exercise 5.3.
Sketch the graphs of $P_{\text{UO}}(0)$ and $P_{\text{UI}}(0)$ as functions of L.
Solution.



Exercise 5.4.
Compute the initial price of an up-and-in put option with price K and barrier L on a stock S.

Solution.

$$P_{\mathrm{UO}}(0) + P_{\mathrm{UI}}(0) = \mathrm{e}^{-rT}\mathbb{E}_Q((K - S(T))^+).$$

Exercise 5.5.
Prove that $P_{\mathrm{UO}}(0) = u(0, S(0))$ where u solves

$$\begin{aligned} u_t + rzu_z + \frac{1}{2}\sigma^2 z^2 u_{zz} &= ru \\ u(T,z) &= (K-z)^+ \quad \text{for all } 0 < x \le L, \\ u(t,L) &= 0 \quad \text{for all } 0 \le t < T. \end{aligned}$$

Solution.
Writing

$$d_1(t,z) = \frac{\ln\frac{K}{z} - \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2(t,z) = \frac{\ln\frac{zK}{L^2} - \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_3(t,z) = \frac{\ln\frac{K}{z} - \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \quad d_4(t,z) = \frac{\ln\frac{zK}{L^2} - \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}},$$

$$\begin{aligned} u(t,z) &= \mathrm{e}^{-r(T-t)}K\left[N(d_1(t,z)) - \left(\frac{L}{z}\right)^{2\frac{r}{\sigma^2}-1} N(d_2(t,z))\right] \\ &\quad -z\left[N(d_3(t,z)) - \left(\frac{L}{z}\right)^{2\frac{r}{\sigma^2}+1} N(d_4(t,z))\right], \end{aligned}$$

the problem boils down to checking that this function solves the PDE.

Exercise 5.6.

Prove that this strategy is admissible and replicates the payoff of Up-and-Out put.

Solution.

The proof for the vanilla options can be repeated. The choice of $x(t), y(t)$ guarantees that

$$V_{(x,y)}(t) = x(t)S(t) + y(t)A(t) = u(t, S(t)).$$

We know that $u(0,S(0)) = P_{\mathrm{UO}}(0)$ and by the same token assuming that t is the initial price, $u(0,S(t)) = P_{\mathrm{UO}}(t)$ arguing and in particular $V_{(x,y)}(T) = u(T,S(T)) = P_{\mathrm{UO}}(T)$. The valueas are non-negative which implies the first condition for admissibility. Since $u(t,S(t)) = P_{\mathrm{UO}}(t)$ and $\tilde{P}_{\mathrm{UO}}(t) = \mathbb{E}_Q(\tilde{P}_{\mathrm{UO}}(T)|\mathcal{F}_t)$ is a martingale, it follows that $\tilde{V}_{(x,y)}(t)$ is a martingale. The self-financing property of (x,y) follows from the relation (3.12) proved in Lemma 3.23

$$\begin{aligned} d\tilde{V}_{(x,y)}(t) &= d\left[\mathrm{e}^{-rt}u(t,S(t))\right] \\ &= \mathrm{e}^{-rt}\sigma S(t)u_z(t,S(t))dW_Q(t) \\ &= x(t)d\tilde{S}_t, \end{aligned}$$

which is equivalent to the self-financing property, as we saw in Proposition 2.9.

Exercise 5.7.

Show that the joint density of $(Y(T), M^Y(T))$ is given by

$$f^{Y,M^Y}(b,c) = \frac{2(2c-b)}{T\sqrt{T}} n(\frac{2c-b)}{\sqrt{T}}) \exp(\nu b - \frac{1}{2}\nu^2 T).$$

Hence find the joint density of $(Z(T), M^Z(T))$, where $Z(t) = \sigma Y(t)$ on $[0,T]$ and use it to show that the premium of the lookback put is

$$P_L(0) = S(0)(N(-d) + \mathrm{e}^{-rT} N(-d+\sqrt{T}) + \frac{\sigma^2}{2r}\mathrm{e}^{-rT}[-N(d - \frac{2r}{\sigma}\sqrt{T}) + \mathrm{e}^{-rT}N(d)]$$

where

$$d = \frac{2r+\sigma^2}{2\sigma\sqrt{T}}.$$

Solution.

On page 115 ν was defined as $\nu = \frac{1}{\sigma}(r - \frac{1}{2}\sigma^2)$. We also know that $Y(t) = \nu t + W^Q(t)$ is a Wiener process under the equivalent probability R defined at (5.3), with $\frac{dR}{dQ}|_{\mathcal{F}(T)} = \exp(-\nu W^Q(T) - \frac{1}{2}\nu^2 T)$. So

$$F^{Y,M^Y}(b,c) = \mathbb{E}_Q[\mathbf{1}_A] = \mathbb{E}_R[e^{\nu Y(T) - \frac{1}{2}\nu^2 T})\mathbf{1}_A].$$

Now recall from the proof of Proposition 5.4 that for a standard Wiener process W the joint distribution of W and its maximum is given by

$$F(b,c) = N(\frac{b}{\sqrt{T}}) - N(\frac{b-2c}{\sqrt{T}}),$$

and the joint density by $f(b,c) = \frac{2}{T}n'(\frac{b-2c}{\sqrt{T}})$. For later reference, note that by definition of n we can also write this as

$$f(b,c) = \frac{2(2c-b)}{T\sqrt{T}} n(\frac{b-2c)}{\sqrt{T}}).$$

Applying this to W^Q under the probability R, employing the Fubini theorem and noting that

$$\int_0^c \frac{2}{T} n'(\frac{x-2y)}{\sqrt{T}} dy = \frac{1}{\sqrt{T}}[n(\frac{x}{\sqrt{T}}) - n(\frac{x-2c}{\sqrt{T}})],$$

we obtain

$$\begin{aligned}
F^{Y,M^Y}(b,c) &= \int_0^c [\int_{-\infty}^b \exp(\nu x - \frac{1}{2}\nu^2 T) f(x,y) dx] dy \\
&= \int_{-\infty}^b \exp(\nu x - \frac{1}{2}\nu^2 T)(\int_0^c f(x,y)dy)dx \\
&= \int_{-\infty}^b \exp(\nu x - \frac{1}{2}\nu^2 T)\frac{1}{\sqrt{T}}(n(\frac{x}{\sqrt{T}}) - n(\frac{x-2c}{\sqrt{T}})dx \\
&= \exp(\nu b - \frac{1}{2}\nu^2 T)\frac{1}{\sqrt{T}}\int_{-\infty}^0 \exp(\nu z)[n(\frac{z+b}{\sqrt{T}}) - n(\frac{z+(b-2c)}{\sqrt{T}})]dz.
\end{aligned}$$

Now for any a,

$$
\begin{aligned}
\frac{1}{\sqrt{T}}\int_{-\infty}^{0}\exp(\nu z)n(\frac{z+a}{\sqrt{T}})dz &= \frac{1}{\sqrt{2\pi T}}\int_{-\infty}^{0}\exp(\nu z-\frac{1}{2}(\frac{z+a}{T})^2)dz \\
&= \exp(-\nu a+\frac{1}{2}\nu^2 T)\int_{-\infty}^{0}\frac{1}{\sqrt{T}}n(\frac{z+a-\nu T}{\sqrt{T}})dz \\
&= [\exp(-\nu a+\frac{1}{2}\nu^2 T)]N(\frac{a-\nu T}{\sqrt{T}})
\end{aligned}
$$

(after adding and subtracting $\exp(-\nu a+\frac{1}{2}\nu^2 T)$ and completing the square in the penultimate step).

Using this with $a=b$ and $a=b-2c$ in the calculation of the joint distribution we have

$$
\begin{aligned}
F^{Y,M^Y}(b,c) &= N(\frac{b-\nu T}{\sqrt{T}}) \\
&\quad -\{\exp(\nu b-\frac{1}{2}\nu^2 T)\exp(-\nu(b-2c)+\frac{1}{2}\nu^2 T)\}N(\frac{b-2c-\nu T}{\sqrt{T}})\} \\
&= N(\frac{b-\nu T}{\sqrt{T}})-e^{2c\nu}N(\frac{b-2c-\nu T}{\sqrt{T}}).
\end{aligned}
$$

Differentiating with respect to b and c yields the desired density:

$$f^{Y,M^Y}(b,c)=\frac{2(2c-b)}{T\sqrt{T}}n(\frac{2c-b}{\sqrt{T}})\exp(\nu b-\frac{1}{2}\nu^2 T).$$

We need to consider $Z(t)=\sigma Y(t)=(r-\frac{1}{2}\sigma^2)t+\sigma W^Q(t)$ and its maximum process M^Z, since the premium of the lookback option is

$$P_L(0)=e^{-rT}S(0)\mathbb{E}_Q[e^{M^Z(T)}-e^{rT}].$$

We need the joint density of $(Z(T),M^Z(T))$. This can be found from the above density for $(Y(T).M^Y(T))$, since $\sigma>0$, so that (where we set $\zeta=r-\frac{1}{2}\sigma^2$ to ease the notation)

$$
\begin{aligned}
P(Z(T) &< b, M^Z(T)<c)=P(Y(T)<\frac{b}{\sigma},M^Y(T)<\frac{c}{\sigma}) \\
&= N(\frac{b-\zeta T}{\sigma\sqrt{T}})-\exp(\frac{2c\zeta}{\sigma^2})N(\frac{b-2c-\zeta T}{\sigma\sqrt{T}}).
\end{aligned}
$$

Again we can find the density by differentiation:

$$f^{Z,M^Z}(b,c)=\frac{2(2c-b)}{\sigma T\sqrt{T}}n(\frac{2c-b)}{\sqrt{T}})\exp(\frac{\zeta b-\frac{1}{2}\zeta^2 T}{\sigma^2}).$$

The density of the maximum M^Z is now found by integrating the joint density over b, to obtain

$$
\begin{aligned}
f^{M^Z}(c) &= \int_{-\infty}^{\infty}f^{Z,M^Z}(b,c)db \\
&= N(\frac{c-\zeta T}{\sigma\sqrt{T}})-\frac{2\zeta}{\sigma^2}\exp(\frac{2\zeta c}{\sigma^2})N(\frac{-c-\zeta T}{\sigma\sqrt{T}})+\exp(\frac{2\zeta c}{\sigma^2})n((\frac{c+\zeta T}{\sigma\sqrt{T}}).
\end{aligned}
$$

Finally,

$$P_L(0) = S(0)(e^{-rT}\int_{-\infty}^{\infty} f^M(c)dc - 1)$$

which can be found by completing the square and integrating, so that, with $d = \frac{2r+\sigma^2}{2\sigma\sqrt{Y}}$, we obtain

$$P_L(0) = S(0)[N(-d)+e^{-rT}N(-d+\sigma\sqrt{T})+\frac{\sigma^2}{2r}e^{-rT}\{-N(d-\frac{2r}{\sigma}\sqrt{T})+e^{-rT}N(d)\}.$$

Exercise 5.8.

Investigate numerically the distance between geometric and arithmetic averages of daily stock prices.

Solution.

For 100 daily steps, some prices simulated with $\mu = 10\%$, $\sigma = 30\%$, $A_{\text{geom}} = 107.5783$, $A_{\text{arithm}} = 107.809$

Exercise 5.9.

Compare the cost of a series of 10 calls for a single share to be exercised over next 10 weeks, with the cost of 10 Asian integral geometric average calls

Solution.

For $\mu = 10\%$, $\sigma = 30\%$, $S(0) = 100$, $K = 100$, the sum of prices of 10 calls to be exercises at $n/52$, $n = 1, \ldots, 10$, is 46.0356. Single Asian integral geometric average call has price 0.8484.

Exercise 5.10.

Compare the cost of a series of 10 calls for a single share to be exercised over next 10 weeks, with the cost of 10 Asian discrete geometric average calls

Solution.

Here with $n = 10$, 10 Asian calls cost 39.4585

Chapter 6

Exercise 6.1.
Show that $aW_1 + bW_2$ is a Wiener process if and only if $a^2 + b^2 = 1$.

Solution.
Write $W(t) = aW_1 + bW_2$. If W is Wiener, it has mean 0, so that its variance is $\mathbb{E}(W^2(t)) = t$. By independence,

$$\mathbb{E}((aW_1 + bW_2)^2) = a^2\mathbb{E}(W_1^2(t)) + b^2\mathbb{E}(W_2^2(t)) = t(a^2 + b^2)$$

which implies that $a^2 + b^2 = 1$ Conversely, if $a^2 + b^2 = 1$, the fact that W is a Wiener process can be proved simply by checking that it satisfies Definition 2.4 in [SCF].

Exercise 6.2.
Prove carefully that $\mathcal{F}_t^{(W_1,W_2)} = \mathcal{F}_t^{(S_1,S_2)}$ if C is invertible.

Solution.
In this case we can invert the relation

$$S_i(t) = S_i(0)\exp\{\mu_i t - \frac{c_{i1}^2 + c_{i2}^2}{2}t + c_{i1}W_1(t) + c_{i2}W_2(t)\}$$

expressing $(W_1(t), W_2(t)) = f(S_1(t), S_2(t))$ where f is continuous (f depends on t). So the fields generated by the vectors $(W_1(t), W_2(t))$ and $(S_1(t), S_2(t))$ coincide for each t. This imples the claim since

$$\begin{aligned}
\mathcal{F}_t^{(W_1,W_2)} &= \sigma(\bigcup_{s\leq t}\mathcal{F}_{(W_1(s),W_2(s))}), \\
\mathcal{F}_t^{(S_1,S_2)} &= \sigma(\bigcup_{s\leq t}\mathcal{F}_{(S_1(s),S_2(s))}).
\end{aligned}$$

Exercise 6.3.
Find the correlation coefficient for $W_1'(t)$ and $W_2'(t)$.

Solution.
We compute the covariance between $W_1'(t)$ and $W_2'(t)$:

$$\begin{aligned}
&\mathrm{Cov}(\frac{c_{11}}{\sqrt{c_{11}^2 + c_{12}^2}}W_1(t) + \frac{c_{12}}{\sqrt{c_{11}^2 + c_{12}^2}}W_2(t), \frac{c_{21}}{\sqrt{c_{21}^2 + c_{22}^2}}W_1(t) + \frac{c_{22}}{\sqrt{c_{21}^2 + c_{22}^2}}W_2(t) \\
= \; & t(\frac{c_{11}}{\sqrt{c_{11}^2 + c_{12}^2}}\frac{c_{21}}{\sqrt{c_{21}^2 + c_{22}^2}} + \frac{c_{12}}{\sqrt{c_{11}^2 + c_{12}^2}}\frac{c_{22}}{\sqrt{c_{21}^2 + c_{22}^2}})
\end{aligned}$$

and the correlation is

$$\frac{c_{11}c_{21} + c_{21}c_{22}}{\sqrt{c_{11}^2 + c_{12}^2}\sqrt{c_{21}^2 + c_{22}^2}}.$$

Exercise 6.4.

Suppose that W_1, W_2 are independent Wiener processes. Show that $\rho \in [-1,1]$ is the correlation coefficient between the random variables $W_1(t)$ and $\rho W_1(t) + \sqrt{1-\rho^2} W_2(t)$ for any t.

Solution.

We compute the covariance: by bilinearity and independence

$$\begin{aligned} & \mathrm{Cov}(W_1(t), \rho W_1(t) + \sqrt{1-\rho^2} W_2(t)) \\ = \; & \rho \mathrm{Cov}(W_1(t), W_1(t)) + \sqrt{1-\rho^2} \mathrm{Cov}(W_1(t), W_2(t)) \\ = \; & \rho t \end{aligned}$$

as claimed. An alternative (direct) argument: Since W_1, W_2 have mean 0,

$$\begin{aligned} \mathrm{Cov}(W_1(t), \rho W_1(t) + \sqrt{1-\rho^2} W_2(t)) &= \mathbb{E}[W_1(t)\{\rho W_1(t) + \sqrt{1-\rho^2} W_2(t)\}] \\ &= \rho \mathbb{E}[W_1^2(t)] + \sqrt{1-\rho^2} \mathbb{E}\{W_1(t) W_2(t) \\ &= \rho t \end{aligned}$$

since $\mathbb{E}[W_1^2(t)] = t$ and the second term is 0 by independence.

To find the correlation, note that, similarly,

$$\mathbb{E}[\{\rho W_1(t) + \sqrt{1-\rho^2} W_2(t)\}^2] = \rho^2 t + (1-\rho^2) t = t,$$

so that

$$\mathrm{Corr}(W_1(t), \rho W_1(t) + \sqrt{1-\rho^2} W_2(t) = \frac{\rho t}{\sqrt{t}\sqrt{t}} = \rho,$$

as claimed.

Alternatively one could use the result of the previous exercise.

Exercise 6.5.

Given $V(0)$ and $x_i(t)$, $i = 1, 2$, find $y(t)$ so that the strategy is self-financing.

Solution.

First we generalise the equivalent formulation of the self-financing property by means of the discounted wealth process. Namely, $(x_1(t), x_2(t), y(t))$ is self-financing if and only if

$$d\tilde{V}(t) = x_1(t) d\tilde{S}_1(t) + x_2(t) d\tilde{S}_2(t)$$

with the same proof as for one asset (Proposition 2.9). Then, following Corollary 2.10, we have

$$y(t) = \frac{1}{A(t)} \left(\mathrm{e}^{rt}[V(0) + \int_0^t x_1(u) d\tilde{S}_1(u) + \int_0^t x_2(u) d\tilde{S}_2(u)] - x(t) S(t) \right).$$

Exercise 6.6.

Prove that if $x_1(t) = \frac{w_1 V(t)}{S_1(t)}$, $x_2(t) = \frac{w_2 V(t)}{S_2(t)}$ is self-financing then

$$\begin{aligned} dV(t) &= [w_1\mu_1 + w_2\mu_2]\, V(t)dt \\ &\quad + [w_1\sigma_{11} + w_2\sigma_{21}]\, V(t)dW_1(t) + [w_1\sigma_{12} + w_2\sigma_{22}]\, V(t)dW_2(t) \end{aligned}$$

Solution.
Direct substitution of the differentials and cancellations give

$$\begin{aligned} dV(t) &= x_1(t)dS_1(t) + x_2(t)dS_2(t) + y(t)dA(t) \\ &= \frac{w_1 V(t)}{S_1(t)}[\mu_1 S_1(t)dt + c_{11}S_1(t)dW_1(t) + c_{12}S_1(t)dW_2(t)] \\ &\quad + \frac{w_2 V(t)}{S_2(t)}[\mu_2 S_2(t)dt + c_{21}S_1(t)dW_1(t) + c_{22}S_2(t)dW_2(t)] \\ &\quad + y(t)rA(t)dt \\ &= w_1\mu_1 V(t)dt + w_2\mu_2 V(t)dt \\ &\quad + c_{11}w_1 V(t)dW_1(t) + c_{21}w_2 V(t)dW_1(t) \\ &\quad + c_{12}w_1 V(t)dW_2(t) + c_{22}w_2 V(t)dW_2(t). \end{aligned}$$

Conversely, inserting $w_i V(t) = x_i(t)S_i(t)$ gives the self-financing condition.

Note that since w_i are constant, the components of the strategy are Ito processes.

Exercise 6.7.
Prove that each discounted stock price process S_i $(i = 1, 2)$ is a martingale with respect to Q (Hint: use independence and Proposition 6.1).

Solution.
Recall, that for $i = 1, 2$, for each $s \leq t$, we have $\mathbb{E}(W_i(t)|\mathcal{F}_s^{(W_1,W_2)}) = W_i(s)$, and $S_i(t) = S_i(0)\exp\{rt - \frac{c_{i1}^2 + c_{i2}^2}{2}t + c_{i1}W_1^Q(t) + c_{i2}W_2^Q(t)\}$, so that

$$\begin{aligned} \mathbb{E}(\tilde{S}_i(t)|\mathcal{F}_s^{(W_1,W_2)}) &= S_i(0)\mathbb{E}(e^{-\frac{1}{2}c_{i1}^2 t + c_{i1}W_1^Q(t)} e^{-\frac{1}{2}c_{i2}^2 t + c_{i2}W_2^Q(t)} |\mathcal{F}_s^{(W_1,W_2)}) \\ &= S_i(0)\mathbb{E}(e^{-\frac{1}{2}c_{i2}^2 t + c_{i2}W_2^Q(t)}|\mathcal{F}_s^{(W_1,W_2)})\mathbb{E}(e^{-\frac{1}{2}c_{i1}^2 t + c_{i1}W_1^Q(t)}|\mathcal{F}_s^{(W_1,W_2)}) \\ &= S_i(0)e^{-\frac{1}{2}c_{i2}^2 s + c_{i2}W_2^Q(s)} e^{-\frac{1}{2}c_{i1}^2 s + c_{i1}W_1^Q(s)} \\ &= \tilde{S}_i(s). \end{aligned}$$

Exercise 6.8.
Show that under Q the process of discounted values of a strategy is a martingale.

Solution.
Since $d\tilde{V}(t) = x_1(t)d\tilde{S}_1(t) + x_2(t)d\tilde{S}_2(t)$ and $d\tilde{S}_1(t) = c_{i2}\tilde{S}_i(t)dW_1^Q(t) + c_{i2}\tilde{S}_i(t)dW_2^Q(t)$, the local martingale property follows but due to the form of the stock prices, these processes are square integrable so $\tilde{V}$ is a martingale.

Exercise 6.9.
Prove that $M^Q(t)$ is not a martingale with respect to $\mathcal{F}_t^{(W_1,W_2)}$ and Q is not risk-neutral with respect to $\mathcal{F}_t^{(W_1,W_2)}$

Solution.
It seems that the claim in the book is incorrect as it stands. In fact,

$$M^Q(T) = \exp\{-\frac{1}{2}\frac{(\mu-r)^2}{c_1^2+c_2^2}T-(\mu-r)\frac{c_1}{\sqrt{c_1^2+c_2^2}}W_1(t)+(\mu-r)\frac{c_2}{\sqrt{c_1^2+c_2^2}}W_2(t)\}$$

is a martingale for the larger filtration, due to the independence of the Wiener processes. However, the risk -neutral property does not hold. Under Q the process $\frac{\mu-r}{\sigma}t+W'(t)$, $\sigma=\sqrt{c_1^2+c_2^2}$, is a Wiener process but this does not imply that the components are: $W'(t) = \frac{c_1}{\sqrt{c_1^2+c_2^2}}W_1(t)+\frac{c_2}{\sqrt{c_1^2+c_2^2}}W_2(t)$ and neither $\frac{c_1}{\sqrt{c_1^2+c_2^2}}\frac{\mu-r}{\sigma}t+\frac{c_1}{\sqrt{c_1^2+c_2^2}}W_1(t)$ nor $\frac{c_2}{\sqrt{c_1^2+c_2^2}}\frac{\mu-r}{\sigma}t+\frac{c_2}{\sqrt{c_1^2+c_2^2}}W_2(t)$ are Wiener processes under Q.

Exercise 6.10.
The random variable $H = W_1(T)$ is not replicable since $x(T)S(T)+y(T)A(T)$ is not $\mathcal{F}_T^{W_1}$-measurable.

Solution.
The random variable $S(T)$ involves $W_2(T)$ which is not $\mathcal{F}_T^{W_1}$-measurable.

Exercise 6.11.
(Corrected formulation; the printed version has t as the variable of integration and as the upper limit of integration.)
Prove that the processes

$$S_i(t) = S_i(0)\exp\{\int_0^t \mu_i(s)ds - \frac{1}{2}\sum_{j,l=1}^{d}\int_0^t \sigma_{ij}(s)\sigma_{lj}(s)ds+\sum_{j=1}^{d}\int_0^t \sigma_{ij}(s)dW_j(s)\}$$

solve (6.3), where $i=1,\ldots,d$.

Solution.
For fixed i the one-dimensional Ito formula does the trick with

$$X(t) = X_i(t) = \int_0^t \mu_i(s)ds - \frac{1}{2}\sum_{j,l=1}^{d}\int_0^t \sigma_{ij}(s)\sigma_{lj}ds+\sum_{j=1}^{d}\int_0^t \sigma_{ij}(s)dW_j(s)$$

and $F(t,x)=e^x$.

Exercise 6.12.
Prove that $[X,Y](t) = \int_0^t b_X(s)b_Y(s)ds$.

Solution.
This follows from the parallelogram identity

$$[X,Y](t) = \frac{1}{4}[X+Y,X+Y](t) - \frac{1}{4}[X-Y,X-Y](t)$$

and the fact that $[X,X](t) = \int_0^t b_X^2(s)ds$, since on the right we have

$$\frac{1}{4}\int_0^t (b_X + b_Y)^2(s)ds - \frac{1}{4}\int_0^t (b_X - b_Y)^2(s)ds$$

and a bit of algebra does the trick.

Exercise 6.13.
Prove that $X(t)Y(t) - [X,Y](t)$ is a martingale.

Solution.
Using $ab = \frac{1}{4}(a+b)^2 - \frac{1}{4}(a-b)^2$ and $[X,Y](t) = \frac{1}{4}[X+Y,X+Y](t) - \frac{1}{4}[X-Y,X-Y](t)$ we find that the process takes the form

$$\frac{1}{4}(X(t)+Y(t))^2 - \frac{1}{4}(X(t)-Y(t))^2 - \frac{1}{4}[X+Y,X+Y](t) + \frac{1}{4}[X-Y,X-Y](t)$$

which is the sum of two martingales.

Exercise 6.14.
(Corrected formulation: in the printed version b_{21} and b_{12} were interchanged in error)
Prove the following formula: $[Y_1,Y_2](t) = \int_0^t (b_{11}(s)b_{12}(s) + b_{21}(s)b_{22}(s))\,ds$, where $Y_k(t) = \int_0^t b_{1k}(s)dW_1(s) + \int_0^t b_{2k}(s)dW_2(s)$.

Solution.
Inserting we have

$$[Y_1,Y_2](t) = [\int_0^t b_{11}(s)dW_1(s) + \int_0^t b_{21}(s)dW_2(s), \int_0^t b_{12}(s)dW_1(s) + \int_0^t b_{22}(s)dW_2(s)](t)$$

so that we get four terms on the right, with

$$\begin{aligned}
& [\int_0^t b_{11}(s)dW_1(s), \int_0^t b_{12}(s)dW_1(s)](t) \\
= \; & \frac{1}{4}[\int_0^t [b_{11}(s)+b_{12}(s)]dW_1(s), \int_0^t [b_{11}(s)+b_{12}(s)]dW_1(s)](t) \\
& -\frac{1}{4}[\int_0^t [b_{11}(s)-b_{12}(s)]dW_1(s), \int_0^t [b_{11}(s)-b_{12}(s)]dW_1(s)](t) \\
= \; & \frac{1}{4}\int_0^t [b_{11}(s)+b_{12}(s)]^2 ds - \frac{1}{4}\int_0^t [b_{11}(s)-b_{12}(s)]^2 ds \\
= \; & \int_0^t (b_{11}(s)b_{12}(s) + b_{21}(s)b_{22}(s))\,ds
\end{aligned}$$

and the same for the term with W_2, so it remains to show that

$$[\int_0^t b_{21}(s)dW_2(s), \int_0^t b_{12}(s)dW_1(s)](t) = 0.$$

One can show that the expectation of the square is zero by approximating the stochastic integral. The square of approximating sums will involve terms of the form

$$\mathbb{E}(b_{21}(t_i)[W_1(t_{i+1})-W_1(t_i)]b_{21}(t_j)[W_1(t_{j+1})-W_1(t_j)]b_{12}(t_k)[W_2(t_{k+1})-W_1(t_k)]b_{12}(t_n)[W_1(t_{n+1})-W_1(t_n)]).$$

A number of cases have to be considered. The extreme case is $i = j = k = n$ and conditioning on $\mathcal{F}_{t_i}$ and using the independence of W_1 and W_2, we can estimate such a term with $(t_{i+1} - t_i)^2$ and the sum can be shown to go to zero. If $i = j < k = n$ then we condition upon $\mathcal{F}_{t_k}$ and get

$$(t_{k+1} - t_k)\mathbb{E}(b^2_{21}(t_i)[W_1(t_{i+1}) - W_1(t_i)]^2 b^2_{12}(t_k)).$$

The sum of random variables under expectation is finite since the b's are bounded and the Wiener process has finite quadratic variation. The remaining cases are easier to handle with the expectation being simply zero at the other extreme, when $i < j < k < n$.

Exercise 6.15.
Verify the uniqueness of Itô process characteristics, i.e. prove that $X_1 = X_2$ implies $a_1 = a_2$, $b_{11} = b_{21}$, $b_{12} = b_{22}$ by applying the Itô formula to find the form of $(X_1(t) - X_2(t))^2$

Solution.
On the one hand, we write $X(t) = (X_1(t) - X_2(t))^2$ and apply the Ito formula: $F(x_1, x_2) = (x_1-x_2)^2$, $F_{x_1} = 2(x_1-x_2)$, $F_{x_2} = -2(x_1-x_2)$, $F_{x_1x_1} = 2$, $F_{x_1x_2} = F_{x_2x_1} - 2$, $F_{x_2x_2} = 2$, and

$$\begin{aligned} dX(t) &= 2X(t)[a_1(t) - a_2(t)]dt \\ &\quad +2X(t)[b_{11}(t) - b_{21}(t)]dW_1(t) + 2X(t)[b_{12}(t) - b_{22}(t)]dW_2(t) \\ &\quad +[b^2_{11}(t) - b_{11}(t)b_{21}(t) + b^2_{21}(t)]dt + [b^2_{12}(t) - b_{12}(t)b_{22}(t) + b^2_{22}(t)]dt \end{aligned}$$

On the other hand, $X(t) = 0$ so $dX(t) = 0$ and by the uniqueness of Ito decomposition we confirm our claim.

Exercise 6.16.
This is identical to Exercise 6.11

Solution. As in Exercise 6.11

Exercise 6.17.

Prove that

$$dS_i(t) = rS_i(t)dt + \sum_{j=1}^{d} c_{ij}(t)S_i(t)dW_j^Q(t), \quad i = 1, \ldots, d.$$

Solution.
Since $W_i^Q(t) = \int_0^t \theta_i(s)ds + W_i(t)$, we have $dW_i^Q(t) = \theta_i(t)dt + dW_i(t)$ so

$$\begin{aligned} & rS_i(t)dt + \sum_{j=1}^{d} c_{ij}(t)S_i(t)dW_j^Q(t) \\ = \quad & rS_i(t)dt + \sum_{j=1}^{d} c_{ij}(t)S_i(t)\theta_i(t)dt + \sum_{j=1}^{d} c_{ij}(t)S_i(t)dW_i(t) \\ = \quad & rS_i(t)dt + [\mu_i(t) - r]S_i(t)dt + \sum_{j=1}^{d} c_{ij}(t)S_i(t)dW_i(t) \\ = \quad & \mu_i(t)S_i(t)dt + \sum_{j=1}^{d} c_{ij}(t)S_i(t)dW_i(t) \end{aligned}$$

which ithe same as $dS_i(t)$.

Exercise 6.18.
Derive the equation for the process $Y(t) = \frac{S_2(t)}{S_1(t)}$ and find the explicit formula for the exchange option.

Solution.
We use the Ito formula with

$$\begin{aligned} F(x_1, x_2) &= \frac{x_2}{x_1}, \\ F_{x_1}(x_1, x_2) &= -\frac{x_2}{x_1^2}, \quad F_{x_2}(x_1, x_2) = \frac{1}{x_1}, \\ F_{x_1x_1}(x_1, x_2) &= 2\frac{x_2}{x_1^3}, \quad F_{x_1x_2}(x_1, x_2) = -\frac{1}{x_1^2}, \quad F_{x_2x_2}(x_1, x_2) = 0, \end{aligned}$$

and employing (we assume constant coefficients for simplicity)

$$dS_i(t) = \mu_i S_i(t)dt + \sum_{j=1}^{d} c_{ij} S_i(t)dW_j(t), \quad i = 1, 2,$$

we find

$$\begin{aligned} dY(t) &= Y(t)[-\mu_1 + \mu_2]dt \\ & \quad Y(t)[-c_{11} + c_{21}]dW_1(t) \\ & \quad Y(t)[-c_{12} + c_{22}]dW_2(t) \\ & \quad +Y(t)[c_{11}^2 - c_{11}c_{21} + c_{12}^2 - c_{12}c_{22}]dt \\ &= \mu_Y Y(t)dt + Y(t)\sigma_1 dW_1 + Y(t)\sigma_2 dW_2(t), \quad \text{say.} \end{aligned}$$

Now introducing a single Wiener process

$$W'(t) = \frac{\sigma_1}{\sqrt{\sigma_1^2 + \sigma_2^2}} W_1(t) + \frac{\sigma_2}{\sqrt{\sigma_1^2 + \sigma_2^2}} W_2(t)$$

we have

$$dY(t) = \mu_Y Y(t) + \sigma Y(t) dW'(t)$$

where $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$. Since $H(0) = S_1(0)\mathbb{E}_{Q_f}(\max\{1-Y(T), 0\})$, we can use the Black-Scholes formula having identified the volatility of Y : we insert the initial value is $\frac{S_2(0)}{S_1(0)}$, the strike $K = 1$, and volatility $\sigma = \sqrt{(c_{21} - c_{11})^2 + (c_{22} - c_{12})^2}$.