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# Problems for Chapter 13 of *Advanced Mathematics for Applications*

## The Legendre Equation

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1. By following the steps indicated in section 13.3.3 p. 322 prove the recurrence relations (13.3.18), (13.3.19) and (13.3.20).

2. Deduce the relations (13.3.21), (13.3.22) and (13.3.23) from (13.3.18), (13.3.19) and (13.3.20).

3. By using the generating function (13.3.13) p. 320 for the Legendre polynomials prove that

$$\int_{-1}^{1} (\cosh 2x - z)^{-1/2} P_n(z)\, \mathrm{d}z \;=\; \frac{\sqrt{2}}{n+\frac{1}{2}} \exp[-(2n+1)x]\,.$$

4. Prove that

$$\int_0^1 P_{2\ell}(\mu)\, \mathrm{d}\mu \;=\; 0 \qquad \text{while} \qquad 2(\ell+1)\int_0^1 P_{2\ell+1}(\mu)\, \mathrm{d}\mu \;=\; P_{2\ell}(0)\,,$$

with the first relation only valid for $\ell \neq 0$.

5. Prove that

$$\int_{x_1}^{x_2} (1-x^2) P'_m(x) P'_n(x)\, \mathrm{d}x \;=\; \left[(1-x^2) P_m P'_n\right]_{x_1}^{x_2} + n(n+1)\int_{x_1}^{x_2} P_m(x) P_n(x)\, \mathrm{d}x\;.$$

From this result deduce the value of $\int_{-1}^{1}(1-x^2)P'_m(x)P'_n(x)\,\mathrm{d}x$.

6. Prove that, if $m$ and $n$ are integers with $m < n$, both being even or odd, then

$$\int_{-1}^{1} P'_m(x) P'_n(x)\, \mathrm{d}x \;=\; m(m+1)\,.$$

7. Show that the integral

$$\int_{-1}^{1} x(1-x^2) P'_m(x) P'_n(x)\, \mathrm{d}x$$

vanishes unless $m = n \pm 1$. Calculate its value in these two particular cases.

8. Show that, if $m > n$ and $m - n$ is even, then

$$\int_0^1 P_m(x) P_n(x)\, \mathrm{d}x \;=\; \frac{1}{2m+1} \qquad \text{for} \qquad m = n$$

and vanishes otherwise.

9. By using results from the theory of the Legendre polynomials show that, for $a > b$,

$$\int_0^{\pi} (a + b\cos x)^n\, \mathrm{d}x \;=\; \pi (a^2-b^2)^{n/2} P_n\left(\frac{a}{\sqrt{a^2-b^2}}\right)\,.$$

10. By using results from the theory of the Legendre polynomials show that

$$\int_0^{\pi} (\cos t + i \sin t \cos x)^n \, dx = \pi P_n(\cos t) .$$

11. Show that the generic coefficient $a_n$ of the expansion $f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$ may be written

$$a_n = \frac{n + \frac{1}{2}}{2^n n!} \int_{-1}^{1} (1 - x^2)^n f^{(n)}(x) \, dx ,$$

provided the function $f$ is sufficiently smooth.

12. Prove the relation

$$(1 - x^2) P_n \left( \frac{1 + x}{1 - x} \right) = \sum_{k=0}^{\infty} \left( \begin{array}{c} n \\ k \end{array} \right)^2 x^k .$$

(Apply Leibnitz's rule for the derivative of a product to Rodriguez's formula and make a suitable change of variables.)

13. Show that

$$P_\ell(z) Q_{\ell-1}(z) - P_{\ell-1}(z) Q_\ell(z) = \frac{1}{\ell} ,$$

and deduce from this relation the expression (13.5.2) p. 327 for $Q_\ell(z)$.

14. In spherical polar coordinates the equation of an axisymmetric, nearly spherical surface of revolution is $r = a[1 + \epsilon P_n(\cos\theta)]$ where $|\epsilon| \ll 1$. Show that, if terms of order $\epsilon^3$ and higher are neglected, the volume enclosed by the surface and the area of the surface are given, respectively, by

$$\frac{4}{3}\pi a^3 \left[ 1 + \frac{3\epsilon^2}{2n + 1} \right] \qquad \text{and} \qquad 4\pi a^2 \left[ 1 + \frac{1}{2} \frac{n^2 + n + 2}{2n + 1} \epsilon^2 \right] .$$

15. Reduce to integrals the calculation of the infinite sums

$$S_1(z) = \sum_{n=0}^{\infty} \frac{1}{n + a} P_n(z) , \qquad S_2(z) = \sum_{n=0}^{\infty} \frac{1}{(n + a)(n + b)} P_n(z) ,$$

where $a$, $b$ are real positive constants. Find also a relation which expresses $S_2$ in terms of $S_1$.

16. Show that, if $J_{n+1/2}$ is a Bessel function of half-integer order $n + 1/2$ (section 12.3 p.309), then

$$\int_{-1}^{1} e^{izt} P_n(t) \, dt = \left( \frac{2\pi}{z} \right)^{1/2} i^n J_{n+1/2}(z) .$$

You can $(a)$ prove the statement by induction starting with $n = 0$, or $(b)$ prove that both sides of the equality satisfy the same differential equation and the same boundary conditions, or $(c)$ that they both satisfy the same recurrence relation and conditions sufficient to make them equal rather than merely proportional.

17. Use the recurrence relation expressing $z P_\ell(z)$ in terms of $P_{\ell\pm1}(z)$ to find the value of $P_{2\ell}(0)$ given on p. 319.

18. The generating function (p. 320) for the Hermite polynomials (section 13.9 p. 334) is given by

$$\exp(2tx - t^2) \;=\; \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x)\, t^n \,.$$

By following a method adapted from that described in section 13.3.3 p. 322 prove the relations

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) \;=\; 0 \qquad (n \geq 1)\,,$$

$$H_n'(x) \;=\; 2nH_{n-1}(x) \qquad (n \geq 1)\,,$$

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) \;=\; 0 \qquad (n \geq 0)\,.$$

19. Consider the eigenvalue problem

$$-\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + x^2 u \;=\; \lambda u \,.$$

Make the substitution $u(x) = e^{-x^2/2} v(x)$ and show, using the recurrence relations of the previous problem, that the resulting equation is solved by the Hermite polynomials provided $\lambda$ takes certain values.

20. Prove that

$$\int_{-\infty}^{\infty} \exp[-(x-y)^2]\, H_n(x)\, \mathrm{d}x \;=\; (2y)^n \sqrt{\pi}\,.$$