Stochastic Interest Rates

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Solutions to Exercises

Chapter 1

1.1. Using (1.2) and (1.3) we can express $R(t,T)$ in terms of $L(t,T)$ to get

$$R(t,T) = \frac{\ln(1 + (T-t)L(t,T))}{T-t}.$$

For $L(0,1) = 5\%$ we find that $R(0,1) = 4.88\%$.

1.2. The present value of all the payments is given by the geometric series

$$\sum_{n=1}^{\infty} B(0,n) = \sum_{n=1}^{\infty} \frac{1}{(1+r)^n} = \frac{1}{r},$$

where we have used the formula $\sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$ for the sum of a geometric series when $|x| < 1$. For $r = 5\%$ the present value of the perpetual bond is \$20.

1.3. Using definitions (1.6) and (1.2) we can write

$$F(0;S,T) = \frac{1}{T-S}\left(\frac{1+TL(0,T)}{1+SL(0,S)} - 1\right)$$

Solving for $L(0,S)$, we get 4.86%.

1.4. Using the formula for the forward rate in Exercise 1.3, we get $F(0;1,2) = 5.769\%$.

1.5. Using the formula

$$R(t;S,T) = \frac{R(t,T)(T-t) - R(t,S)(S-t)}{T-S}.$$

for $t = 0$, $S = 1$ and $T = 2$, we find that $R(0,2) = 4.75\%$.

1.6. Equating formulae (1.3) and (1.11), we can see that

$$\int_t^T f(t,u)du = (T-t)R(t,T).$$

Differentiating both sides with respect to T shows that

$$f(t,T) = R(t,T) + (T-t)\frac{\partial}{\partial T}R(t,T).$$

1.7. The floating coupons

$$C_n = N\,(T_i - T_{i-1})\,L(T_i, T_{i-1})$$

were paid at times T_i for $i = 1,\ldots,6$. In addition, the notional amount N was paid at time T_6. As a result, the cash flow of the floating-coupon bond was

i	T_i	C_i
1	1 Feb 2010	0.039 60
2	1 Mar 2010	0.039 84
3	1 Apr 2010	0.045 86
4	4 May 2010	0.049 50
5	1 Jun 2010	0.042 55
6	1 Jul 2010	100.046 52

1.8. Use formula (1.15) along with the data provided to get

$$S_{1,3}(0) = \frac{B(0,1) - B(0,3)}{\sum_{i=3}^{6} 0.5B(0,0.5i)} \approx 5\%.$$

1.9. Using formula (1.15) we can write

$$S_{0,1}(0) = \frac{1 - B(0,T_1)}{\tau_1 B(0,T_1)},$$

for the case $i = 1$. Re-arranging we get

$$B(0,T_1) = \frac{1}{\tau_1 S_{0,1}(0) + 1}.$$

We now proceed by induction. For the case $i = j$ assume we can write $B(0,T_i)$ in terms of the co-initial swap rates for $i = 1,\ldots,j$. Using formula (1.15) we have

$$S_{0,j+1}(0) = \frac{1 - B(0,T_{j+1})}{\sum_{i=1}^{j+1} \tau_i B(t,T_i)},$$

for the case $i = j + 1$. Re-arranging we get

$$B(0, T_{j+1}) = \left(S_{0,j+1}(0) \sum_{i=1}^{j} \tau_i B(0, T_i) + 1 \right)^{-1}.$$

1.10. Using the bootstrapping formula (1.18), we see that

$$B(T_0, T_1) = \frac{1}{1 + \tau_1 r_1} = 0.99631.$$

Having solved for $B(T_0, T_1)$, we can write

$$B(T_0, T_2) = \frac{1 - r_2 \tau_1 B(T_0, T_1)}{1 + \tau_2 r_2} = 0.98898.$$

Repeating, we get $B(T_0, T_3) = 0.97720$, $B(T_0, T_4) = 0.96135$ and $B(T_0, T_5) = 0.94198$.

Chapter 2

2.1. Let $V(t)$ be the price process of the derivative security. At the exercise time it is equal to the payoff, $V(T) = X$. Moreover, just like for any other security, the price process $\frac{V(t)}{B(t)}$ discounted by the money market account is a martingale under the risk-neutral measure Q; see Assumption 2.1. It follows by Proposition 2.2 that $\frac{V(t)}{A(t)}$ is a martingale under P_A for any choice of numeraire $A(t)$. We can conclude that

$$\frac{V(t)}{A(t)} = \mathbb{E}_{P_A}\left(\frac{V(T)}{A(T)} \middle| \mathcal{F}_t \right) = \mathbb{E}_{P_A}\left(\frac{X}{A(T)} \middle| \mathcal{F}_t \right)$$

for each t such that $0 \leq t \leq T$. This implies, in particular, that

$$V(0) = A(0)\mathbb{E}_{P_A}\left(\frac{X}{A(T)} \right).$$

2.2. First observe that $\frac{dP_S}{dP_T}$ is $\mathcal{F}_{\min\{S,T\}}$-measurable since P_S is a measure defined on $\mathcal{F}_S$ and P_T on $\mathcal{F}_T$. Now consider the case when $S \leq T$. For any $A \in \mathcal{F}_S$ we have

$$P_S(A) = \mathbb{E}_Q\left(\mathbf{1}_A \frac{dP_S}{dQ} \right) = \mathbb{E}_Q\left(\mathbf{1}_A \frac{1}{B(S)B(0, S)} \right).$$

On the other hand,

$$\begin{aligned}P_S(A) &= \mathbb{E}_{P_T}\left(\mathbf{1}_A \frac{dP_S}{dP_T}\right) = \mathbb{E}_Q\left(\mathbf{1}_A \frac{dP_S}{dP_T}\frac{dP_T}{dQ}\right) = \mathbb{E}_Q\left(\mathbf{1}_A \frac{dP_S}{dP_T}\frac{1}{B(T)B(0,T)}\right)\\ &= \mathbb{E}_Q\left(\mathbf{1}_A \frac{dP_S}{dP_T}\frac{B(T,T)}{B(T)B(0,T)}\right) = \mathbb{E}_Q\left(\mathbf{1}_A \frac{dP_S}{dP_T}\mathbb{E}_Q\left(\frac{B(T,T)}{B(T)B(0,T)}\middle|\mathcal{F}_S\right)\right)\\ &= \mathbb{E}_Q\left(\mathbf{1}_A \frac{dP_S}{dP_T}\frac{B(S,T)}{B(S)B(0,T)}\right).\end{aligned}$$

This is because $\mathbf{1}_A$ and $\frac{dP_S}{dP_T}$ are $\mathcal{F}_S$-measurable and $\frac{B(t,T)}{B(t)}$ is a martingale under Q. The above holds for any $A \in \mathcal{F}_S$, so

$$\frac{1}{B(S)B(0,S)} = \frac{dP_S}{dP_T}\frac{B(S,T)}{B(S)B(0,T)}.$$

It follows that

$$\frac{dP_S}{dP_T} = \frac{B(0,T)}{B(0,S)B(S,T)}.$$

Finally, consider $S \geq T$. For any $A \in \mathcal{F}_T$ we have

$$\begin{aligned}P_S(A) &= \mathbb{E}_Q\left(\mathbf{1}_A \frac{dP_S}{dQ}\right) = \mathbb{E}_Q\left(\mathbf{1}_A \frac{1}{B(S)B(0,S)}\right) = \mathbb{E}_Q\left(\mathbf{1}_A \frac{B(S,S)}{B(S)B(0,S)}\right)\\ &= \mathbb{E}_Q\left(\mathbf{1}_A \mathbb{E}_Q\left(\frac{B(S,S)}{B(S)B(0,S)}\middle|\mathcal{F}_T\right)\right) = \mathbb{E}_Q\left(\mathbf{1}_A \frac{B(T,S)}{B(T)B(0,S)}\right)\end{aligned}$$

since $\mathbf{1}_A$ is $\mathcal{F}_T$-measurable and $\frac{B(t,S)}{B(t)}$ is a martingale under Q. On the other hand,

$$P_S(A) = \mathbb{E}_{P_T}\left(\mathbf{1}_A \frac{dP_S}{dP_T}\right) = \mathbb{E}_Q\left(\mathbf{1}_A \frac{dP_S}{dP_T}\frac{dP_T}{dQ}\right) = \mathbb{E}_Q\left(\mathbf{1}_A \frac{dP_S}{dP_T}\frac{1}{B(T)B(0,T)}\right).$$

Because the above holds for any $A \in \mathcal{F}_T$, it follows that

$$\frac{B(T,S)}{B(T)B(0,S)} = \frac{dP_S}{dP_T}\frac{1}{B(T)B(0,T)},$$

so

$$\frac{dP_S}{dP_T} = \frac{B(0,T)B(T,S)}{B(0,S)}.$$

2.3. Since

$$\begin{aligned}dB(t,T) &= B(t,T)\mu(t,T)dt + B(t,T)\Sigma(t,T)dW(t),\\ dB(t) &= B(t)r(t)dt,\end{aligned}$$

we also have

$$d\frac{1}{B(t)} = -\frac{1}{B(t)}r(t)dt$$

and can use the Itô formula to compute

$$\begin{aligned} d\frac{B(t,T)}{B(t)} &= \frac{1}{B(t)}dB(t,T) + B(t,T)d\frac{1}{B(t)} + dB(t,T)d\frac{1}{B(t)} \\ &= \frac{B(t,T)}{B(t)}\left(\mu(t,T) - r(t)\right)dt + \frac{B(t,T)}{B(t)}\Sigma(t,T)dW(t). \end{aligned}$$

Since $\frac{B(t,T)}{B(t)}$ is a martingale under the risk-neutral measure Q, we can conclude that the term next to dt is zero, so $\mu(t,T) = r(t)$ for each $t \in [0,T]$.

2.4. Since (2.8) is a linear SDE, we know that its solution with initial value $B(0,T)$ can be written as

$$B(t,T) = B(0,T)\exp\left(\int_0^t \Sigma(u,T)dW(u) + \int_0^t \left(r(u) - \frac{1}{2}\Sigma(u,T)^2\right)du\right).$$

For $t = T$ we have $B(T,T) = 1$, so

$$B(0,T) = \exp\left(-\int_0^T \Sigma(u,T)dW(u) - \int_0^T \left(r(u) - \frac{1}{2}\Sigma(u,T)^2\right)du\right).$$

Combining these two formulae, we get

$$B(t,T) = \exp\left(-\int_t^T \Sigma(u,T)dW(u) - \int_t^T \left(r(u) - \frac{1}{2}\Sigma(u,T)^2\right)du\right).$$

2.5. This follows from the solution to Exercise 2.3. With $\mu(t,T) = r(t)$ we get

$$d\frac{B(t,T)}{B(t)} = \frac{B(t,T)}{B(t)}\Sigma(t,T)dW(t)$$

and since $\xi(t) = \left.\frac{dP_T}{dQ}\right|_t = \frac{1}{B(0,T)}\frac{B(t,T)}{B(t)}$, we have

$$d\xi(t) = \xi(t)\Sigma(t,T)dW(t).$$

2.6. Given that

$$\begin{aligned} dB(t,T) &= B(t,T)r(t)dt + B(t,T)\sum_{i=1}^{n}\Sigma_i(t,T)dW_i(t), \\ dB(t) &= B(t)r(t)dt, \end{aligned}$$

we can use the Itô formula to check that the Radon–Nikodym density

$$\xi(t) = \left.\frac{dP_T}{dQ}\right|_t = \frac{B(t,T)}{B(t)B(0,T)}$$

satisfies the SDE

$$d\xi(t) = \xi(t)\sum_{i=1}^{n}\Sigma_i(t,T)dW_i(t).$$

Solving this SDE with the initial condition $\xi(0) = 1$, we find that

$$\xi(t) = \left.\frac{dP_T}{dQ}\right|_t = \exp\left(\int_0^t \sum_{i=1}^{n}\Sigma_i(u,T)dW_i(u) - \frac{1}{2}\int_0^t \sum_{i=1}^{n}\Sigma_i(u,T)^2 du\right).$$

For $t = T$ this gives

$$\frac{dP_T}{dQ} = \exp\left(\int_0^T \sum_{i=1}^{n}\Sigma_i(u,T)dW_i(u) - \frac{1}{2}\int_0^T \sum_{i=1}^{n}\Sigma_i(u,T)^2 du\right).$$

2.7. Using the Itô formula and (2.8), we get

$$\begin{aligned} d\left(\frac{1}{B(t,S)}\right) &= -\frac{dB(t,S)}{B(t,S)^2} + \frac{dB(t,S)dB(t,S)}{B(t,S)^3} \\ &= \frac{\Sigma(t,S)^2dt - r(t)dt - \Sigma(t,S)dW(t)}{B(t,S)} \end{aligned}$$

and

$$\begin{aligned} d\left(\frac{B(t,T)}{B(t,S)}\right) &= B(t,T)d\left(\frac{1}{B(t,S)}\right) + \frac{dB(t,T)}{B(t,S)} + dB(t,T)d\left(\frac{1}{B(t,S)}\right) \\ &= \frac{B(t,T)}{B(t,S)}\Sigma(t,S)\left(\Sigma(t,S) - \Sigma(t,T)\right)dt \\ &\qquad + \frac{B(t,T)}{B(t,S)}\left(\Sigma(t,T) - \Sigma(t,S)\right)dW(t). \end{aligned}$$

Since $\mathbf{FP}(t;S,T) = \frac{B(t,T)}{B(t,S)}$, this verifies (2.13).

2.8. Because $\ln\frac{B(S,T)}{\mathbf{FP}(t;S,T)}$ is independent of $\mathcal{F}_t$ and $\mathbf{FP}(t;S,T)$ is $\mathcal{F}_t$-measurable, we have

$$\begin{aligned} P_S\left(B(S,T) \geq K|\mathcal{F}_t\right) &= P_S\left(\left.\frac{B(S,T)}{\mathbf{FP}(t;S,T)} \geq \frac{K}{\mathbf{FP}(t;S,T)}\right|\mathcal{F}_t\right) \\ &= P_S\left(\ln\frac{B(S,T)}{\mathbf{FP}(t;S,T)} \geq \ln\frac{K}{\mathbf{FP}(t;S,T)}\right) \\ &= P_S\left(X \geq -d_-\right) \\ &= N(d_-), \end{aligned}$$

where

$$X = \frac{\ln \frac{B(S,T)}{\mathbf{FP}(t;S,T)} + \frac{1}{2}\nu(t,S)}{\nu(t,S)}$$

has the normal distribution $N(0,1)$ with mean 0 and variance 1 under P_S and where

$$d_- = \frac{\ln \frac{\mathbf{FP}(t;S,T)}{K} - \frac{1}{2}\nu(t,S)}{\nu(t,S)}.$$

Likewise,

$$\begin{aligned} P_T\left(B(S,T) \geq K | \mathcal{F}_t\right) &= P_T\left(\frac{B(S,T)}{\mathbf{FP}(t;S,T)} \geq \frac{K}{\mathbf{FP}(t;S,T)} \middle| \mathcal{F}_t\right) \\ &= P_T\left(\ln \frac{B(S,T)}{\mathbf{FP}(t;S,T)} \geq \ln \frac{K}{\mathbf{FP}(t;S,T)}\right) \\ &= P_T\left(Y \geq -d_+\right) \\ &= N(d_+), \end{aligned}$$

where

$$Y = \frac{\ln \frac{B(S,T)}{\mathbf{FP}(t;S,T)} - \frac{1}{2}\nu(t,S)}{\nu(t,S)}$$

has the normal distribution $N(0,1)$ with mean 0 and variance 1 under P_T and where

$$d_+ = \frac{\ln \frac{\mathbf{FP}(t;S,T)}{K} + \frac{1}{2}\nu(t,S)}{\nu(t,S)}.$$

2.9. Since

$$\mathbf{BC}(t;S,T,K) = B(t,T)N(d_+) - KB(t,S)N(d_-),$$

from the put-call parity relationship

$$\mathbf{BC}(t;S,T,K) - \mathbf{BP}(t;S,T,K) = B(t,T) - KB(t,S)$$

we get

$$\begin{aligned} \mathbf{BP}(t;S,T,K) &= \mathbf{BC}(t;S,T,K) - B(t,T) + KB(t,S) \\ &= B(t,T)N(d_+) - KB(t,S)N(d_-) - B(t,T) + KB(t,S) \\ &= -B(t,T)\left(1 - N(d_+)\right) + KB(t,S)\left(1 - N(d_-)\right) \\ &= KB(t,S)N(-d_-) - B(t,T)N(-d_+). \end{aligned}$$

Chapter 3

3.1. From (3.4) we have

$$F(t,r;T) = \exp\left(-r(T-t) - \frac{1}{2}\alpha(T-t)^2 + \frac{1}{6}\sigma^2(T-t)^3\right).$$

We compute the partial derivatives

$$\begin{aligned}\frac{\partial F(t,r;T)}{\partial t} &= \left(r + \alpha(T-t) - \frac{1}{2}\sigma^2(T-t)^2\right)F(t,r;T),\\ \frac{\partial F(t,r;T)}{\partial r} &= -(T-t)F(t,r;T),\\ \frac{\partial^2 F(t,r;T)}{\partial r^2} &= (T-t)^2 F(t,r;T).\end{aligned}$$

This gives

$$\begin{aligned}&\frac{\partial F(t,r;T)}{\partial t} + \alpha\frac{\partial F(t,r;T)}{\partial r} + \frac{1}{2}\sigma^2\frac{\partial^2 F(t,r;T)}{\partial r^2}\\ &= \left(r + \alpha(T-t) - \frac{1}{2}\sigma^2(T-t)^2 - \alpha(T-t) + \frac{1}{2}\sigma^2(T-t)^2\right)F(t,r;T)\\ &= rF(t,r;T),\end{aligned}$$

which means that $F(t,r;T)$ satisfies the term structure equation (3.2).

3.2. According to (3.5),

$$D(t,T) = \int_t^T e^{-\alpha(s-t)}ds = \frac{1-e^{-\alpha(T-t)}}{\alpha}.$$

The mean of the random variable

$$X = \theta\int_t^T D(u,T)du + \sigma\int_t^T D(u,T)dW(u)$$

under the risk-neutral probability Q is

$$\begin{aligned}m = \mathbb{E}_Q(X) &= \theta\int_t^T D(u,T)du = \theta\int_t^T \frac{1-e^{-\alpha(T-u)}}{\alpha}du\\ &= \frac{\theta}{\alpha^2}\left(\alpha(T-t) + e^{-\alpha(T-t)} - 1\right).\end{aligned}$$

The variance of X is

$$\begin{aligned}s^2 = \mathrm{Var}(X) &= \sigma^2\int_t^T D(u,T)^2du = \sigma^2\int_t^T\left(\frac{1-e^{-\alpha(T-u)}}{\alpha}\right)^2du\\ &= \frac{\sigma^2}{2\alpha^3}\left(2\alpha(T-t) - 3 + 4e^{-\alpha(T-t)} - e^{-2\alpha(T-t)}\right).\end{aligned}$$

As a result, the zero-coupon bond price in the Vasiček model can be written as

$$\begin{aligned} B(t,T) &= \exp\bigg(- r(t)D(t,T) - \theta \int_t^T D(u,T)du \\ &\qquad + \frac{1}{2}\sigma^2 \int_t^T D(u,T)^2 du \bigg) \\ &= \exp\bigg(- r(t)\frac{1}{\alpha}\left(1 - \mathrm{e}^{-\alpha(T-t)}\right) - \frac{\theta}{\alpha^2}\left(\alpha\,(T-t) + \mathrm{e}^{-\alpha(T-t)} - 1\right) \\ &\qquad + \frac{\sigma^2}{4\alpha^3}\left(2\alpha\,(T-t) - 3 + 4\mathrm{e}^{-\alpha(T-t)} - \mathrm{e}^{-2\alpha(T-t)}\right)\bigg). \end{aligned}$$

3.3. From the solution to Exercise 3.2 we have

$$\begin{aligned} F(t,r;T) &= \exp\bigg(- rD(t,T) - \theta \int_t^T D(u,T)du \\ &\qquad + \frac{1}{2}\sigma^2 \int_t^T D(u,T)^2 du \bigg) \\ &= \exp\bigg(- r\frac{1}{\alpha}\left(1 - \mathrm{e}^{-\alpha(T-t)}\right) - \frac{\theta}{\alpha^2}\left(\alpha\,(T-t) + \mathrm{e}^{-\alpha(T-t)} - 1\right) \\ &\qquad + \frac{\sigma^2}{4\alpha^3}\left(2\alpha\,(T-t) - 3 + 4\mathrm{e}^{-\alpha(T-t)} - \mathrm{e}^{-2\alpha(T-t)}\right)\bigg). \end{aligned}$$

We compute the partial derivatives

$$\begin{aligned} \frac{\partial F(t,r;T)}{\partial t} &= -\frac{\sigma^2\left(1 - \mathrm{e}^{-\alpha(T-t)}\right)^2 - 2\theta\alpha\left(1 - \mathrm{e}^{-\alpha(T-t)}\right) - 2r\alpha^2\mathrm{e}^{-\alpha(T-t)}}{2\alpha^2} F(t,r;T), \\ \frac{\partial F(t,r;T)}{\partial r} &= -\frac{1 - \mathrm{e}^{-\alpha(T-t)}}{\alpha} F(t,r;T), \\ \frac{\partial^2 F(t,r;T)}{\partial r^2} &= \left(\frac{1 - \mathrm{e}^{-\alpha(T-t)}}{\alpha}\right)^2 F(t,r;T). \end{aligned}$$

This gives

$$\frac{\partial F(t,r;T)}{\partial t} + (\theta - \alpha r)\,\frac{\partial F(t,r;T)}{\partial r} + \frac{1}{2}\sigma^2 \frac{\partial^2 F(t,r;T)}{\partial r^2}$$
$$= \Bigg(-\frac{\sigma^2\left(1-\mathrm{e}^{-\alpha(T-t)}\right)^2 - 2\theta\alpha\left(1-\mathrm{e}^{-\alpha(T-t)}\right) - 2r\alpha^2\mathrm{e}^{-\alpha(T-t)}}{2\alpha^2}$$
$$-(\theta-\alpha r)\,\frac{1-\mathrm{e}^{-\alpha(T-t)}}{\alpha} + \frac{1}{2}\sigma^2\left(\frac{1-\mathrm{e}^{-\alpha(T-t)}}{\alpha}\right)^2\Bigg)F(t,r;T)$$
$$= rF(t,r;T)$$

which means that $F(t,r;T)$ satisfies the term structure equation (3.2).

3.4. First we integrate $r(s)$ given by (3.9) from t to T,

$$\int_t^T r(s)ds = r(t)\int_t^T \mathrm{e}^{-\alpha(s-t)}ds$$
$$+\int_t^T\left(\int_t^s \theta(u)\mathrm{e}^{-\alpha(T-u)}du\right)ds + \int_t^T\left(\int_t^s \sigma(u)\mathrm{e}^{-\alpha(T-u)}dW(u)\right)ds.$$

The first term on the right-hand side is equal to $r(t)D(t,T)$, where $D(t,T)$ is given by (3.5). To compute the second and third terms observe that

$$d\left(\int_t^s \theta(u)\mathrm{e}^{-\alpha(s-u)}du\right) = \theta(s)ds - \alpha\left(\int_t^s \theta(u)\mathrm{e}^{-\alpha(s-u)}du\right)ds,$$
$$d\left(\int_t^s \sigma(u)\mathrm{e}^{-\alpha(s-u)}dW(u)\right) = \sigma(s)dW(s) - \alpha\left(\int_t^s \sigma(u)\mathrm{e}^{-\alpha(s-u)}dW(u)\right)ds.$$

Hence

$$\left(\int_t^s \theta(u)\mathrm{e}^{-\alpha(s-u)}du\right)ds = d\left(\int_t^s \theta(u)\frac{1-\mathrm{e}^{-\alpha(s-u)}}{\alpha}du\right)$$
$$= d\left(\int_t^s \theta(u)D(u,s)du\right),$$
$$\left(\int_t^s \sigma(u)\mathrm{e}^{-\alpha(s-u)}dW(u)\right)ds = d\left(\int_t^s \sigma(u)\frac{1-\mathrm{e}^{-\alpha(s-u)}}{\alpha}dW(u)\right)$$
$$= d\left(\int_t^s \sigma(u)D(u,s)dW(u)\right).$$

As a result, integrating from t to T, we find that

$$\int_t^T r(s)ds = r(t)D(t,T) + \int_t^T \theta(u)D(u,T)du + \int_t^T \sigma(u)D(u,T)dW(u).$$

The random variable

$$X = \int_t^T \theta(u)D(u,T)du + \int_t^T \sigma(u)D(u,T)dW(u)$$

is independent of $\mathcal{F}_t$ and normally distributed with mean

$$m = \int_t^T \theta(u)D(u,T)du$$

and variance given by the Itô isometry as

$$s^2 = \int_t^T \sigma(u)^2 D(u,T)^2 du$$

under the risk-neutral measure $\mathbb{Q}$. Because the expectation of e^{-X} is $\mathrm{e}^{-m+\frac{1}{2}s^2}$, by (3.3) this proves that

$$B(t,T) = \exp\left(-r(t)D(t,T) - \int_t^T \theta(u)D(u,T)du + \frac{1}{2}\int_t^T \sigma(u)^2 D(u,T)^2 du\right).$$

3.5. Using (3.11), we can write (3.14) as

$$\begin{aligned}
r(s) &= f(0,s) + (r(t) - f(0,t))\,\mathrm{e}^{-\alpha(s-t)} + \int_0^s \sigma(u)^2 D(u,s)\mathrm{e}^{-\alpha(s-u)}du \\
&\quad - \int_0^t \sigma(u)^2 D(u,t)\mathrm{e}^{-\alpha(s-u)}du + \int_t^s \sigma(u)\mathrm{e}^{-\alpha(s-u)}dW(u) \\
&= f(0,s) + (r(t) - f(0,t))\,\mathrm{e}^{-\alpha(s-t)} + \int_t^s \sigma(u)^2 D(u,s)\mathrm{e}^{-\alpha(s-u)}du \\
&\quad + \int_0^t \sigma(u)^2 (D(u,s) - D(u,t))\mathrm{e}^{-\alpha(s-u)}du + \int_t^s \sigma(u)\mathrm{e}^{-\alpha(s-u)}dW(u) \\
&= f(0,s) + (r(t) - f(0,t))\,\mathrm{e}^{-\alpha(s-t)} + \int_t^s \sigma(u)^2 D(u,s)\mathrm{e}^{-\alpha(s-u)}du \\
&\quad + \int_0^t \sigma(u)^2 D_t(u,t)D(t,s)\mathrm{e}^{-\alpha(s-u)}du + \int_t^s \sigma(u)\mathrm{e}^{-\alpha(s-u)}dW(u).
\end{aligned}$$

Computing the integral of $r(s)$ from t to T, changing the order of integration and making use of (3.15) gives

$$\begin{aligned}
&\int_t^T r(s)ds \\
&= \int_t^T f(0,s)ds + (r(t) - f(0,t)) \int_t^T e^{-\alpha(s-t)} ds + \int_t^T \left(\int_t^s \sigma(u)^2 D(u,s) e^{-\alpha(s-u)} du \right) ds \\
&\quad + \int_t^T \left(\int_0^t \sigma(u)^2 D_t(u,t) D(t,s) e^{-\alpha(s-u)} du \right) ds + \int_t^T \left(\int_t^s \sigma(u) e^{-\alpha(s-u)} dW(u) \right) ds \\
&= \int_t^T f(0,s)ds + (r(t) - f(0,t)) \int_t^T e^{-\alpha(s-t)} ds + \int_t^T \sigma(u)^2 \left(\int_u^T D(u,s) e^{-\alpha(s-u)} ds \right) du \\
&\quad + \int_0^t \sigma(u)^2 D_t(u,t) e^{-\alpha(t-u)} \left(\int_t^T D(t,s) e^{-\alpha(s-t)} ds \right) du + \int_t^T \sigma(u) \left(\int_u^T e^{-\alpha(s-u)} ds \right) dW(u) \\
&= \ln \frac{B(0,T)}{B(0,t)} + (r(t) - f(0,t)) D(t,T) + \frac{1}{2} \int_t^T \sigma(u)^2 D(u,T)^2 du \\
&\quad + \frac{1}{2} D(t,T)^2 \int_0^t \sigma(u)^2 D_t(u,t)^2 du + \int_t^T \sigma(u) D(u,T) dW(u)
\end{aligned}$$

Finally, substituting the above expression into the pricing formula (3.3), we recover (3.13).

3.6. In the Merton model

$$\ln B(S,T) = -r(S)(T-S) - \frac{1}{2}\alpha (T-S)^2 + \frac{1}{6}\sigma^2 (T-S)^2$$

with

$$r(S) = r(0) + \alpha S + \sigma W(S).$$

As a result, the variance of $\ln B(S,T)$ is

$$v(0,S) = \mathrm{Var}\,(\sigma(T-S)W(S)) = \sigma^2 (T-S)^2 S.$$

Formulae (3.16)–(3.18) for the call and put prices apply with $v(0,S)$ given by this expression.

3.7. In the Vasiček model

$$\ln B(S,T) = -r(S)D(S,T) - \theta^2 \int_S^T D(u,T)du + \frac{1}{2}\sigma^2 \int_S^T D(u,T)^2 du$$

with

$$D(t,T) = \int_t^T e^{-\alpha(s-t)} ds = \frac{1 - e^{-\alpha(T-t)}}{\alpha}$$

and

$$r(S) = r(0)e^{-\alpha S} + \theta \int_0^S e^{-\alpha(S-u)} du + \sigma \int_0^S e^{-\alpha(S-u)} dW(u).$$

Therefore, the variance of $\ln B(S,T)$ is

$$\begin{aligned} v(0,S) &= \mathrm{Var}\left(\sigma D(S,T) \int_0^S e^{-\alpha(S-u)} dW(u)\right) \\ &= \sigma^2 D(S,T)^2 \int_0^S e^{-2\alpha(S-u)} du \\ &= \frac{\sigma^2}{2\alpha^3}\left(1 - e^{-\alpha(T-S)}\right)^2 \left(1 - e^{-2\alpha S}\right). \end{aligned}$$

Formulae (3.16)–(3.18) for the call and put prices apply with $v(0,S)$ given by this expression.

The formulae are the same as those for the Hull–White model with constant $\sigma(t)$ because they do not depend on the mean of $\ln B(S,T)$ but only on the variance of $\ln B(S,T)$, which is the same as in the Vasiček model.

3.8. By definition, $K_i = B(T_0, T_i)$ for $i = 1, \ldots, n$ when $r(T_0) = \tilde{r}$. Therefore, by using Proposition 3.5, we can write

$$\begin{aligned} K_i = &\frac{B(0,T_i)}{B(0,T_0)} \exp\Bigg(- (\tilde{r} - f(0,T_0))D(T_0,T_i) \\ &\qquad - \frac{1}{2} D(T_0,T_i)^2 \int_0^{T_0} \sigma(u)^2 D_{T_0}(u,T_0)^2 du\Bigg). \end{aligned}$$

Substituting this expression into (3.24), we can use a numerical root finding algorithm such as the bisection method to solve for the unknown critical value $\tilde{r}$ and hence find K_i for $i = 1, \ldots, n$.

3.9. By applying the Itô formula to (3.10) and using (3.8), we get

$$\begin{aligned} dB(t,T) &= (\ldots)\, dt - B(t,T)D(t,T)dr(t) \\ &= (\ldots)\, dt - B(t,T)D(t,T)\sigma(t)dW(t). \end{aligned}$$

Comparing this with (2.8), we can see that

$$\Sigma(t,T) = -\sigma(t)D(t,T).$$

Substituting

$$\begin{aligned} W^T(t) &= W(t) - \int_0^t \Sigma(u,T)du \\ &= W(t) + \int_0^t \sigma(u)D(u,T)du \end{aligned}$$

into (3.8), we therefore get

$$\begin{aligned} dr(t) &= (\theta(t) - \alpha r(t))\,dt + \sigma(t)dW(t) \\ &= \Big(\theta(t) - \alpha r(t) - \sigma(t)^2 D(t,T)\Big)\,dt + \sigma(t)dW^T(t). \end{aligned}$$

3.10. Observe that

$$\phi(t) = r(0)\mathrm{e}^{-\alpha t} + \int_0^t \theta(s)\mathrm{e}^{-\alpha(t-s)}ds$$

satisfies

$$\begin{aligned} d\phi(t) &= \left(-\alpha r(0)\mathrm{e}^{-\alpha t} - \alpha \int_0^t \theta(s)\mathrm{e}^{-\alpha(t-s)}ds + \theta(t)\right)dt \\ &= (-\alpha\phi(t) + \theta(t))\,dt. \end{aligned}$$

Since

$$y(t) = \frac{1}{\alpha - \beta}u(t),$$

we find that (3.36) holds:

$$\begin{aligned} dy(t) &= \frac{du(t)}{\alpha - \beta} = \frac{-\beta u(t)dt + \varepsilon dZ(t)}{\alpha - \beta} \\ &= -\beta y(t)dt + \frac{\varepsilon}{\alpha - \beta}dZ(t) = -\beta y(t)dt + \eta dV(t), \end{aligned}$$

where we put

$$\eta = \frac{\varepsilon}{\alpha - \beta}, \quad V(t) = Z(t).$$

Moreover, since

$$x(t) = r(t) - \phi(t) - \frac{u(t)}{\alpha - \beta} = r(t) - \phi(t) - y(t),$$

we find that (3.34) holds, and (3.35) is also satisfied:

$$\begin{aligned} dx(t) &= dr(t) - d\phi(t) - dy(t) \\ &= (\theta(t) + u(t) - \alpha r(t))\, dt + \delta dW(t) - (-\alpha\phi(t) + \theta(t))\, dt \\ &\quad - (-\beta y(t)dt + \eta dV(t)) \\ &= -\alpha\,(r(t) - \phi(t) - y(t))\, dt + \delta dW(t) - \eta dV(t) \\ &= -\alpha x(t)dt + \sigma dU(t), \end{aligned}$$

where we put

$$\sigma = \sqrt{\delta^2 + \eta^2 - 2\sigma\eta\varrho}$$

and define a Brownian motion $U(t)$ by

$$U(t) = \frac{\delta}{\sigma}W(t) - \frac{\eta}{\sigma}V(t).$$

Finally, we consider (3.37):

$$dU(t)dV(t) = d\left(\frac{\delta}{\sigma}W(t) - \frac{\eta}{\sigma}V(t)\right)dV(t) = \frac{\delta\varrho - \eta}{\sigma}dt = \rho dt,$$

where

$$\rho = \frac{\delta\varrho - \eta}{\sigma}.$$

3.11. The integral of the short rate (3.38) is

$$\begin{aligned} \int_t^T r(s)ds &= \int_t^T \phi(s)ds + x(t)\int_t^T \mathrm{e}^{-\alpha(s-t)}ds + y(t)\int_t^T \mathrm{e}^{-\beta(s-t)}ds \\ &\quad +\sigma\int_t^T \left(\int_t^s \mathrm{e}^{-\alpha(s-u)}dU(u)\right)ds + \eta\int_t^T \left(\int_t^s \mathrm{e}^{-\beta(s-u)}dV(u)\right)ds \\ &= \int_t^T \phi(s)ds + x(t)\frac{1-\mathrm{e}^{-\alpha(T-t)}}{\alpha} + y(t)\frac{1-\mathrm{e}^{-\beta(T-t)}}{\beta} \\ &\quad +\sigma\int_t^T \frac{1-\mathrm{e}^{-\alpha(T-u)}}{\alpha}dU(u) + \eta\int_t^T \frac{1-\mathrm{e}^{-\beta(T-u)}}{\beta}dV(u), \end{aligned}$$

where the double integrals can be computed in a similar way as in the derivation of formula (3.6). It follows that

$$\begin{aligned} &\int_t^T r(s) - \int_t^T \phi(s)ds - x(t)\frac{1-\mathrm{e}^{-\alpha(T-t)}}{\alpha} - y(t)\frac{1-\mathrm{e}^{-\beta(T-t)}}{\beta} \\ &= \sigma\int_t^T \frac{1-\mathrm{e}^{-\alpha(T-u)}}{\alpha}dU(u) + \eta\int_t^T \frac{1-\mathrm{e}^{-\beta(T-u)}}{\beta}dV \end{aligned}$$

is independent of $\mathcal{F}_t$ and normally distributed under the risk-neutral measure Q with mean 0 and variance

$$\begin{aligned}V(t,T) &= \sigma^2 \int_t^T \left(\frac{1-\mathrm{e}^{-\alpha(T-u)}}{\alpha}\right)^2 du + \eta^2 \int_t^T \left(\frac{1-\mathrm{e}^{-\beta(T-u)}}{\beta}\right)^2 du \\ &\quad +2\sigma\eta\rho \int_t^T \frac{1-\mathrm{e}^{-\alpha(T-u)}}{\alpha}\frac{1-\mathrm{e}^{-\beta(T-u)}}{\beta} du \\ &= \frac{\sigma^2}{2\alpha^3}\left(2\alpha(T-t) - 3 + 4\mathrm{e}^{-\alpha(T-t)} - \mathrm{e}^{-2\alpha(T-t)}\right) \\ &\quad +\frac{\eta^2}{2\beta^3}\left(2\beta(T-t) - 3 + 4\mathrm{e}^{-\beta(T-t)} - \mathrm{e}^{-2\beta(T-t)}\right) \\ &\quad +\frac{2\sigma\eta\rho}{\alpha\beta}\left((T-t) - \frac{1-\mathrm{e}^{-\beta(T-t)}}{\beta} - \frac{1-\mathrm{e}^{-\alpha(T-t)}}{\alpha} + \frac{1-\mathrm{e}^{-(\alpha+\beta)(T-t)}}{\alpha+\beta}\right).\end{aligned}$$

Hence, applying (3.3), we obtain formula (3.39) for the bond price.

3.12. Solving the SDEs (3.35), (3.36) with the initial conditions $x(0) = 0$ and $y(0) = 0$, we get

$$x(t) = \sigma \int_0^t \mathrm{e}^{-\alpha(t-u)} dU(u), \quad y(t) = \eta \int_0^t \mathrm{e}^{-\beta(t-u)} dV(u).$$

Because

$$dU(t)dU(t) = dt, \quad dV(t)dV(t) = dt, \quad dU(t)dV(t) = \rho dt,$$

by (3.42) the variance of $\ln B(S,T)$ is

$$\begin{aligned}v(0,S) &= \mathrm{Var}\left(\frac{1-\mathrm{e}^{-\alpha(T-S)}}{\alpha}x(S) + \frac{1-\mathrm{e}^{-\beta(T-S)}}{\beta}y(S)\right) \\ &= \sigma^2\left(\frac{1-\mathrm{e}^{-\alpha(T-S)}}{\alpha}\right)^2 \int_0^S \mathrm{e}^{-2\alpha(S-u)} du + \eta^2\left(\frac{1-\mathrm{e}^{-\beta(T-S)}}{\beta}\right)^2 \int_0^S \mathrm{e}^{-2\beta(S-u)} du \\ &\quad +2\sigma\eta\rho\frac{1-\mathrm{e}^{-\alpha(T-S)}}{\alpha}\frac{1-\mathrm{e}^{-\beta(T-S)}}{\beta}\int_0^S \mathrm{e}^{-(\alpha+\beta)(S-u)} du \\ &= \frac{\sigma^2}{2\alpha^3}\left(1-\mathrm{e}^{-\alpha(T-S)}\right)^2\left(1-\mathrm{e}^{-2\alpha S}\right) + \frac{\eta^2}{2\beta^2}\left(1-\mathrm{e}^{-\beta(T-S)}\right)^2\left(1-\mathrm{e}^{-2\beta S}\right) \\ &\quad +\frac{2\sigma\eta\rho}{\alpha\beta(\alpha+\beta)}\left(1-\mathrm{e}^{-\alpha(T-S)}\right)\left(1-\mathrm{e}^{-\beta(T-S)}\right)\left(1-\mathrm{e}^{-(\alpha+\beta)S}\right).\end{aligned}$$

Chapter 4

4.1. In Example 4.2 we found that the instantaneous forward rate in the Ho–Lee model can be written as

$$f(t,T) = f(0,T) + \frac{1}{2}\sigma^2 t\,(2T - t) + \sigma W(t)$$

and the short rate as

$$r(t) = f(t,t) = f(0,t) + \frac{1}{2}\sigma^2 t^2 + \sigma W(t).$$

It follows that

$$\begin{aligned}\int_t^T f(t,u)du &= \int_t^T \left(f(0,u) + \frac{1}{2}\sigma^2 t\,(2u - t) + \sigma W(t)\right) du \\ &= \int_t^T f(0,u)du + \frac{1}{2}\sigma^2 tT\,(T-t) + \sigma\,(T-t)\,W(t) \\ &= (T-t)\,(r(t) - f(0,t)) + \int_t^T f(0,u)du + \frac{1}{2}\sigma^2 t\,(T-t)^2\,.\end{aligned}$$

As a result,

$$\begin{aligned}B(t,T) &= \exp\left(-\int_t^T f(t,u)du\right) \\ &= \exp\left(-\,(T-t)\,(r(t) - f(0,t)) - \int_t^T f(0,u)du - \frac{1}{2}\sigma^2 t\,(T-t)^2\right).\end{aligned}$$

4.2. Setting $\xi(t) = \sigma(t)\mathrm{e}^{\alpha t}$ and $\eta(t) = \mathrm{e}^{-\alpha t}$, we have

$$I(t,T) = \frac{1}{\eta(t)}\int_t^T \eta(u)du = \int_t^T \mathrm{e}^{-\alpha(u-t)}du = D(t,T),$$

where $D(t,T)$ is given by (3.5), and

$$\frac{1}{2}I(t,T)^2 \int_0^t \xi(s)^2\eta(t)^2 ds = \frac{1}{2}D(t,T)^2 \int_0^t \sigma(s)^2\mathrm{e}^{-2\alpha(t-s)}ds.$$

Noting that

$$\exp\left(-\int_t^T f(0,u)du\right) = \frac{B(0,T)}{B(0,t)},$$

we recover (3.13), the Hull–White zero-coupon bond price at time $t \geq 0$ that gives an exact fit to the term structure of interest rates at time 0.

4.3. The short rate $r(t)$ in the Gaussian HJM model with separable volatility is given by

$$r(t) = f(0,t) + \int_0^t \xi(s)\eta(t)\left(\int_s^t \xi(s)\eta(u)du\right)ds + \int_0^t \xi(s)\eta(t)dW(s).$$

We take the stochastic differential

$$dr(t) = \left(\frac{\partial f(0,t)}{\partial t} + \eta'(t)\int_0^t \xi(s)^2\left(\int_s^t \eta(u)du\right)ds + \eta(t)\int_0^t \xi(s)^2\eta(t)ds + \eta'(t)\int_0^t \xi(s)dW(s)\right)dt + \xi(t)\eta(t)dW(t).$$

Using the above expression for $r(t)$, we can write

$$\int_0^t \xi(s)dW(s) = \frac{r(t) - f(0,t)}{\eta(t)} - \int_0^t \xi(s)^2\left(\int_s^t \eta(u)du\right)ds.$$

Substituting this into the stochastic increment for $r(t)$ and noting that $\eta'(t) = \alpha(t)\eta(t)$, we arrive at

$$dr(t) = \left(\frac{\partial f(0,t)}{\partial t} + \phi(t) + \alpha(t)(f(0,t) - r(t))\right)dt + \sigma(t)dW(t),$$

where

$$\phi(t) = \int_0^t \sigma(s)^2 \exp\left(-2\int_s^t \alpha(u)du\right)ds.$$

4.4. We have $\sigma(t,T) = f(T-t)$, where

$$f(x) = \sigma(\gamma x + 1)\mathrm{e}^{-\frac{\lambda}{2}x}.$$

The derivative of this function is

$$f'(x) = \sigma\left(\gamma - \frac{\lambda}{2}(\gamma x + 1)\right)\mathrm{e}^{-\frac{\lambda}{2}x},$$

which is positive for $x < \frac{2\gamma-\lambda}{\lambda\gamma}$, zero for $x = \frac{2\gamma-\lambda}{\lambda\gamma}$ and negative for $\frac{2\gamma-\lambda}{\lambda\gamma} < x$. This shows that $\sigma(t,T)$ as a function of T has a maximum at $T = t + \frac{2\gamma-\lambda}{\lambda\gamma}$, is increasing to the left of this value, and decreasing to the right.

4.5. By Theorem 4.1,

$$dB(t,T) = r(t)B(t,T)dt + \Sigma(t,T)B(t,T)dW(t)$$

with deterministic log-volatility

$$\Sigma(t,T) = -\int_t^T \sigma(t,u)du = -\sigma\int_t^T (\gamma(u-t)+1)e^{-\frac{\lambda}{2}(u-t)}du$$
$$= \frac{2\sigma}{\lambda^2}\left((\gamma\lambda(T-t)+\lambda+2\gamma)\,e^{-\frac{\lambda}{2}(T-t)} - (\lambda+2\gamma)\right).$$

It follows by Theorem 2.4 and Exercise 2.9 that the call and put prices are given by (2.21), (2.22), (2.18) and (2.20) with

$$v(t,S) = \int_t^S (\Sigma(u,T)-\Sigma(u,S))^2\,du$$
$$= \frac{4\sigma^2}{\lambda^4}\int_t^S \left((\gamma\lambda(T-u)+\lambda+2\gamma)\,e^{-\frac{\lambda}{2}(T-u)} - (\gamma\lambda(S-u)+\lambda+2\gamma)\,e^{-\frac{\lambda}{2}(S-u)}\right)^2 du$$
$$= \frac{4\sigma^2}{\lambda^4}\int_t^S (A-Bu)^2\,e^{\lambda u}du,$$

where

$$A = (\gamma\lambda T+\lambda+2\gamma)\,e^{-\frac{\lambda}{2}T} - (\gamma\lambda S+\lambda+2\gamma)\,e^{-\frac{\lambda}{2}S},$$
$$B = \gamma\lambda\left(e^{-\frac{\lambda}{2}T} - e^{-\frac{\lambda}{2}S}\right).$$

Integrating by parts, we obtain

$$v(t,S) = \frac{4\sigma^2}{\lambda^5}B^2\left(S^2e^{\lambda S} - t^2e^{\lambda t}\right) - \frac{8\sigma^2}{\lambda^6}B(B+A\lambda)\left(S\,e^{\lambda S} - te^{\lambda t}\right)$$
$$+ \frac{4\sigma^2}{\lambda^7}\left(2B^2+2AB\lambda+A^2\lambda^2\right)\left(e^{\lambda S} - e^{\lambda t}\right).$$

4.6. When $U = T$, (4.14) becomes

$$df(t,T) = \sum_{i=1}^n \sigma_i(t,T)dW_i^T(t),$$

where $W^T(t) = \left(W_1^T(t),\ldots,W_n^T(t)\right)$ is a Brownian motion under the forward measure P_T. If

$$\mathbb{E}_{P_T}\left(\int_0^T \sum_{i=1}^n \sigma_i(t,T)^2dt\right) < \infty,$$

it means that $f(t,T)$ is a martingale under P_T. Since $f(T,T) = r(T)$, it follows that

$$\mathbb{E}_{P_T}\left(r(T)|\mathcal{F}_t\right) = \mathbb{E}_{P_T}\left(f(T,T)|\mathcal{F}_t\right) = f(t,T).$$

4.7. The choice of $\sigma(t,T) = \sigma(t)e^{-\alpha(T-t)}$ is of the form (4.6), where $\xi(t) = \sigma(t)e^{\alpha t}$ and $\eta(T) = e^{-\alpha T}$. Therefore, by (4.7), $f(t,T)$ can be written in terms of the short rate $r(t)$. For our choice of volatility we have

$$f(t,T) = f(0,T) + (r(t) - f(0,t))e^{-\alpha(T-t)} + \int_0^t \sigma^2(s)e^{-\alpha(T-s)} \int_t^T e^{-\alpha(u-s)} du ds.$$

Substituting this into the one-factor equivalent of (4.15) and calculating the integral with respect to du, we get (3.29).

Chapter 5

5.1. First we show that $Z_i^j(t) = \sum_{l=1}^n \eta_{i,l} W_l^j(t)$ is a Brownian motion under the forward measure P_{T_j} for each $i = 1, \ldots, n$. To this end, we use Lévy's characterisation of Brownian motion, according to which it is enough to verify that $Z_i^j(t)$ and $Z_i^j(t)^2 - t$ are martingales under P_{T_j} and have continuous paths. The continuity of paths follows from that of $W_l^j(t)$. Next, for any $0 \leq s \leq t$, we compute the following conditional expectations by using the fact that $W_1^j(t), \ldots, W_n^j(t)$ are independent Brownian motions:

$$\begin{aligned}
\mathbb{E}_{P_{T_j}}\left(Z_i^j(t)\middle|\mathcal{F}_s\right) &= \sum_{l=1}^n \eta_{i,l}\mathbb{E}_{P_{T_j}}\left(W_l^j(t)\middle|\mathcal{F}_s\right) \\
&= \sum_{l=1}^n \eta_{i,l} W_l^j(s) = Z_l^j(s)
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}_{P_{T_j}}\left(Z_i^j(t)^2\middle|\mathcal{F}_s\right) &= \sum_{l,m=1}^n \eta_{i,l}\eta_{i,m}\mathbb{E}_{P_{T_j}}\left(W_l^j(t)W_m^j(t)\middle|\mathcal{F}_s\right) \\
&= \sum_{l,m=1}^n \eta_{i,l}\eta_{i,m}\left(\delta_{l,m}(t-s) + W_l^j(s)W_m^j(s)\right) \\
&= \sum_{l=1}^n \eta_{i,l}\eta_{i,l}(t-s) + \sum_{l,m=1}^n \eta_{i,l}\eta_{i,m}W_l^j(s)W_m^j(s) \\
&= t - s + Z_i^j(s)^2,
\end{aligned}$$

where $\delta_{i,h} = 1$ if $i = h$ and 0 if $i \neq h$, and where we apply the equality $\sum_{l=1}^n \eta_{i,l}\eta_{i,l} = \rho_{i,i} = 1$. We can see that $Z_i^j(t)$ and $Z_i^j(t)^2 - t$ are indeed martingales under P_{T_j} with continuous paths, hence $Z_i^j(t)$ is a Brownian motion under P_{T_j}.

Moreover,

$$\begin{aligned} dZ_k^j(t)dZ_l^j(t) &= \sum_{i=1}^n \eta_{k,i} dW_i^j(t) \sum_{m=1}^n \eta_{l,m} dW_m^j(t) \\ &= \sum_{i=1}^n \sum_{m=1}^n \eta_{k,i}\eta_{l,m}\delta_{i,m} dt = \sum_{i=1}^n \eta_{k,i}\eta_{l,i} dt = \rho_{k,l} dt. \end{aligned}$$

5.2. It was shown in Section 5.2 that

$$\begin{aligned} F_i(T_{i-1}) &= F_i(t) \exp\left(\int_t^{T_{i-1}} \sigma_i(s) dZ_i^i(s) - \frac{1}{2} \int_t^{T_{i-1}} \sigma_i(s)^2 ds \right) \\ &= F_i(t) \exp\left(-s_i X_i - \frac{1}{2} s_i^2 \right), \end{aligned}$$

with

$$s_i = \sqrt{\int_t^{T_{i-1}} \sigma_i(s)^2 ds}$$

and

$$X_i = -\frac{1}{s_i} \int_t^{T_{i-1}} \sigma_i(s) dZ_i^i(s),$$

where X_i is independent of $\mathcal{F}_t$ and normally distributed with mean 0 and variance 1. Observe that

$$F_i(T_{i-1}) \geq K \iff X_i \leq d_-,$$

where

$$d_- = \frac{\ln \frac{F_i(t)}{K} - \frac{1}{2} s_i^2}{s_i}.$$

Since $F_i(t)$ is $\mathcal{F}_t$-measurable, it follows that

$$\begin{aligned} \mathbb{E}_{P_{T_i}}\left(\mathbf{1}_{\{F_i(T_{i-1}) \geq K\}} \middle| \mathcal{F}_t \right) &= \mathbb{E}_{P_{T_i}}\left(\mathbf{1}_{\{X_i \leq d_-\}} \middle| \mathcal{F}_t \right) \\ &= \int_{-\infty}^{d_-} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = N(d_-) \end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}_{P_{T_i}}\Big(F_i(T_{i-1})\mathbf{1}_{\{F_i(T_{i-1})\geq K\}}\Big|\mathcal{F}_t\Big) &= F_i(t)\mathbb{E}_{P_{T_i}}\Big(\mathrm{e}^{-s_iX_i-\frac{1}{2}s_i^2}\mathbf{1}_{\{X_i\leq d_-\}}\Big|\mathcal{F}_t\Big)\\
&= F_i(t)\int_{-\infty}^{d_-}\mathrm{e}^{-s_ix-\frac{1}{2}s_i^2}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{x^2}{2}}\,dx\\
&= F_i(t)\int_{-\infty}^{d_-}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{(x+s_i)^2}{2}}\,dx\\
&= F_i(t)\int_{-\infty}^{d_-+s_i}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{y^2}{2}}\,dy = F_i(t)N(d_+),
\end{aligned}$$

where

$$d_+ = d_- + s_i = \frac{\ln\frac{F_i(t)}{K}+\frac{1}{2}s_i^2}{s_i}.$$

5.3. Let $Z_1^j(t),\ldots,Z_n^j(t)$ be correlated Brownian motions under the forward measure P_{T_j} that satisfy (5.1) and let $i\geq j$. Then we have $T_i\geq T_j$. The Radon–Nikodym derivative of P_{T_i} with respect to P_{T_j} is

$$\frac{dP_{T_i}}{dP_{T_j}} = \frac{B(0,T_j)}{B(0,T_i)}B(T_j,T_i),$$

which corresponds to the change of numeraire from $B(t,T_j)$ to $B(t,T_i)$. The associated Radon–Nikodym density is

$$\begin{aligned}
\xi_j^i(t) &= \mathbb{E}_{P_{T_j}}\left(\frac{dP_{T_i}}{dP_{T_j}}\middle|\mathcal{F}_t\right)\\
&= \frac{B(0,T_j)}{B(0,T_i)}\mathbb{E}_{P_{T_j}}\left(\frac{B(T_j,T_i)}{B(T_j,T_j)}\middle|\mathcal{F}_t\right) = \frac{B(0,T_j)}{B(0,T_i)}\frac{B(t,T_i)}{B(t,T_j)}.
\end{aligned}$$

It can be written as

$$\xi_j^i(t) = \frac{B(0,T_j)}{B(0,T_i)}\prod_{k=j+1}^{i}\frac{B(t,T_k)}{B(t,T_{k-1})} = \frac{B(0,T_j)}{B(0,T_i)}\prod_{k=j+1}^{i}\frac{1}{1+\tau_kF_k(t)}.$$

By the Itô formula, we get

$$d\xi_j^i(t) = -\xi_j^i(t)\sum_{k=j+1}^{i}\frac{\tau_k dF_k(t)}{1+\tau_kF_k(t)} + (\cdots)\,dt.$$

The explicit expressions for the terms with dt will not be needed. We substitute

$$dF_k(t) = \mu_k^j(t)F_k(t)dt + \sigma_k(t)F_k(t)dZ_k^j(t)$$

using the SDE (5.2) and collect the terms with $dZ_k^j(t)$ and dt separately. Because $\xi_j^i(t)$ is a martingale under P_{T_j}, the terms with dt cancel out, so

$$d\xi_j^i(t) = -\xi_j^i(t) \sum_{k=j+1}^{i} \frac{\tau_k \sigma_k(t) F_k(t)}{1 + \tau_k F_k(t)} dZ_k^j(t).$$

Next, substituting for $Z_k^j(t)$ from (5.4), we get

$$d\xi_j^i(t) = -\xi_j^i(t) \sum_{k=i+1}^{j} \sum_{l=1}^{n} \frac{\tau_k \sigma_k(t) F_k(t)}{1 + \tau_k F_k(t)} \eta_{k,l} dW_l^j(t),$$

and solve this SDE to obtain

$$\xi_j^i(t) = \exp\Bigg(- \int_0^t \sum_{k=i+1}^{j} \sum_{l=1}^{n} \frac{\tau_k \sigma_k(s) F_k(s)}{1 + \tau_k F_k(s)} \eta_{k,l} dW_l^j(s) \\ - \frac{1}{2} \int_0^t \sum_{k=i+1}^{j} \sum_{l=i+1}^{j} \frac{\tau_k \sigma_k(s) F_k(s)}{1 + \tau_k F_k(s)} \frac{\tau_l \sigma_l(s) F_l(s)}{1 + \tau_l F_l(s)} \rho_{k,l} ds \Bigg).$$

Given that $W_l^j(t)$ for $l = 1, \ldots, n$ are the components of an n-dimensional Brownian motion under the forward measure P_{T_j}, we can apply the Girsanov theorem (see [BSM]) to conclude that

$$W_l^i(t) = W_l^j(t) + \int_0^t \sum_{k=i+1}^{j} \frac{\tau_k \sigma_k(s) F_k(s)}{1 + \tau_k F_k(s)} \eta_{k,l} ds$$

for $l = 1, \ldots, n$ are the components of an n-dimensional Brownian motion under P_{T_i}. Now we apply (5.4) once again together with (5.5) to finally find by using Exercise 5.1 that

$$Z_l^i(t) = \sum_{m=1}^{n} \eta_{l,m} W_m^i(t) = \sum_{m=1}^{n} \eta_{l,m} \left(W_m^j(t) + \int_0^t \sum_{k=i+1}^{j} \frac{\tau_k \sigma_k(s) F_k(s)}{1 + \tau_k F_k(s)} \eta_{k,m} ds \right) \\ = Z_l^j(t) + \int_0^t \sum_{k=i+1}^{j} \frac{\tau_k \sigma_k(s) F_k(s)}{1 + \tau_k F_k(s)} \rho_{k,l} ds$$

for $l = 1, \ldots, n$ are Brownian motions under P_{T_i} correlated so that

$$dZ_k^i(t) dZ_l^i(t) = \rho_{k,l} dt.$$

5.4. Let $i > j$. We put $l = i$ in Exercise 5.3. This gives

$$Z_i^i(t) = Z_i^j(t) + \int_0^t \sum_{k=j+1}^{i} \frac{\tau_k \sigma_k(s) F_k(s)}{1 + \tau_k F_k(s)} \rho_{k,i} ds.$$

Substituting this into (5.3), we get

$$\begin{aligned} dF_i(t) &= \sigma_i(t)F_i(t)dZ_i^i(t) \\ &= \sigma_i(t)F_i(t)dZ_i^j(t) + \sigma_i(t)F_i(t)\sum_{k=j+1}^{i}\frac{\tau_k\sigma_k(t)F_k(t)}{1+\tau_kF_k(t)}\rho_{k,i}dt. \end{aligned}$$

Comparing this with (5.2), we obtain

$$\mu_i^j(t) = \sum_{k=j+1}^{i}\frac{\tau_k\rho_{k,i}\sigma_i(t)\sigma_k(t)F_k(t)}{1+\tau_kF_k(t)}.$$

5.5. Let $t \in [0, T_n]$. The discrete money market account can be expressed as

$$L(t) = B(t, T_{\alpha(t)})\prod_{k=0}^{\alpha(t)}\frac{1}{B(T_{k-1}, T_k)} = B(t, T_{\alpha(t)})L(T_{\alpha(t)}),$$

where $\alpha(t)$ is given by (5.15). Since $L(T_{\alpha(t)})$ is $\mathcal{F}_{T_{\alpha(t)-1}}$-measurable and $T_{\alpha(t)-1} < t$, we have

$$\begin{aligned} \mathbb{E}_Q\left(\left.\frac{L(T_{\alpha(t)})}{B(T_{\alpha(t)})}\right|\mathcal{F}_t\right) &= L(T_{\alpha(t)})\mathbb{E}_Q\left(\left.\frac{1}{B(T_{\alpha(t)})}\right|\mathcal{F}_t\right) = L(T_{\alpha(t)})\mathbb{E}_Q\left(\left.\frac{B(T_{\alpha(t)}, T_{\alpha(t)})}{B(T_{\alpha(t)})}\right|\mathcal{F}_t\right) \\ &= L(T_{\alpha(t)})\frac{B(t, T_{\alpha(t)})}{B(t)} = \frac{L(t)}{B(t)}. \end{aligned}$$

It follows that for any $t \in [0, T_n]$

$$\begin{aligned} \mathbb{E}_Q\left(\left.\frac{L(T_n)}{B(T_n)}\right|\mathcal{F}_t\right) &= \mathbb{E}_Q\left(\left.\mathbb{E}_Q\left(\left.\frac{L(T_n)}{B(T_n)}\right|\mathcal{F}_{T_{n-1}}\right)\right|\mathcal{F}_t\right) = \mathbb{E}_Q\left(\left.\frac{L(T_{n-1})}{B(T_{n-1})}\right|\mathcal{F}_t\right) \\ &= \mathbb{E}_Q\left(\left.\mathbb{E}_Q\left(\left.\frac{L(T_{n-1})}{B(T_{n-1})}\right|\mathcal{F}_{T_{n-2}}\right)\right|\mathcal{F}_t\right) = \mathbb{E}_Q\left(\left.\frac{L(T_{n-2})}{B(T_{n-2})}\right|\mathcal{F}_t\right) \\ &\vdots \\ &= \mathbb{E}_Q\left(\left.\mathbb{E}_Q\left(\left.\frac{L(T_{\alpha(t)+1})}{B(T_{\alpha(t)+1})}\right|\mathcal{F}_{T_{\alpha(t)}}\right)\right|\mathcal{F}_t\right) = \mathbb{E}_Q\left(\left.\frac{L(T_{\alpha(t)})}{B(T_{\alpha(t)})}\right|\mathcal{F}_t\right) \\ &= \frac{L(t)}{B(t)}, \end{aligned}$$

hence $\frac{L(t)}{B(t)}$ is a martingale under the risk-neutral measure Q. Taking $B(t, T_n)$ as numeraire, we can conclude that $\frac{L(t)}{B(t,T_n)}$ is a martingale under the corresponding measure P_{T_n}; see Section 2.1.

5.6. According to Theorem 4.6, in a multi-factor HJM model the bond prices satisfy the SDE

$$dB(t,T) = r(t)B(t,T)dt + \sum_{i=1}^{n} \Sigma_i(t,T)B(t,T)dW_i(t),$$

where $(W_1(t), \ldots, W_n(t))$ is an n-dimensional Brownian motion under the risk-neutral measure Q, and where

$$\Sigma_i(t,T) = -\int_t^T \sigma_i(t,u)du$$

with $\sigma_i(t,T)$ for $i = 1, \ldots, n$ being the volatilities for the instantaneous forward rate $f(t,T)$. Solving the SDE with a final condition $B(T,T) = 1$, we get

$$\begin{aligned} B(t,T) &= \exp\left(-\int_t^T \sum_{i=1}^{n} \Sigma_i(u,T)dW_i(u) + \frac{1}{2}\int_t^T \sum_{i=1}^{n} \Sigma_i(u,T)^2 du - \int_t^T r(u)du \right) \\ &= \frac{B(t)}{B(T)} \exp\left(-\int_t^T \sum_{i=1}^{n} \Sigma_i(u,T)dW_i(u) + \frac{1}{2}\int_t^T \sum_{i=1}^{n} \Sigma_i(u,T)^2 du \right), \end{aligned}$$

where

$$B(t) = \exp\left(\int_0^t r(u)du \right).$$

This enables us to derive the formula

$$\begin{aligned} \frac{dP_S}{dQ} &= \frac{1}{B(S)B(0,S)} \\ &= \exp\left(\int_0^S \sum_{i=1}^{n} \Sigma_i(u,S)dW_i(u) - \frac{1}{2}\int_0^S \sum_{i=1}^{n} \Sigma_i(u,S)^2 du \right) \end{aligned}$$

for the Radon–Nikodym derivative (see also Exercise 2.6). Hence, by the Girsanov theorem, the process $(W_1^S(t), \ldots, W_n^S(t))$, where

$$W_i^S(t) = W_i(t) - \int_0^t \Sigma_i(u,S)du$$

for $i = 1, \ldots, n$ and $t \in [0,S]$, is an n-dimensional Brownian motion under the forward measure P_S.

5.7. In Section 5.5 we saw that

$$\frac{dP_L}{dP_{T_n}} = B(0,T_n)L(T_n).$$

In the same manner we can show that

$$\frac{dP_L}{dP_{T_i}} = B(0,T_i)L(T_i)$$

for any $i = 1, \ldots, n$. The corresponding Radon–Nikodym density is

$$\begin{aligned}\xi_i^L(t) &= \left.\frac{dP_L}{dP_{T_i}}\right|_t = \mathbb{E}_{P_{T_i}}\left(B(0,T_i)L(T_i)|\mathcal{F}_t\right)\\ &= B(0,T_i)\mathbb{E}_{P_{T_i}}\left(\left.\frac{L(T_i)}{B(T_i,T_i)}\right|\mathcal{F}_t\right) = B(0,T_i)\frac{L(t)}{B(t,T_i)}\end{aligned}$$

since $\frac{L(t)}{B(t,T_i)}$ is a martingale under P_{T_i}. Putting $\alpha(t) = \min\{j : t \le T_j\}$, we have $L(t) = L(T_{\alpha(t)})B(t,T_{\alpha(t)})$ and so

$$\xi_i^L(t) = B(0,T_i)L(T_{\alpha(t)})\frac{B(t,T_{\alpha(t)})}{B(t,T_i)}.$$

Using the Itô formula and SDE for bond prices in a multi-factor HJM model given in Theorem 4.6, we find that $\xi_i^L(t)$ satisfies an SDE of the form

$$\begin{aligned}d\xi_i^L(t) &= \xi_i^L(t)\sum_{k=1}^{n}(\Sigma_k(t,T_{\alpha(t)}) - \Sigma_k(t,T_i))dW_k(t) + (\cdots)dt\\ &= \xi_i^L(t)\sum_{k=1}^{n}(\Sigma_k(t,T_{\alpha(t)}) - \Sigma_k(t,T_i))dW_k^i(t),\end{aligned}$$

where $(W_1(t),\ldots,W_n(t))$ and $(W_1^i(t),\ldots,W_n^i(t))$ are n-dimensional Brownian motions under the risk-neutral measure Q and under the forward measure P_{T_i}, respectively. The second equality holds because $\xi_i^L(t)$ is a martingale under P_{T_i}. The precise form of the expression in front of dt is not important in this argument and is omitted for brevity. This SDE for $\xi_i^L(t)$ can be solved with the initial condition $\xi_i^L(0) = 1$ to get

$$\begin{aligned}&\frac{dP_L}{dP_{T_i}} = \xi_i^L(T_i)\\ &= \exp\left(\sum_{k=1}^{n}\int_0^{T_i}(\Sigma_k(t,T_{\alpha(t)}) - \Sigma_k(t,T_i))\,dW_k^i(t) - \frac{1}{2}\sum_{k=1}^{n}\int_0^{T_i}(\Sigma_k(t,T_{\alpha(t)}) - \Sigma_k(t,T_i))^2\,dt\right).\end{aligned}$$

It follows by the Girsanov theorem that $(W_1^L(t),\ldots,W_n^L(t))$, where

$$W_k^L(t) = W_k^i(t) - \int_0^t(\Sigma_k(s,T_{\alpha(t)}) - \Sigma_k(s,T_i))\,ds$$

for $k = 1, \ldots, n$, is an n-dimensional Brownian motion under the spot LIBOR measure P_L.

5.8. By (5.20),

$$\Sigma_k(s, T_{\alpha(t)}) - \Sigma_k(s, T_i) = \sum_{m=\alpha(t)+1}^{i} \frac{\tau_m \lambda_{m,k}(t) F_m(t)}{1 + \tau_m F_m(t)}$$
$$= \sum_{m=\alpha(t)+1}^{i} \frac{\tau_m \eta_{m,k} \sigma_m(t) F_m(t)}{1 + \tau_m F_m(t)}.$$

Substituting this into (5.22) gives

$$W_k^i(t) = W_k^L(t) + \int_0^t \left(\Sigma_k(s, T_{\alpha(t)}) - \Sigma_k(s, T_i)\right) ds$$
$$= W_k^L(t) + \int_0^t \sum_{m=\alpha(t)+1}^{i} \frac{\tau_m \eta_{m,k} \sigma_m(s) F_m(s)}{1 + \tau_m F_m(s)} ds.$$

As a result,

$$Z_j^i(t) = \sum_{k=1}^{n} \eta_{j,k} W_k^i(t)$$
$$= \sum_{k=1}^{n} \eta_{j,k} \left(W_k^L(t) + \int_0^t \sum_{m=\alpha(t)+1}^{i} \frac{\tau_m \eta_{m,k} \sigma_m(s) F_m(s)}{1 + \tau_m F_m(s)} ds \right)$$
$$= Z_j^L(t) + \int_0^t \sum_{m=\alpha(t)+1}^{i} \frac{\tau_m \rho_{i,m} \sigma_m(s) F_m(s)}{1 + \tau_m F_m(s)} ds.$$

5.9. Substituting (5.23) with $i = j$ into (5.3), we obtain

$$dF_i(t) = \sigma_i(t) F_i(t) dZ_i^i(t)$$
$$= \sum_{m=\alpha(t)+1}^{i} \frac{\tau_m \rho_{i,m} \sigma_i(t) \sigma_m(t) F_m(t)}{1 + \tau_m F_m(t)} F_i(t) dt + \sigma_i(t) F_i(t) dZ_i^L(t).$$

5.10. The critical values of (5.25) are found by solving

$$\frac{d\sigma_i(t)}{dt} = \left(ac - b + bc\,(T_{i-1} - t)\right) \mathrm{e}^{-c(T_{i-1}-t)} = 0$$

to get

$$T_{i-1} - t = \frac{1}{c} - \frac{a}{b}.$$

The second derivative of (5.25) at that time is

$$\left.\frac{d^2\sigma_i(t)}{dt^2}\right|_{T_{i-1}-t=\frac{1}{c}-\frac{a}{b}} = -bc\mathrm{e}^{-c(T_{i-1}-t)}.$$

Since $c > 0$, the extremum is a maximum when $b > 0$.

5.11. Solving the SDE (5.33), we get

$$\begin{aligned} S_{0,n}(T_0) &= S_{0,n}(t)\exp\left(\int_t^{T_0}\sigma_{0,n}(s)dW^A(s) - \frac{1}{2}\int_t^{T_0}\sigma_{0,n}(s)^2ds\right) \\ &= S_{0,n}(t)\exp\left(-s_{0,n}X - \frac{1}{2}s_{0,n}^2\right), \end{aligned}$$

where

$$s_{0,n} = \sqrt{\int_t^{T_0}\sigma_{0,n}(s)^2ds}$$

and

$$X = -\frac{1}{s_{0,n}}\int_t^{T_0}\sigma_{0,n}(s)dW^A(s)$$

is independent of $\mathcal{F}_t$ and normally distributed with mean 0 and variance 1. We have

$$\begin{aligned} \mathbf{PSwpt}_{0,n}(t) &= A_{0,n}(t)\mathbb{E}_{P_A}\left((S_{0,n}(T_0)-K)^+\middle|\mathcal{F}_t\right) \\ &= A_{0,n}(t)\left(\mathbb{E}_{P_A}\left(S_{0,n}(T_0)\mathbf{1}_{\{S_{0,n}(T_0)\geq K\}}\middle|\mathcal{F}_t\right) - K\mathbb{E}_{P_A}\left(\mathbf{1}_{\{S_{0,n}(T_0)\geq K\}}\middle|\mathcal{F}_t\right)\right). \end{aligned}$$

Observe that

$$S_{0,n}(T_0) \geq K \iff X \leq \frac{\ln\frac{S_{0,n}(t)}{K} - \frac{1}{2}s_{0,n}^2}{s_{0,n}} = d_-$$

As a result,

$$\begin{aligned} \mathbb{E}_{P_A}\left(\mathbf{1}_{\{S_{0,n}(T_0)\geq K\}}\middle|\mathcal{F}_t\right) &= \mathbb{E}_{P_A}\left(\mathbf{1}_{\{X\leq d_-\}}\middle|\mathcal{F}_t\right) \\ &= \int_{-\infty}^{d_-}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{x^2}{2}}dx = N(d_-) \end{aligned}$$

and

$$
\begin{aligned}
\mathbb{E}_{P_A}\left(S_{0,n}(T_0)\mathbf{1}_{\{S_{0,n}(T_0)\geq K\}} \middle| \mathcal{F}_t \right) &= S_{0,n}(t)\mathbb{E}_{P_A}\left(e^{-s_{0,n}X-\frac{1}{2}s_{0,n}^2}\mathbf{1}_{\{X\leq d_-\}} \middle| \mathcal{F}_t \right) \\
&= S_{0,n}(t)\int_{-\infty}^{d_-} e^{-s_{0,n}x-\frac{1}{2}s_{0,n}^2}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx \\
&= S_{0,n}(t)\int_{-\infty}^{d_-} \frac{1}{\sqrt{2\pi}}e^{-\frac{\left(x+s_{0,n}\right)^2}{2}}\,dx \\
&= S_{0,n}(t)\int_{-\infty}^{d_-+s_{0,n}} \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\,dy = S_{0,n}(t)N(d_+),
\end{aligned}
$$

where

$$
d_+ = d_- + s_{0,n} = \frac{\ln\frac{S_{0,n}(t)}{K} + \frac{1}{2}s_{0,n}^2}{s_{0,n}}.
$$

It follows that

$$
\mathbf{PSwpt}_{0,n}(t) = A_{0,n}(t)\left(S_{0,n}(t)N(d_+) - KN(d_-)\right).
$$

5.12. We need to switch from the terminal measure P_{T_n} to the swap measure P_A to transform the SDE (5.12) for the forward rates $F_i(t)$ under P_{T_n} into the SDE (5.35) under P_A. The numeraires associated with P_{T_n} and P_A are $B(t,T_n)$ and $A_{0,n}(t)$, respectively, so the Radon–Nikodym derivative of P_A with respect to P_{T_n} is

$$
\frac{dP_A}{dP_{T_n}} = \frac{B(0,T_n)}{A_{0,n}(0)}\frac{A_{0,n}(T_n)}{B(T_n,T_n)},
$$

and the corresponding Radon–Nikodym density process is

$$
\begin{aligned}
\xi_n^A(t) = \mathbb{E}_{P_{T_n}}\left(\frac{dP_A}{dP_{T_n}} \middle| \mathcal{F}_t \right) &= \frac{B(0,T_n)}{A_{0,n}(0)}\frac{A_{0,n}(t)}{B(t,T_n)} \\
&= \frac{B(0,T_n)}{A_{0,n}(0)}\sum_{j=1}^{n}\tau_j\frac{B(t,T_j)}{B(t,T_n)}.
\end{aligned}
$$

In Section 5.3 we derived the SDE (5.10) for $\xi_n^k(t) = \frac{B(0,T_n)}{B(0,T_k)}\frac{B(t,T_k)}{B(t,T_n)}$, which implies that

$$
d\left(\frac{B(t,T_k)}{B(t,T_n)}\right) = \frac{B(t,T_k)}{B(t,T_n)}\sum_{l=k+1}^{n}\frac{\tau_l\sigma_l(t)F_l(t)}{1+\tau_lF_l(t)}dZ_l^n(t).
$$

As a result,

$$\begin{aligned} d\xi_n^A(t) &= \frac{B(0,T_n)}{A_{0,n}(0)} \sum_{k=1}^{n-1} \tau_k d\left(\frac{B(t,T_k)}{B(t,T_n)}\right) \\ &= \frac{B(0,T_n)}{A_{0,n}(0)} \sum_{k=1}^{n-1} \tau_k \frac{B(t,T_k)}{B(t,T_n)} \sum_{l=k+1}^{n} \frac{\tau_l \sigma_l(t) F_l(t)}{1+\tau_l F_l(t)} dZ_l^n(t) \\ &= \frac{B(0,T_n)}{A_{0,n}(0)} \sum_{l=2}^{n} \sum_{k=1}^{l-1} \tau_k \frac{B(t,T_k)}{B(t,T_n)} \frac{\tau_l \sigma_l(t) F_l(t)}{1+\tau_l F_l(t)} dZ_l^n(t) \\ &= \xi_n^A(t) \sum_{l=2}^{n} \sum_{k=1}^{l-1} \tau_k \frac{B(t,T_k)}{A_{0,n}(t)} \frac{\tau_l \sigma_l(t) F_l(t)}{1+\tau_l F_l(t)} dZ_l^n(t). \end{aligned}$$

By following the same argument as in Section 5.3 involving orthogonal Brownian motions, we can conclude by using the Girsanov theorem that

$$Z_i^A(t) = Z_i^n(t) - \int_0^t \sum_{l=2}^{n} \sum_{k=1}^{l-1} \tau_k \frac{B(s,T_k)}{A_{0,n}(s)} \frac{\tau_l \sigma_l(s) F_l(s)}{1+\tau_l F_l(s)} \rho_{l,i} ds$$

for $i = 1, \ldots, n$ are correlated Brownian motions under the swap measure P_A such that

$$dZ_i^A(t) dZ_j^A(t) = \rho_{i,j} dt$$

for all $i, j = 1, \ldots, n$. Substituting this into the SDE (5.12), we obtain

$$dF_i(t) = \mu_i^A(t) F_i(t) dt + \sigma_i(t) F_i(t) dZ_i^A(t)$$

with

$$\mu_i^A(t) = \mu_i^n(t) + \sum_{k=2}^{n} \sum_{l=1}^{k-1} \tau_l \frac{B(t,T_l)}{A_{0,n}(t)} \frac{\tau_k \rho_{k,i} \sigma_i(t) \sigma_k(t) F_k(t)}{1+\tau_k F_k(t)}.$$

Using (5.13) for $\mu_i^n(t)$, we therefore get

$$\begin{aligned}
\mu_i^A(t) &= -\sum_{k=i+1}^{n} \frac{\tau_k \rho_{k,i}\sigma_i(t)\sigma_k(t)F_k(t)}{1+\tau_k F_k(t)} \\
&\quad + \sum_{k=2}^{n}\sum_{l=1}^{k-1} \tau_l \frac{B(t,T_l)}{A_{0,n}(t)} \frac{\tau_k \rho_{k,i}\sigma_i(t)\sigma_k(t)F_k(t)}{1+\tau_k F_k(t)} \\
&= \sum_{k=2}^{i}\sum_{l=1}^{k-1} \tau_l \frac{B(t,T_l)}{A_{0,n}(t)} \frac{\tau_k \rho_{k,i}\sigma_i(t)\sigma_k(t)F_k(t)}{1+\tau_k F_k(t)} \\
&\quad + \sum_{k=i+1}^{n}\left(\sum_{l=1}^{k-1} \tau_l \frac{B(t,T_l)}{A_{0,n}(t)} - 1\right) \frac{\tau_k \rho_{k,i}\sigma_i(t)\sigma_k(t)F_k(t)}{1+\tau_k F_k(t)} \\
&= \sum_{k=2}^{i}\sum_{l=1}^{k-1} \tau_l \frac{B(t,T_l)}{A_{0,n}(t)} \frac{\tau_k \rho_{k,i}\sigma_i(t)\sigma_k(t)F_k(t)}{1+\tau_k F_k(t)} \\
&\quad - \sum_{k=i+1}^{n}\sum_{l=k}^{n} \tau_l \frac{B(t,T_l)}{A_{0,n}(t)} \frac{\tau_k \rho_{k,i}\sigma_i(t)\sigma_k(t)F_k(t)}{1+\tau_k F_k(t)}
\end{aligned}$$

since

$$1 = \sum_{l=1}^{n} \tau_l \frac{B(t,T_l)}{A_{0,n}(t)} = \sum_{l=1}^{k-1} \tau_l \frac{B(t,T_l)}{A_{0,n}(t)} + \sum_{l=k}^{n} \tau_l \frac{B(t,T_l)}{A_{0,n}(t)}.$$

Changing the order of summation, we can also write this as

$$\begin{aligned}
\mu_i^A(t) &= \sum_{l=1}^{i-1} \tau_l \frac{B(t,T_l)}{A_{0,n}(t)} \sum_{k=l+1}^{i} \frac{\tau_k \rho_{k,i}\sigma_i(t)\sigma_k(t)F_k(t)}{1+\tau_k F_k(t)} \\
&\quad - \sum_{l=i+1}^{n} \tau_l \frac{B(t,T_l)}{A_{0,n}(t)} \sum_{k=i+1}^{l} \frac{\tau_k \rho_{k,i}\sigma_i(t)\sigma_k(t)F_k(t)}{1+\tau_k F_k(t)}.
\end{aligned}$$

Chapter 6

6.1. We proceed by induction on m. Denote by $\eta^{(m)}(\theta)$ the $n \times m$ matrix with entries

$$\begin{aligned}
\eta_{i,j}^{(m)}(\theta) &= \cos\theta_{i,j} \prod_{k=1}^{j-1} \sin\theta_{i,k} \quad \text{for } 1 \le j < m, \\
\eta_{i,m}^{(m)}(\theta) &= \prod_{k=1}^{m-1} \sin\theta_{i,k}.
\end{aligned}$$

For $m = 2$ the ith row of $\eta^{(2)}(\theta)$ is represented by the vector

$$\left[\begin{array}{cc} \eta_{i,1}^{(2)}(\theta) & \eta_{i,2}^{(2)}(\theta) \end{array}\right] = \left[\begin{array}{cc} \cos\theta_{i,1} & \sin\theta_{i,1} \end{array}\right],$$

whose squared Euclidean norm is

$$\left(\eta_{i,1}^{(2)}\right)^2 + \left(\eta_{i,2}^{(2)}\right)^2 = (\cos\theta_{i,1})^2 + (\sin\theta_{i,1})^2 = 1.$$

Now suppose that we have already proved for some $m \geq 2$ that the squared Euclidean norm of the ith row

$$\left[\begin{array}{cccc} \eta_{i,1}^{(m)}(\theta) & \eta_{i,2}^{(m)}(\theta) & \cdots & \eta_{i,m}^{(m)}(\theta) \end{array}\right]$$

of $\eta^{(m)}(\theta)$ is equal to 1, that is,

$$\left(\eta_{i,1}^{(m)}(\theta)\right)^2 + \cdots + \left(\eta_{i,m}^{(m)}(\theta)\right)^2 = 1.$$

Observe that

$$\begin{aligned}
\eta_{i,j}^{(m+1)}(\theta) &= \eta_{i,j}^{(m)}(\theta), \quad \text{for } 1 \leq j < m, \\
\eta_{i,m}^{(m+1)}(\theta) &= \cos\theta_{i,m}\eta_{i,m}^{(m)}(\theta), \\
\eta_{i,m}^{(m+1)}(\theta) &= \sin\theta_{i,m}\eta_{i,m}^{(m)}(\theta),
\end{aligned}$$

It follows by the induction hypothesis that the square of the Euclidean norm of the ith row

$$\left[\begin{array}{cccc} \eta_{i,1}^{(m+1)}(\theta) & \eta_{i,2}^{(m+1)}(\theta) & \cdots & \eta_{i,m+1}^{(m+1)}(\theta) \end{array}\right]$$

of $\eta^{(m+1)}(\theta)$ is

$$\begin{aligned}
&\left(\eta_{i,1}^{(m+1)}(\theta)\right)^2 + \cdots + \left(\eta_{i,m+1}^{(m+1)}(\theta)\right)^2 \\
&= \left(\eta_{i,1}^{(m)}(\theta)\right)^2 + \cdots + \left(\eta_{i,m-1}^{(m)}(\theta)\right)^2 + (\cos\theta_{i,m})^2\left(\eta_{i,m}^{(m)}(\theta)\right)^2 + (\sin\theta_{i,m})^2\left(\eta_{i,m}^{(m)}(\theta)\right)^2 \\
&= \left(\eta_{i,1}^{(m)}(\theta)\right)^2 + \cdots + \left(\eta_{i,m-1}^{(m)}(\theta)\right)^2 + \left(\eta_{i,m}^{(m)}(\theta)\right)^2 \\
&= 1,
\end{aligned}$$

completing the induction argument.

6.2. Since

$$\rho(\theta) = \eta(\theta)\eta(\theta)^\top,$$

it follows that for any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$

$$x\rho(\theta)x^\top = x\eta(\theta)\eta(\theta)^\top x^\top = (x\eta(\theta))\,(x\eta(\theta))^\top = \|x\eta(\theta)\|^2 \geq 0,$$

where $\|\cdot\|$ represents the Euclidean norm in $\mathbb{R}^n$. This means that $\rho(\theta)$ is a positive semidefinite $n \times n$ matrix.

6.3. To verify the inequality

$$n\,(m-1)-\frac{m\,(m-1)}{2}\le\frac{n\,(n-1)}{2}$$

we move all terms to one side and factorise:

$$\frac{n\,(n-1)}{2}-n\,(m-1)+\frac{m\,(m-1)}{2}=\frac{(n+1-m)\,(n-m)}{2}\ge 0.$$

The expression is non-negative for any $m\le n$, which proves the above inequality.

Chapter 7

7.1. By the Bayes formula for conditional expectation (see [PF]),

$$\mathbb{E}_{P_{T_{i-1}}}(L(T_{i-1},T_i)|\mathcal{F}_t)\,\mathbb{E}_{P_{T_i}}\left(\frac{dP_{T_{i-1}}}{dP_{T_i}}\middle|\mathcal{F}_t\right)=\mathbb{E}_{P_{T_i}}\left(L(T_{i-1},T_i)\frac{dP_{T_{i-1}}}{dP_{T_i}}\middle|\mathcal{F}_t\right).$$

The Radon–Nikodym derivative of $P_{T_{i-1}}$ with respect to P_{T_i} is

$$\frac{dP_{T_{i-1}}}{dP_{T_i}}=\frac{B(0,T_i)}{B(0,T_{i-1})}\frac{B(T_{i-1},T_{i-1})}{B(T_{i-1},T_i)}=\frac{B(0,T_i)}{B(0,T_{i-1})}\left(1+\tau_iL(T_{i-1},T_i)\right).$$

It follows that

$$\begin{aligned}\mathbb{E}_{P_{T_i}}\left(\frac{dP_{T_{i-1}}}{dP_{T_i}}\middle|\mathcal{F}_t\right)&=\frac{B(0,T_i)}{B(0,T_{i-1})}\mathbb{E}_{P_{T_i}}\left(\frac{B(T_{i-1},T_{i-1})}{B(T_{i-1},T_i)}\middle|\mathcal{F}_t\right)\\&=\frac{B(0,T_i)}{B(0,T_{i-1})}\frac{B(t,T_{i-1})}{B(t,T_i)}\end{aligned}$$

since $\frac{B(t,T_{i-1})}{B(t,T_i)}$ is a martingale under P_{T_i}. Moreover,

$$\mathbb{E}_{P_{T_i}}\left(L(T_{i-1},T_i)\frac{dP_{T_{i-1}}}{dP_{T_i}}\middle|\mathcal{F}_t\right)=\frac{B(0,T_i)}{B(0,T_{i-1})}\mathbb{E}_{P_{T_i}}(L(T_{i-1},T_i)\,(1+\tau_iL(T_{i-1},T_i))|\mathcal{F}_t).$$

Since $\frac{\mathbf{Lia}_i(t)}{B(t,T_{i-1})}$ is a martingale under $P_{T_{i-1}}$, as a result we have

$$\begin{aligned}\mathbf{Lia}_i(t)&=B(t,T_{i-1})\mathbb{E}_{P_{T_{i-1}}}(\mathbf{Lia}_i(T_{i-1})|\mathcal{F}_t)\\&=\tau_iB(t,T_{i-1})\mathbb{E}_{P_{T_{i-1}}}(L(T_{i-1},T_i)|\mathcal{F}_t)\\&=\tau_iB(t,T_i)\mathbb{E}_{P_{T_i}}(L(T_{i-1},T_i)\,(1+\tau_iL(T_{i-1},T_i))|\mathcal{F}_t)\\&=\tau_iB(t,T_i)\mathbb{E}_{P_{T_i}}\left(L(T_{i-1},T_i)+\tau_iL(T_{i-1},T_i)^2\middle|\mathcal{F}_t\right)\\&=\tau_iB(t,T_i)\left(F_i(t)+\tau_i\mathbb{E}_{P_{T_i}}\left(L(T_{i-1},T_i)^2\middle|\mathcal{F}_t\right)\right)\\&=\tau_iB(t,T_i)\bar{F}_i(t).\end{aligned}$$

Finally, because $\mathbb{E}_{P_{T_i}}\left(L(T_{i-1},T_i)^2\middle|\mathcal{F}_t\right)$ is a martingale under P_{T_i} and so is $F_i(t)$, it follows that $\bar{F}_i(t)$ is also is a martingale under P_{T_i}.

7.2. Solving the SDE

$$dF_i(t) = \sigma_i(t)F_i(t)dZ_i^i(t),$$

we obtain

$$F_i(t) = F_i(0)\exp\left(\int_0^t \sigma_i(s)dZ_i^i(s) - \frac{1}{2}\int_0^t \sigma_i(s)^2 ds\right).$$

It follows that

$$L(T_{i-1},T_i)^2 = F_i(T_{i-1})^2 = F_i(0)^2\exp\left(2\int_0^{T_{i-1}} \sigma_i(s)dZ_i^i(s) - \int_0^{T_{i-1}} \sigma_i(s)^2 ds\right).$$

Since the volatility $\sigma_i(t)$ is deterministic and $Z_i^i(t)$ is a Brownian motion under the forward measure P_{T_i}, it follows that $\int_0^{T_{i-1}} \sigma_i(s)dZ_i^i(s)$ has the normal distribution with mean 0 and variance

$$\int_0^{T_{i-1}} \sigma_i(s)^2 ds = v_i^2 T_{i-1}$$

under P_{T_i}. As a result,

$$\begin{aligned}\mathbb{E}_{P_{T_i}}\left(L(T_{i-1},T_i)^2\right) &= F_i(0)^2\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi v_i^2 T_{i-1}}}\mathrm{e}^{-\frac{x^2}{2v_i^2T_{i-1}}}\mathrm{e}^{2x-v_i^2T_{i-1}}dx\\ &= F_i(0)^2\mathrm{e}^{v_i^2T_{i-1}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi v_i^2 T_{i-1}}}\mathrm{e}^{-\frac{\left(x-2v_i^2T_{i-1}\right)^2}{2v_i^2T_{i-1}}}dx\\ &= F_i(0)^2\mathrm{e}^{v_i^2T_{i-1}}.\end{aligned}$$

7.3. Using the expression (7.5) of the bond $B(T_i,T_j)$ discounted by the annuity $A_{i,k}(T_i)$ is by linear function of the swap rate, namely

$$\frac{B(T_i,T_j)}{A_{i,k}(T_i)} = \alpha + \beta_j S_{i,k}(T_i),$$

we can write

$$\sum_{j=i+1}^{k}\tau_j\frac{B(T_i,T_j)}{A_{i,k}(T_i)} = \alpha\sum_{j=i+1}^{k}\tau_j + S_{i,k}(T_i)\sum_{j=i+1}^{k}\tau_j\beta_j = 1.$$

Since by (7.6) we have

$$\beta_j = \frac{1}{S_{i,k}(0)}\left(\frac{B(0,T_j)}{A_{i,k}(0)} - \alpha\right),$$

it follows that

$$\alpha \sum_{j=i+1}^{k} \tau_j + \frac{S_{i,k}(T_i)}{S_{i,k}(0)} \left[1 - \alpha \sum_{j=i+1}^{k} \tau_j \right] = 1.$$

For this to be valid we must have

$$\alpha \sum_{j=i+1}^{k} \tau_j = 1,$$

i.e. (7.7) must hold true.

7.4. To approximate the price at time 0 of a set-in-arrears CMS we use (7.8) and (7.11) with $l = i$ to get

$$\begin{aligned}
\mathbf{CMSia}(0) &= \sum_{i=0}^{n-1} \tau_{i+1} B(0, T_i) \mathbb{E}_{P_{T_i}} (S_{i,i+m}(T_i) - K) \\
&= \sum_{i=0}^{n-1} \tau_{i+1} A_{i,i+m}(0) \left(\alpha S_{i,i+m}(0) + \beta_i \mathbb{E}_{P_A} \left(S_{i,i+m}(T_i)^2 \right) \right) \\
&\qquad - K \sum_{i=0}^{n-1} \tau_{i+1} B(0, T_i) \\
&\approx \sum_{i=0}^{n-1} \tau_{i+1} A_{i,i+m}(0) \left(\alpha S_{i,i+m}(0) + \beta_i S_{i,i+m}(T_i)^2 \mathrm{e}^{v_{i,i+m}^2 T_i} \right) \\
&\qquad - K \sum_{i=0}^{n-1} \tau_{i+1} B(0, T_i),
\end{aligned}$$

where

$$\alpha = \left(\sum_{j=i+1}^{i+m} \tau_j \right)^{-1}, \quad \beta_i = \frac{1}{S_{i,i+m}(0)} \left(\frac{B(0, T_i)}{A_{i,i+m}(0)} - \alpha \right).$$

7.5. The payment at time T_i is

$$c_i = \tau_i (\alpha F_i(T_{i-1}) + X - K) \mathbf{1}_{\{\max\{F(s_i^j; s_i^j; s_i^j + \delta) : j = 1, \ldots, n_i\} < u\}}$$

for $i = 1, \ldots, n$. In particular, if the trigger level has been hit in any of the previous accrual periods, the indicator function and hence the payment c_i will be zero. The value of the trigger swap at time 0 is

$$\sum_{i=1}^{n} \mathbb{E}_{P_L} \left(\frac{c_i}{L(T_i)} \right),$$

where the discrete money market $L(t)$ is used as numeraire and the expectation is taken under the spot LIBOR measure P_L.