The origins and development of the logit model

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Abstract

This is and updated and somewhat extended version of Chapter 9 of *Logit Models from Economics and Other Fields* (Cambridge University Press, 2003) which includes additional material obtained since the completion of that book. The text has been adapted so that this paper can be read independently.

The paper describes the origins of the logistic function and its history up to the adoption of the logit in bio-assay and the beginning of its wider acceptance in statistics. Its roots spread back to the 19th century, when the function was invented to describe population growth and given its name by the Belgian mathematician Verhulst. Subsequent events have been determined decisively by the individual actions and personal histories of a few scholars: the rediscovery of the growth function is due to Pearl and Reed, the survival of the term *logistic* to Yule, and the introduction of the function in bio-assay (and hence in statistics in general) to Berkson.

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1 Introduction

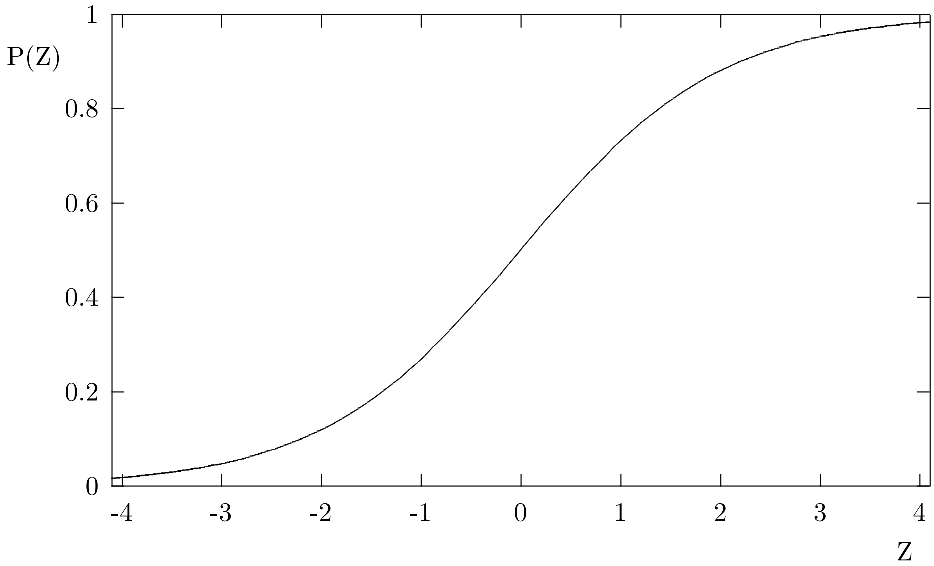

Figure 1. The logistic curve P(Z)

The sigmoid curve of Figure 1 is traced by the *logistic function*

$$P(Z) = \frac{expZ}{1 + expZ}. \tag{1}$$

P behaves like the distribution function of a symmetrical density, with midpoint zero; as Z moves through the real number axis, P rises monotonically between the bounds of zero and 1. The meaning of this function varies according to the the definition of the variables. In the logit version of bio-assay, P is the probability of a binary outcome (the survival or death of an organism), and $Z = \alpha + \beta X$, with X a continuous stimulus or exposure variable (like the dosage of an insecticide); α determines the location of the curve on the X-axis, and β its slope. In logistic regression there are several determinants of P, and $Z = x^T\beta$, with x a vector of covariates (including a unit constant) and β their coefficients. But the logistic function was originally designed to describe the course of a *proportion* P over time t, with $Z = \alpha + \beta t$; it is a *growth curve*, since $P(t)$ rises monotonically with t.

Over a fairly wide central range, for values of P from .3 to .7, the shape of the logistic curve closely resembles the normal probability distribution function. The two functions

$$P_l(x) = \frac{exp(\beta x)}{1 + exp(\beta x)}. \tag{2}$$

and

$$P_n(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\inf}^{x} \exp\{-1/2(u/\sigma)^2\} du. \tag{3}$$

both pass through the point $(0, .5)$, and they can be made almost to coincide upon a suitable adjustment of β and σ. This is a sheer algebraic coincidence, for there appears to be no intrinsic relation between the two forms.

2 The origins of the logistic function

The logistic function was invented in the 19th century for the description of the growth of populations and the course of autocatalytic chemical reactions, or *chain reactions*. In either case we consider the time path of a quantity $W(t)$ and its growth rate

$$\dot{W}(t) = \mathrm{d}W(t)/\mathrm{d}t. \tag{4}$$

The simplest assumption is that $\dot{W}(t)$ is proportional to $W(t)$

$$\dot{W}(t) = \beta W(t),\ \beta = \dot{W}(t)/W(t), \tag{5}$$

with β the constant rate of growth. This leads of course to exponential growth

$$W(t) = A \exp \beta t,$$

where A is sometimes replaced by the initial value $W(0)$. With $W(t)$ the human population of a country, this is a model of unopposed growth; as Malthus (1789) put it, a human population, left to itself, will increase *in geometric progression.* It is a reasonable model for a young and empty country like United States in its early years[1]. Like many others, Alphonse Quetelet (1795–1874), the Belgian astronomer turned statistician, was well aware that the indiscriminate extrapolation of exponential growth must lead to impossible values. He experimented with several adjustments of (5) and also asked

[1]Two hundred years later exponential growth played a major part in the *Report to the Club of Rome* of Meadows, Meadows, Randers, and Behrens (1972), and it is still implicit in many economic analyses.

his pupil, the mathematician Pierre–François Verhulst (1804–1849), to look into the problem.

Like Quetelet, Verhulst approached the problem by adding an extra term to (5) to represent the increasing resistance to further growth, as in

$$\dot{W}(t) = \beta W(t) - \phi(W(t)). \tag{6}$$

and then experimenting with various forms of ϕ. The logistic appears when this is a simple quadratic, for in that case we may rewrite (6) as

$$\dot{W}(t) = \beta W(t)(\Omega - W(t)) \tag{7}$$

where Ω denotes the upper limit or *saturation level* of W, its asymptote as $t \to \infty$. Growth is now proportional both to the population already attained $W(t)$ and to the remaining room for further expansion $\Omega - W(t)$. If we express $W(t)$ as a proportion $P(t) = W(t)/\Omega$ this gives

$$P(t) = \beta P(t)\{1 - P(t)\}, \tag{8}$$

and the solution of this differential equation is

$$P(t) = \frac{\exp(\alpha + \beta t)}{1 + \exp(\alpha + \beta t)}, \tag{9}$$

which Verhulst named the *logistic* function. The population $W(t)$ then follows

$$W(t) = \Omega \frac{\exp(\alpha + \beta t)}{1 + \exp(\alpha + \beta t)}. \tag{10}$$

Verhulst published his suggestions between 1838 and 1847 in three papers. The first is a brief note in the *Correspondance Mathématique et Physique*, edited by Quetelet, in 1838. It contains the essence of the argument in four small pages, followed by a demonstration that the curve agrees very well with the actual course of the population of France, Belgium, Essex and Russia for periods up to 1833; Verhulst explains that he did his research a couple of years before, that he did not have the time for an update and that he publishes this note only at the insistence of Quetelet. He does not say how he fitted the curves. The second paper, in the *Proceedings* of the Belgian Royal Academy of 1845, is a much fuller account of the function and its properties. Here Verhulst names it the logistic, without further explanation: in a neat diagram, the *courbe logistique* is drawn alongside the *courbe logarithmique*, which we would nowadays call the exponential. Verhulst also determines the three parameters Ω, α and β of (10) by making the curve pass through three observed points. With data for some twenty or thirty years only this is a hazardous method, as is borne out by the resulting estimates of the limiting population Ω. Employing the known values of the Belgian population in 1815, 1830 and 1845 Verhulst finds a limiting population of 6.6 million for that country, and in a similar exercise 40 million for France: at present these populations number 10.2 and 58.7 million. In 1847 there followed a second paper in the *Proceedings*, which is chiefly notable for an adjustment of the correction term that leads to a much better estimate of 9.5 millions for the Belgian Ω.

Verhulst was in poor health and died in 1849. He was primarily a mathematician - professor of mathematics at the Belgian Military Academy - but sensitive to social and political issues. In his obituary of Verhulst, Quetelet (1850) attributes his early death to overwork and, rather curiously, to his great stature, as Verhulst was 1.89 meters or six feet tall. His discovery of the logistic curve was not taken up with much enthusiasm by Quetelet; as Vanpaemel (1987) has shown, the two men did not see eye to eye on the question of population growth. This may in part account for some curious elements in Quetelet's obituary; while ostensibly praising his lamented pupil, Quetelet stresses his impulsive nature and depicts him as a somewhat silly man. Quetelet recounts at length Verhulst's adventures in Rome. Verhulst was staying in that city in the summer of 1830, when the news broke of the revolution in Paris and of the Belgian secession from the Netherlands. These events moved him strongly and set him drafting a democratic constitution for the Papal State. He submitted this document to some cardinals he had met, who expressed great interest; still the police were called in, and Verhulst banished from Rome. He left under somewhat dramatic circumstances, having at first barricaded his apartment with the intention of withstanding a siege by the forces of law and order. But then he was only 26 years old at the time.

Quetelet did not pay much attention to the logistic curve in his writings; it is barely mentioned, in an aside, in Quetelet (1848). But Verhulst's work was quoted with approval by Liagre (1852), his colleague at the Military Academy, and in the second edition of this textbook Camille Peney repeats the estimation of Ω for Belgium on the basis of more recent population figures, arriving at a value of 13.7 millions.

As a model of population growth the logistic function was discovered anew in 1920 by Pearl and Reed. They were unaware of Verhulst's work (though not of the curves for autocatalytic reactions discussed presently), and they arrived independently at the logistic curve of (10). When this was fitted to Census figures of the U.S., again by making the curve pass through three points, it gave a good fit for the period from 1790 to 1910. But the estimate of Ω of 197 millions once more compares badly with the present value of about 270 millions. Along with the pursuit of many other interests, Pearl and his collaborators in the next twenty years went on to apply the logistic growth curve to almost any living population from fruit flies to the human population of the French colonies in North Africa as well as to the growth of cantaloupes.

In 1920, Raymond Pearl (1879–1940) had just been appointed Director of the Department of Biometry and Vital Statistics at Johns Hopkins University, and Lowell J. Reed (1886–1966) was his deputy (and his successor when a few years later Pearl was promoted to Professor of Biology). Pearl was trained as a biologist and acquired his statistics as a young man by spending the year 1905–1906 in London with Karl Pearson (and later quarrelling with him). He became a prodigious investigator and a prolific writer on a wide variety of phenomena like longevity, fertility, contraception, and the effects of alcohol and tobacco consumption on health, all subsumed under the heading of human biology. During World War I Pearl worked in the U.S. Food Administration, and this may account for his preoccupation with the food needs of a growing population in the 1920 paper. Reed, who was trained as a mathematician, made a quiet career in biostatistics; he excelled as a teacher and as an administrator, and was brought back in 1953 from retirement to serve as President of Johns Hopkins. Among his publications in the aftermath of the 1920 paper with Pearl is an application of the logistic curve to autocatalytic reactions, Reed and Berkson (1929). We shall hear more about this co-author in the next section.

Verhulst's work was rediscovered soon after Pearl and Reed's first paper of 1920. The immediate sequel, Pearl and Reed (1922), does not mention it; Verhulst's priority is first acknowledged in a footnote in Pearl (1922), and, at greater length, in Pearl and Reed (1923). In this paper, Pearl and Reed call Verhulst's papers *"long since forgotten"*, except for a single article by Du Pasquier (1918), and they then go out of their way to criticize that author for an *"entirely unjustified and in practice usually incorrect modification"* of Verhulst's formula, without substantiating this harsh judgment. In fact Du

Pasquier's paper is a harmless reflection on four mathematical theories of population, of a very formal and abstract character to the point of inanity. The four theories are ascribed to Halley, de Moivre, Euler and Verhulst, and these authors are briefly introduced; Halley, for example, as *"the famous astronomer"*, and Verhulst, rather oddly, as *"a Belgian who died in 1847"*. No references are given.

Louis–Gustave Du Pasquier (1876–1957), Professor of Mathematics at the University of Neuchâtel, took his degrees in mathematics in Zürich, but followed courses in the social sciences as well and spent the year 1900–1901 in Paris, taking courses at a variety of academic institutions. He may well have read about Verhulst in Liagre or elsewhere in the French literature, but I have been unable to find a useful reference to this effect in his textbook of probability (1926) . It is also not clear how Pearl learned about Verhulst, or, for that matter, about Du Pasquier[2].

The next important publication is Yule's Presidential Address to the Royal Statistical Society of 1925. Yule, who says he owes the reference to Pearl (1922), treats Verhulst much more handsomely than Pearl and Reed did, devoting an appendix to his work. Yule is also the first author to revive the name *logistic*, which is not used by Liagre or Du Pasquier nor by Pearl and Reed in their earlier references. By 1924, however, "logistic" is used as a commonplace term in the correspondence between Pearl and Yule, who were lifelong friends. It would take until 1933 for Miner (a collaborator of Pearl) to pay tribute to Verhulst, if in an oblique way: instead of reproducing at least one of Verhulst's papers, Miner translates Quetelet's obituary, and emphasises Verhulst's Roman imbroglio by adding an extract from the memoirs of Queen Hortense de Beauharnais recording this episode.

As we have already hinted there is another early root of the logistic function in chemistry, where it was employed (again with some variations) to describe the course of autocatalytic or chain reactions, where the product itself acts as a catalyst for the process while the supply of raw material is fixed. This leads naturally to a differential equation like (8) and hence to the logistic function for the time path of the amount of the reaction product. The review of the application of logistic curves to a number of such processes by Reed and Berkson (1929) quotes work of the German professor of chemistry

[2]The Pearl archives at the American Philosophical Society in Philadelphia contain Pearl's correspondence with several hundred individuals, but Du Pasquier is not among them.

Wilhelm Ostwald of 1883. Authors like Yule (1925) and Wilson (1925) were well aware of this strand of the literature.

The basic idea of logistic growth is simple and effective, and it is used to this day to model population growth and market penetration of new products and technologies. The introduction of mobile telephones is an autocatalytic process, and so is the spread of many new products and techniques in industry.

3 The invention of the probit and the advent of the logit in bio-assay

The invention of the probit model is usually credited to Gaddum (1933) and Bliss (1934a,1934b), but one look at the historical section of Finney (1971) or indeed at Gaddum's paper and his references will show that this is too simple. The roots of the method and in particular the transformation of frequencies to equivalent normal deviates can be traced to the German scholar Fechner (1801–1887). Stigler (1986) recounts how Fechner was drawn to study human responses to external stimuli by experimental test of the ability to distinguish differences in weight. The issue of the variability of human responses had been raised by astronomers, who relied on human observers of celestial phenomena and found that their readings showed much unaccountable variation. Fechner recognized that human response to an identical stimulus is not uniform, and he was the first to transform observed differences to equivalent normal deviates. The historical sketches of Finney (1971), Ch. 3.6, and of Aitchison and Brown (1957), Ch. 1.2, record a long line of largely independent rediscoveries of this approach that spans the seventy years from Fechner (1860) to the early 1930's when Gaddum and Bliss published their contributions. Both authors regard the assumption of a normal distribution as commonplace, and attach more importance to the logarithmic transformation of the stimulus. Their papers contain no major innovations, but they mark the emergence of a standard paradigm of bio-assay and of a new terminology. Gaddum wrote a comprehensive and authoritative report with the emphasis on practical aspects of the experiments and on the statistical interpretation of bio-assay, giving several worked examples from the medical and pharmaceutical literature. Bliss published two brief notes in *Science*, introducing the term *probit*; he followed this up with a series of articles setting out the maximum likelihood estimation of the probit curve, in one instance with assistance from R.A. Fisher, Bliss (1935). Both Gaddum and Bliss set standards of estimation; until the 1930's this was largely a matter of ad hoc numerical and graphical adjustment of curves to categorical data.

John Henry Gaddum (1900–1965) studied medicine at Cambridge but failed in his final examinations. He turned to pharmacology and worked under Trevan at the Wellcome Laboratories, then transferred to the National Institute for Medical Research (where he wrote the 1933 report) before embarking on an academic career of professorships in pharmacology in Cairo, London and Edinburgh. He was elected to the Royal Society in 1945 and knighted in 1964. To this day the British Pharmacological Society awards an annual Gaddum Memorial Prize for pharmaceutical research. Charles Ittner Bliss (1899–1979) studied as an entomologist at Ohio State University and was a field worker with the U.S. Department of Agriculture until this employment was terminated in 1933. He then spent two years in London studying statistics with R.A. Fisher, and Fisher found him a job as a statistician in Leningrad where he lived from 1936 and 1938. The political conditions were not propitious for serious research. Bliss returned to the Connecticut Agricultural Experiment Station, combining his work as a practising statistician with a Lecturership at Yale from 1942 until his retirement. He played an important role in the founding of the Biometric Society.

In their early writings on bio-assay both authors adhere firmly to the classical model of bio-assay, where the stimulus is determinate and responses are random because of the variability of individual tolerance levels. Bliss introduced the term *probit* (short for 'probability unit') originally as a convenient *scale* for normal deviates, but abandoned this within a year in favour of a different definition which has since been generally accepted. For any (relative) frequency f there is an equivalent normal deviate $\tilde{Z}$ such that the cumulative normal distribution at $\tilde{Z}$ equals f; $\tilde{Z}$ is the solution of

$$f = \frac{1}{\sqrt{2\pi}} \int_{-\inf}^{\tilde{Z}} \exp\{-(1/2)u^2\} du, \tag{11}$$

and this can be read off from a table of the normal distribution. The probit of f is this equivalent normal deviate $\tilde{Z}$, or initially $\tilde{Z}$ increased by 5; this ensures that the probit is almost always positive, which facilitates calculation. In the 1930's such additive constants were a common device. In the probit method probits of relative frequencies or of probabilities f are linearly related to (the logarithm of the) stimulus.

The acceptance of the probit method was aided by the articles of Bliss, who published regularly in this field until the 1950's, and by Finney and others (Gaddum returned to pharmacology). The full flowering of this school probably coincides with the first edition of Finney's monograph in 1947. Without the underlying theory of bio-assay, probit analysis was quickly used for any relation of a discrete binary outcome to one or more determinants. In economics and market research, for example, the first applications appear already in the 1950's: Farrell (1954) uses a probit model for the ownership of cars of different vintage as a function of household income, and Adam (1958) fits lognormal demand curves to survey data of the willingness to buy cigarette lighters and the like at various prices. The classic monograph on the lognormal distribution of Aitchison and Brown (1957) brought probit analysis to the notice of a wider audience of economists.

As far as I can see the introduction of the logistic as an alternative to the normal probability function is the work of a single person, namely Joseph Berkson (1899–1982), Reed's co-author of the paper on autocatalytic functions of 1929. Berkson read physics at Columbia, then went to Johns Hopkins for his M.D. and a doctorate in statistics in 1928. He stayed on as an assistant for three years and this is when he collaborated with Reed on autocatalytic functions. Berkson then moved to the Mayo Clinic where he remained for the rest of his working life as chief statistician. In the 1930's he published numerous papers on medical and public health matters, but in 1944 he turned his attention to the statistical methodology of bio-assay and proposed the use of the logistic instead of the normal probability function of (11), coining the term 'logit' by analogy to the 'probit' of Bliss (for which he was initially much derided). As we have indicated earlier the two functions are almost indistinguishable. By the inverse of the logistic function (1) we have

$$logit(P) = log\frac{P}{1-P} = Z, \tag{12}$$

which is of course much simpler than the definition of the probit of (11). The issue of logit versus probit was tangled by Berkson's simultaneous attacks on the method of maximum likelihood and his advocacy of minimum chi-squared estimation instead. Between 1944 and 1980 he wrote a large number of papers on both issues; examples are Berkson (1951) and Berkson (1980). He often adopted a somewhat provocative style, and much controversy ensued.

The close resemblance of the logistic to the normal distribution function must have been common knowledge among those who were familiar with the logistic; it had been demonstrated by Wilson (1925) and written up by Winsor (1932) (another collaborator of Pearl). Wilson was probably the first to publish an application of the logistic curve in bio-assay in Wilson and Worcester (1943), just before Berkson (1944). But it was Berkson who persisted and fought a long and spirited campaign which lasted for several decades.

Berkson's suggestion was not well received by the biometric establishment. In the first place, the logit was regarded as somewhat inferior and disreputable because unlike the probit it can not be related to an underlying (normal) distribution of tolerance levels. Aitchison and Brown (1957) dismiss the logit in a single sentence, because it *"lacks a well-recognized and manageable frequency distribution of tolerances which the probit curve does possess in a natural way"* (p.72). Berkson was aware of this defect and tried to remedy it by adapting the autocatalytic argument, in Berkson (1951), but this did not convince as this argument essentially deals with a process over time. In retrospect it is surprising that so much importance was attached to these somewhat ideological points of interpretation. At the time no one (not even Berkson) seems to have recognized the formidable power of the logistic's analytical properties. In the second place, Berkson's case for the logit was not helped by his simultaneous attacks on the established wisdom of maximum likelihood estimation and his advocacy of minimum chi-squared. The unpleasant atmosphere in which this discussion was conducted can be gauged from the acrimonious exchanges between R.A. Fisher and Berkson in Fisher (1954).

In the practical aspect of ease of computation the logit had a clear advantage over the probit, even with maximum likelihood estimation. To quote Cochran (from his comments on Fisher (1954), p.147) *".. the speed with which a new technique becomes widely used is considerably influenced by the simplicity or otherwise of the calculations that it requires. Next door to the lecture room in which the probit method is expounded one may still find the laboratory in which the workers compute their LD 50s by the* [much less sophisticated] *Behrens (Reed–Muench) method ..".* On this count the logit spread much more quickly in workfloor practice than in the academic discourse. Until the advent of the computer and the pocket calculator, some twenty years later, all numerical work was done by hand, that is with pencil and paper, sometimes aided by graphical inspection of 'freehand curves', 'fitted by eye'. For probit and logit analyses of grouped data or class frequencies there was graph paper with a special grid on which a probit or logit curve would appear as a straight line. Wilson (1925) had introduced the logistic (or 'autocatalytic') grid, and examples of lognormal paper can be found in

Aitchison and Brown (1957) and Adam (1958);[3] Berkson himself had designed logistic graph paper as well as several nomograms.[4] Numerical work was supported rather feebly by the slide rule and by mechanical calculating machines, driven by hand or powered by a small electric motor, which were capable of addition and multiplication; punched card equipment was helpful if numerous data had to be analysed. Values of the normal distribution (and of exponentials and logarithms) were obtained from printed tables like Pearson's *Biometrika* Tables or the *Statistical Tables* of Fisher and Yates (1938). From the first edition the latter carried specially designed tables for probit analysis (with auxiliary tables contributed by Bliss and by Finney), and from the fifth edition of 1957 onwards they also included special tables for logit analysis.

In time, the ideological conflict over bio-assay abated. Finney, who had ignored the logit in the second edition of his textbook of 1952, made amends in the third edition of 1970, recognizing (somewhat belatedly) that *"what matters is the dependence of P on dose and the unknown parameters, and the tolerance distribution is merely a substructure leading to this"* (p.47). In fact the narrow conflict between probit and logit in bio-assay had long been overtaken by independent developments in statistics and biometrics.

4 The ascent of the logit

When the ideological debate about logit and probit in bio-assay had abated, around 1960, the logit terminology and the logit transformation of (12) were soon much more widely adopted, and their origins forgotten. An accurate history of the adoption and further development of the logit would require an intimate knowledge of several quite distinct disciplines, for many new generalizations were introduced independently and in almost complete isolation in completely unrelated applied work. We shall here only briefly touch upon some major movements in statistics, in epidemiology, and in the social sciences and econometrics, without attempting a systematic treatment.

The earliest developments took place in the late 1950's and the 1960's in statistics and epidemiology. In statistics, the analytical advantages of the logit transformation as a means of dealing with discrete binary outcomes

[3]Finney (1971) traces the invention of the probability grid to a French artilleryman of the late 1890's.

[4]A nomogram is a graph from which one can read off a transformations, as from a table; sophisticated nomograms may permit the quick solution of more complicated equations.

were soon recognized. Cox was among the first to explore (and exploit) these possibilities; he wrote a series of papers between around 1960, and followed these up with an influential textbook in 1969. The logit model of bio-assay is easily generalized to logistic regression where binary outcomes are related to a number of determinants, without a specific theoretical background, and this statistical model proved as fertile as linear regression in an earlier era. Later, the link of the logistic model with discriminant analysis was recognized, and its ready association with loglinear models in general. In epidemiology, case-control studies began even earlier, and since these are directly concerned with odds, and odds ratios, the log-odds or logit transformation arises naturally. The practice had already called for a theoretical justification, especially of the sampling aspects, from an early date; see, for example, the work of Cornfield in the early 1950's.

Table 1. Number of articles in statistical journals containing the word 'probit' or 'logit'.

	probit	logit
1935 – 39	6	-
1940 – 44	3	1
1945 – 49	22	6
1950 – 54	50	15
1955 – 59	53	23
1960 – 64	41	27
1965 – 69	43	41
1970 – 74	48	61
1975 – 79	45	72
1980 – 84	93	147
1985 – 89	98	215
1990 – 94	127	311

The ascent of the logit in the statistical literature is illustrated in Table 1, which is drawn from the JSTOR electronic repertory of major statistical journals in the english language[5]. The table shows the number of articles which contain the word "probit" or "logit". It must be borne in mind that

[5]These are all the journals of the Royal Statistical Society and of the American Statistical Association; the *Annals of Applied Probability, Annals of (Mathematical) Statistics, Annals of Probability*, *Biometrics, Biometrika* and *Statistical Science*.

the overall number of articles in these journals increases substantially over time; from 1935 to 1985 it increased about eightfold. Up to around 1970 the relative numbers show the predominance of probit in bio-assay; then logit soars ahead - not because of the after-effects of a victory over probit in bio-assay, but because of its much wider use in statistical theory and applications generally.

Until about 1980 computational effort was still an important issue in the discussion of statistical techniques, but by then the computer revolution put an end to this. On the specific issue of estimating logit and probit analyses, maximum likelihood estimation became the norm when routines for this method, applicable to individual data, were included in commercial statistical program packages. This facility was probably first offered by the BMDP (or BIOMEDICAL DATA PROCESSING) program of 1977. By the time the first comprehensive textbook with medical applications of Hosmer and Lemeshow (1989) was published the use of such routines was taken for granted. Of the two causes Berkson advocated, minimum chi-squared estimation was effectively overtaken by the computer revolution, while the logit transformation of (12) was triumphant.

We conclude with some remarks on contributions from econometrics and the social sciences. We have earlier indicated that the probit model of bio-assay was readily adopted in these disciplines. The theoretical justification of bio-assay in terms of determinate stimulus and random thresholds was first jettisoned in the change to logistic regression, and then retrieved in the form of the latent regression equation model that is still dear to the behavioural sciences. This is probably due to McKelvey and Zavoina (1975), who introduce it in an ordered probit analysis of the voting behaviour of US Congressmen. An example of simultaneous independent discoveries is the generalization of logistic regression to the multinomial or polychotomous case. This was first set out, at some length, by Gurland, Lee, and Dahm (1960). Several years later it was put forward quite independently by the statistician Cox (1966) and by the biometric statistician Mantel (1966). And some years later again it was once more rediscovered independently by the econometrician Theil (1969), who arrived at it from the general perspective of modelling shares.

For a long time, logistic regression, whether in the binary or the multinomial context, was principally used as a technique, a simple tool without a specific underlying process and therefore without a characteristic interpretation. But in 1973 McFadden, working as a consultant for a Californian public transportation project, linked the multinomial logit to the theory of discrete choice from mathematical psychology. This provided a theoretical foundation of the logit model that is much more profound than any theory put forward for the use of the probit in bio-assay. It earned its author the Nobel prize in economics in 2000.

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